Equipartitioning Common Domains of Non-Atomic Measures

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§ 1. Introduction

Let $h(a_1, \dots, a_n)$ denote the *harmonic mean* $n(a_1^{-1} + \dots + a_n^{-1})^{-1}$ of positive numbers $a_1, \dots, a_n$, and extend this definition (by continuity) to the non-negative case by setting $h(a_1, \dots, a_n) = 0$ if $a_i = 0$ for some $i \leq n$. Recall that a measure μ is *non-atomic* if every measurable set A of strictly positive measure has a measurable subset $B \subseteq A$ satisfying $\mu(A) > \mu(B) > 0$. A *measurable partition* of a measure space $(X, \mathcal{F})$ is a collection of disjoint subsets $A_1, \dots, A_n$ of X satisfying $A_i \in \mathcal{F}$ for all i , and $\bigcup_{i=1}^n A_i = X$. Throughout this paper, “measure” means “countably additive, non-negative measure”, and $\|\mu\|$ is the total mass (or total variation) of the measure μ ; the main purpose is to prove the following theorem.

Theorem 1.1. *Let $\mu_1, \mu_2, \dots, \mu_n$ be non-atomic finite measures on a measure space $(X, \mathcal{F})$. Then there is a measurable partition $A_1, \dots, A_n$ of X satisfying*

$$\mu_i(A_i) \geq n^{-1} h(\|\mu_1\|, \dots, \|\mu_n\|) \quad \text{for all } i, \quad (1)$$

and this bound is best possible. Moreover, if $\|\mu_i\| > 0$ for all i , then:

(2a) *if $\mu_i/\|\mu_i\| = \mu_j/\|\mu_j\|$ for all $i, j \leq n$, then every partition satisfying (1) holds with equality for all i ; and*

(2b) *otherwise, the $\{A_i\}$ may be chosen so that strict inequality holds in (1) for all i .*

Theorem 1.1 has an interesting “distribution of value” or “fair-division” interpretation. Suppose an object, such as a cake or piece of land, is to be divided among n people whose values of the object may differ. What is the largest c possible so that each person may receive a piece he values at least c , according to his own values? Theorem 1.1 essentially says that if the values vary continuously (e.g., pieces of zero volume are worth nothing), then c is at least $1/n$ times the harmonic mean of the total values assigned by each person, and that in general it is not possible to do better.

Corollary 1.2 (Dubins and Spanier [2]). *Let $\mu_1, \dots, \mu_n$ be non-atomic probability measures on a measure space $(X, \mathcal{F})$. Then there is a measurable partition $A_1, \dots, A_n$ of X satisfying*

$$\mu_i(A_i) \geq 1/n \quad \text{for all } i, \quad (3)$$

and if $\mu_i \neq \mu_j$ for some $i \neq j$, then strict inequality in (3) for all i may be attained.

(Actually, Dubins and Spanier showed more: for example, $\mu_i(A_i) \geq p_i$ may be attained for any $p_i \geq 0$, $\sum_{i=1}^n p_i = 1$.)

The question underlying (3) dates back to Steinhaus [7], and the reader is referred to [2] for a number of interesting related results many of which are based on Lyapounov's convexity theorem. A proof of Theorem 1.1 can also be given using the convexity theorem, but the argument below is based on a recent $L_1(0, 1)$ inequality of Dor.

§ 2. Proof of Main Theorem

For measures $m_1, m_2, \dots, m_n$ on $(X, \mathcal{F})$, let $\bigvee_{i=1}^n m_i$ denote the smallest measure which dominates each of the $\{m_i\}$, that is, for each $A \in \mathcal{F}$,

$$\left(\bigvee_{i=1}^n m_i \right) (A) = \sup \left\{ \sum_{i=1}^n m_i(B_i) : B_1, \dots, B_n \text{ is a measurable partition of } A \right\}.$$

The next proposition provides the key step in the proof; it is a standard generalization, via the non-atomic separable-measure-algebra isomorphism theorem (e.g. [4] Theorem C, p. 173), of a result of Dor ([1], p. 472) who stated it only for $L_1(0, 1)$.

Proposition 2.1. *Let $\mu_1, \mu_2, \dots, \mu_n$ be non-atomic finite measures on a measure space $(X, \mathcal{F})$, and let $c \geq 0$. If*

$$\left\| \bigvee_{i=1}^n \alpha_i \mu_i \right\| \geq c \sum_{i=1}^n \alpha_i \quad \text{for all } \alpha_1, \dots, \alpha_n \geq 0, \quad (4)$$

then there are disjoint measurable sets $A_1, \dots, A_n \in \mathcal{F}$ satisfying

$$\mu_i(A_i) \geq c \quad \text{for all } i \leq n. \quad (5)$$

The proof of Theorem 1.1 will also use three short lemmas; the first lemma must be known, but as no reference is known to the author, a proof is given.

