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A STRONGER FORM OF THE BOREL-CANTELLI LEMMA

BY

THEODORE P. HILL[1]

1. Introduction

Let $A_1, A_2, \ldots$ be a sequence of measurable sets in a probability space (X, A, P), let $p_1 = P(A_1) > 0$, and, for $n > 1$, let p_n be the conditional probability of A_n given F_{n-1} (the σ-field generated by $A_1, \ldots, A_{n-1}$). Let $\chi(A)$ denote the indicator function of the set A, and, for $n \geqslant 1$, let

$$S_n = \sum_{j=1}^{n} \chi(A_j) \quad \text{and} \quad s_n = \sum_{j=1}^{n} p_j.$$

Let $\log_k r$ denote the *k-th* iterated logarithm of r (for example, $\log_3 r = \log\log\log r$). The main objective of this paper is to prove:

THEOREM 1. *For any positive integer k, both*

(1)

$$\lim_{n\to\infty} (S_n - s_n)/[s_n \log_1(s_n) \cdots \log_{k-1}(s_n) \log_k^2(s_n)]^{1/2} = 0$$

and

(2)

$$\lim_{n\to\infty} (S_n - s_n)/[S_n \log_1(S_n) \cdots \log_{k-1}(S_n) \log_k^2(S_n)]^{1/2} = 0$$

a.s. on the set where $\sum_1^\infty p_j = \infty$.

Theorem 1 brings the classical Borel-Cantelli lemma much closer to the central limit theorem and law of the iterated logarithm, without any additional assumptions concerning the divergence of the sums of the variances of the random variables in question, assumptions quite essential in both latter results. It sharpens Levy's conditional form of the Borel-Cantelli lemma [5, Corollary 68, p. 249], and an improved version due to Dubins and Freedman ([2, Theorem 1] or [6, Corollary VII-2-6, p. 152]) which is stated

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here for ease of reference:

(3) $\lim_{n\to\infty} S_n/s_n < \infty$ a.s., and equals 1 a.s. where $\Sigma_1^\infty p_j = \infty$.

A rather different generalization of the Borel-Cantelli Lemma based on the unconditional hypothesis $\lim\sup(ES_n)^2/E(S_n^2) > 0$ was given by Kochen and Stone [4].

2. Proof of Theorem 1

The following result for infinite series, a simultaneous generalization of [3, p. 293] and a theorem of Dini [3, p. 290], will be used in the proof of Theorem 1.

LEMMA 1. *Let $d_1, d_2, \ldots$ be a sequence of positive numbers with $\Sigma_1^\infty d_j = \infty$, and let $D_n = \Sigma_1^n d_j$. For every positive integer k,*

(4)
$$\sum_{n=N}^{\infty} d_n/[(D_n \log_1(D_n) \cdots \log_{k-1}(D_n) \log_k^2(D_n)] < \infty \quad \textit{for some } N \geqslant 1.$$

DEFINITION. For each positive integer k, let ϕ_k denote the function
$$\phi_k(r) = r \log_1(r) \cdots \log_{k-1}(r) \log_k^2(r).$$

Proof of Lemma 1. Fix $k \geqslant 1$, and N so that $\log_k D_{N-1} > 0$. Let $g:[1, \infty) \to \mathbf{R}$ be the step function $g(x) = d_n$ for all $x \in [n, n+1)$, and let $f(x) = \int_1^x g(t)\,dt$. Then for all $n \geqslant N$, and all $x \in [n, n+1)$,

$$d_n/\phi_k(D_n) \leqslant g(x)/\phi_k(f(x)) = f'(x)/\phi_k(f(x))$$

(where f' denotes the ordinary derivative of f). Since $\lim_{n\to\infty} D_n = \infty$, it follows that

$$\sum_{n=N}^{\infty} [d_n/\phi_k(D_n)] \leqslant \int_N^\infty [f'(x)/\phi_k(f(x))]\,dx = 1/\log_k(D_{N-1}) < \infty. \qquad \blacksquare$$

Proof of Theorem 1. Fix $k \geqslant 1$. To establish (1), consider the set B where $\Sigma_1^\infty p_j$ is infinite, and let $R_n = (\chi(A_n) - p_n)/(\phi_k(s_n))^{1/2}$. By Kronecker's lemma, in order to show that $(S_n - s_n)/(\phi_k(s_n))^{1/2}$ converges to zero almost surely on B, it is sufficient to show that the series $\Sigma_1^\infty R_n$ converges a.s. on B. By the well-known conditional form of Kolmogorov's Three-Series Theorem, the series $\Sigma_1^\infty R_n$ converges almost surely on the set where the following three series all converge:

(5) $\Sigma_1^\infty [P(|R_n| \geqslant 1|F_{n-1})$;
(6) $\Sigma_1^\infty E[R_n \cdot \chi(|R_n| \leqslant 1)|F_{n-1}]$;
(7) $\Sigma_1^\infty \{E[R_n^2 \cdot \chi(|R_n| \leqslant 1)|F_{n-1}] - E^2[R_n \cdot \chi(|R_n| \leqslant 1)|F_{n-1}]\}$.

The series in (5) and (6) converge a.s. on B since on the set where $\log_k s_n > 1$, both

$$P(|R_n| \geqslant 1|F_{n-1}) \quad \text{and} \quad E[R_n \cdot \chi(|R_n| \leqslant 1)|F_{n-1}]$$

are almost surely zero. To establish (1), it therefore suffices to show that the series $\Sigma_1^\infty E[R_n^2 \cdot \chi(|R_n| \leqslant 1)|F_{n-1}]$ converges a.s. on B.

On the set where $\log_k s_n > 1$ it follows that

$$E[R_n^2 \cdot \chi(|R_n| \leqslant 1)|F_{n-1}] = E[R_n^2|F_{n-1}] = (p_n - p_n^2)/\phi_k(s_n)$$

almost surely. But Lemma 1 implies that the series $\Sigma_1^\infty p_n/\phi_k(s_n)$, and thus the series $\Sigma_1^\infty(p_n - p_n^2)/\phi_k(s_n)$, converges (almost surely) on B. This completes the proof of (1).

To establish (2), observe that

$$\lim_{n\to\infty}(S_n - s_n)/(\phi_k(S_n))^{1/2} = \lim_{n\to\infty}[(S_n - s_n)/(\phi_k(s_n))^{1/2}] \cdot [(\phi_k(s_n))^{1/2}/(\phi_k(S_n))^{1/2}],$$

and recall that by (1) the first term converges to zero a.s. on B, and by (3) the second converges to one. ■

For an alternative proof of the first part of Theorem 1, one could use Lemma 1 and (in place of the basic Kronecker Lemma conditional three-series theorem argument given above) a result of Chow on martingale difference sequences [1].

The exponent of $\log_k$ in both Theorem 1 and Lemma 1 may easily be reduced from 2 to $1 + \varepsilon$ for any $\varepsilon > 0$, but the resulting conclusions are seen to be no stronger. The denominator in (1) is close to being sharp, for if the $\{A_j\}$ happen to be independent and equiprobable with $0 < p \equiv P(A_j) < 1$, the law of the iterated logarithm implies that $\lim\sup(S_n - s_n)/(s_n \log_2 s_n)^{1/2} = (2 - 2p)^{1/2}$ almost surely.

It is an easy exercise to extend Theorem 1 to include uniformly bounded random variables and increasing σ-fields F_n (to which the p_n are adapted).

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Georgia Institute of Technology
Atlanta, Georgia