For each positive integer n , let $\prod_n$ denote the space of probabilities on n points, that is $\prod_n = \{ \vec{p} = (p_1, \dots, p_n) \in \mathbb{R}^n : p_i \geq 0 \text{ for all } i, \text{ and } \sum_{i=1}^n p_i = 1 \}.$

Lemma 2.2. If $a_1, \dots, a_n$ are strictly positive, then

$$\inf_{\vec{p} \in \prod_n} \max_{i \leq n} \{p_i a_i\} = n^{-1} h(a_1, \dots, a_n) = \sup_{\vec{p} \in \prod_n} \min_{i \leq n} \{p_i a_i\}, \quad (6)$$

and both equalities are attained only for $\vec{p} = (p_1^*, \dots, p_n^*)$ where $p_i^* = a_i^{-1} n^{-1} h(a_1, \dots, a_n)$.

Proof. By the compactness of $\prod_n$ and the continuity in $\vec{p}$ of both $\min_{i \leq n} \{p_i a_i\}$ and $\max_{i \leq n} \{p_i a_i\}$, the inf and sup in (6) are both attained. An easy redistribution of mass argument then shows the inf and sup are each attained only when $p_1 a_1 = \dots = p_n a_n$; substitution into the equation $\sum_{i=1}^n p_i = 1$ yields the desired conclusion. $\square$

Lemma 2.3. Let $\mu_1, \dots, \mu_n$ be finite measures. Then

$$\left\| \bigvee_{i=1}^n p_i \mu_i \right\| \geq \max_{i \leq n} \|p_i \mu_i\| \geq n^{-1} h(\|\mu_1\|, \dots, \|\mu_n\|) \quad \text{for all } (p_1, \dots, p_n) \in \prod_n. \quad (7)$$

Proof. The first inequality in (7) is elementary, and the second follows by Lemma 2.2 with $a_i = \|\mu_i\|$. $\square$

Lemma 2.4. Suppose $\|\mu_i\| > 0$ for all $i \leq n$. If $\left\| \bigvee_{i=1}^n p_i \mu_i \right\| = n^{-1} h(\|\mu_1\|, \dots, \|\mu_n\|)$ for some $(p_1, \dots, p_n) \in \prod_n$, then $p_i = \|\mu_i\|^{-1} n^{-1} h(\|\mu_1\|, \dots, \|\mu_n\|)$ for all $i \leq n$, and $p_i \mu_i = p_j \mu_j$ for all $i, j \leq n$.

Proof. By (7), $\max_{i \leq n} \|p_i \mu_i\| = n^{-1} h(\|\mu_1\|, \dots, \|\mu_n\|)$, and by Lemma 2.2 this implies $p_i = \|\mu_i\|^{-1} n^{-1} h(\|\mu_1\|, \dots, \|\mu_n\|)$ for all $i \leq n$. Hence

$$\left\| \bigvee_{i=1}^n p_i \mu_i \right\| = \|p_1 \mu_1\| = \dots = \|p_n \mu_n\|,$$

which in turn implies $p_i \mu_i = p_j \mu_j$ for all $i, j \leq n$. $\square$

Proof of Theorem 1.1. If $\|\mu_i\| = 0$ for some i , then $h(\|\mu_1\|, \dots, \|\mu_n\|) = 0$, and (1) follows immediately; that this bound is best possible in this case is clear.

For the remainder of the proof, suppose $\|\mu_i\| > 0$ for all $i = 1, \dots, n$. Lemma 2.3 implies that (4) is satisfied with $c = n^{-1} h(\|\mu_1\|, \dots, \|\mu_n\|)$, (the case $\alpha_1 = \dots = \alpha_n = 0$ being trivial), and (1) follows from Proposition 2.1; that this bound is best possible will follow from (2a).

Suppose $\mu_i / \|\mu_i\| = \mu_j / \|\mu_j\|$ for all $i, j \leq n$, and let $A_1, \dots, A_n$ be a measurable partition of X satisfying (1). Then for all $i \leq n$,

$$\begin{aligned}
\mu_i(A_i) &= \sum_{j=1}^n \mu_i(A_j) - \sum_{j \neq i}^n \mu_i(A_j) \\
&= \|\mu_i\| - \sum_{j \neq i}^n \|\mu_i\| \|\mu_j\|^{-1} \mu_j(A_j) \\
&\leq \|\mu_i\| (1 - n^{-1} h(\|\mu_1\|, \dots, \|\mu_n\|) \sum_{j \neq i}^n \|\mu_j\|^{-1}) \\
&= n^{-1} h(\|\mu_1\|, \dots, \|\mu_n\|),
\end{aligned}$$

so by (1) equality holds throughout, which establishes (2a).

Finally, suppose $\mu_i/\|\mu_i\| \neq \mu_j/\|\mu_j\|$ for some $i \neq j$. By Lemmas 2.3 and 2.4, $\left\| \bigvee_{i=1}^n p_i \mu_i \right\| > n^{-1} h(\|\mu_1\|, \dots, \|\mu_n\|)$ for all $(p_1, \dots, p_n) \in \prod_n$. The compactness of $\prod_n$ and continuity of $\left\| \bigvee_{i=1}^n p_i \mu_i \right\|$ in $\vec{p}$ then imply the existence of a $c > n^{-1} h(\|\mu_1\|, \dots, \|\mu_n\|)$ satisfying $\left\| \bigvee_{i=1}^n p_i \mu_i \right\| > c$ for all $(p_1, \dots, p_n) \in \prod_n$, which by Proposition 2.1 completes the proof of (2b). $\square$

§3. Structured Domains

When $\mu_1, \dots, \mu_n$ are all probability measures, applications of inequalities of the type in Corollary 1.2 have been made to statistical hypotheses testing (see for example [3, 6]). If the domain in question has a particular structure (e.g., as in [5]), then one may ask whether there is a measurable partition $\{A_i\}$ with the same structure satisfying (1) or (3). The following theorem is a typical result along these lines; its proof uses an idea in [5] and is given here primarily as an application of the strict inequality conclusion (2b) of Theorem 1.1.

Theorem 3.1. *Suppose $f_1, \dots, f_n$ are continuous non-negative functions defined on a polygon $P \subset \mathbb{R}^2$. Then there is a polygonal decomposition $P_1, \dots, P_n$ of P satisfying*

$$\int_{P_i} f_i \geq [(\int_P f_1)^{-1} + \dots + (\int_P f_n)^{-1}]^{-1} \quad \text{for all } i \leq n, \quad (8)$$

and this bound is best possible.

Sketch of Proof. If $f_i \equiv 0$ for some i , any polygonal decomposition of P will suffice, so assume $\int_P f_i > 0$ for all $i \leq n$. Suppose first that $f_i/\int_P f_i = f_j/\int_P f_j$ for all $i, j \leq n$. Take as P_1 any polygon contained in P for which $P \setminus P_1$ is also a polygon and $\int_{P_1} f_1 = [(\int_P f_1)^{-1} + \dots + (\int_P f_n)^{-1}]^{-1}$. Continue with P_2 in $P \setminus P_1$, and so on; since the functions are proportional, (8) is seen to be satisfied with equality throughout. On the other hand, if $f_i/\int_P f_i \neq f_j/\int_P f_j$ for some $i \neq j$, then use

Theorem 1.1 (2b) to infer the existence of a *Borel* partition $A_1, \dots, A_n$ of P satisfying

$$\int_{A_i} f_i > [(\int_P f_1)^{-1} + \dots + (\int_P f_n)^{-1}]^{-1} \quad \text{for all } i \leq n.$$

Using this strict inequality, approximate the $\{A_i\}$ simultaneously by disjoint polygons $\{P_{ij}\}$ in P in such a way as to retain the inequality. $\square$

§4. Remarks

Clearly the conclusion of Theorem 1.1 may fail if the measures have atoms (the worst case being when each μ_i is the dirac delta measure on the same point $x \in X$), and as the next example shows, the conclusion (1) may also fail if the requirement of finiteness of the measures is dropped (where h is defined using $0^{-1} = \infty$ and $\infty^{-1} = 0$).

Example 4.1. Let μ_1 be Lebesgue measure on $[0, 1]$, and let μ_2 be given by $\mu_2(A) = \int_A f d\mu_1$ where $f(x) = 1/x$ for all $x \in (0, 1)$. Then $\|\mu_1\| = 1$, $\|\mu_2\| = \infty$, and $(\|\mu_1\|^{-1} + \|\mu_2\|^{-1})^{-1} = 1$, but for any measurable set A , either $\mu_1(A) < 1$ or $\mu_2(A^c) = 0$, so (1) fails.

If the requirement of non-negativity of the measures is dropped, it is also easy to see that inequality (1) may fail, whether $\|\mu\|$ is interpreted either as the total variation or as the total mass of μ .

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