SPECIAL RELATIVITY L

Bell's spaceships: a useful relativistic paradox

Bell's spaceship 'paradox' [1] in special relativity is a particularly good one to examine with students, because although it deals with accelerated motions, it can be dissolved with elementary space–time diagrams. Furthermore, it forces us to be very clear about the relativity of simultaneity, proper length, and the 'reality' of the Lorentz contraction.

Bell's spaceships: the paradox

Bell asks us to imagine three spaceships *A*, *B* and *C* that drift freely in a region of space distant from other matter. The three spaceships are originally in a state of relative rest. *B* and *C* are equidistant from *A*. When *B* and *C* receive a light signal from *A*, they begin to accelerate gently. The ships *B* and *C* are assumed to be identical in all relevant respects and to have 'identical acceleration programmes'. Suppose we tie a string between *B* and *C* just long enough to span the distance between them before they begin accelerating. The apparent paradox concerns whether the string breaks. Since *B* and *C*, and hence the string, are speeding up relative to *A*, the length of the string should Lorentz contract. So it seems the string should break. However, measurements by an observer O_A in *A* reveal that *B* and *C* remain equidistant, since *B* and *C* have the same velocity at every instant. So doesn't this show that the string should *not* break?

Bell's solution is that 'as the rockets speed up, it [the string] will become too short, because of its need to Fitzgerald contract, and must finally break. It must break when, at sufficiently high velocity, the artificial prevention of the natural contraction imposes intolerable stress'. Notice the two elements of the paradox. First, one wants to show that indeed the string breaks, which is enough of a lesson itself, as the string would not break if the spaceships were in Newtonian spacetime. Second, one wants to explain *why* the string breaks. It is here that we part ways with Bell, for his subtle explanation requires familiarity with relativistic electrodynamics and computer integration, and weighty assumptions about the constitution of matter.

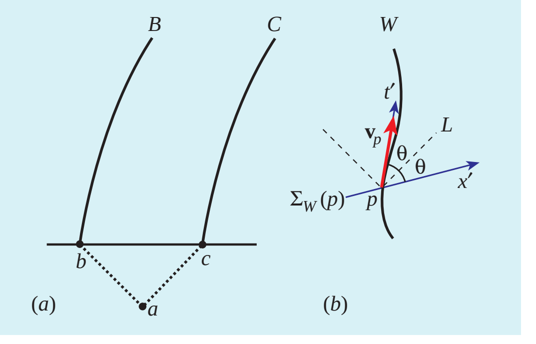

Figure 1. *Elementary constructions.* **Figure 1(*a*)** *Space–time diagram showing the configuration of the spaceships for Bell's paradox. Event* a *is the emission of the light pulse from ship* A *(whose worldline is not shown). Events* b *and* c *are the reception events, when ships* B *and* C *begin to accelerate gently.* **Figure 1(*b*)**. *Construction of co-moving inertial co-ordinate system* K′ *at a point* p *on an arbitrary non-inertial worldline* W.

Bell's spaceships: dissolving the paradox

One can show, using only elementary space–time diagrams, that the following three statements are true:

(**1**) Observers O_B in *B* and O_C in *C* each conclude that the string must break because (a) O_C measures *B* lagging further and further behind and (b) O_B measures *C* pulling further and further ahead.

(**2**) The distance between the spaceships as measured by O_A does *not* change.

(**3**) O_A judges that the string will break.

We make three assumptions to draw all our space–time diagrams. First, we suppress two spatial dimensions. Second, we treat the spaceships as idealized point particles and label their worldlines *A*, *B* and *C* respectively. Third, we interpret Bell's requirement that *B* and *C* have 'identical acceleration programmes' as the condition that the worldlines *B* and *C* are parallel but non-inertial paths in space–time. For example, in the reference frame in which the spaceships are originally at rest, the worldline *C* is

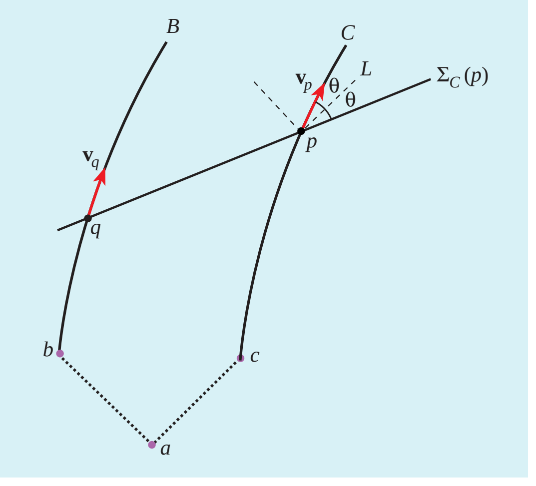

Figure 2(*a*). *Space–time diagrams showing that* O_B *and* O_C *both judge the string breaks. Here* O_C *judges that the trailing spaceship* B *is falling further and further behind.* $\mathbf{v}_p$ *is inclined more toward the light cone* L *than* $\mathbf{v}_q$.

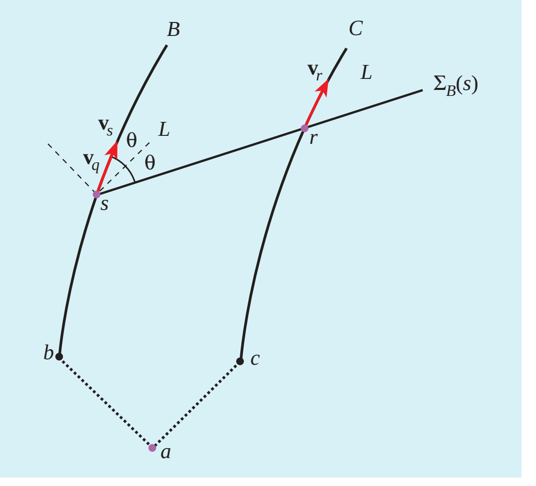

Figure 2(*b*). O_B *judges that* O_C *is pulling further and further ahead.* $\mathbf{v}_r$ *is inclined more toward the light cone* L *than* $\mathbf{v}_s$.

the same as the worldline B but it is shifted some co-ordinate distance Δx in the positive x-direction.

All of our constructions begin with a base diagram (figure 1(*a*)), which illustrates the worldlines B and C, the event a when the light is emitted from spaceship A, and the events b and c when the light is received by spaceships B and C respectively. Our demonstration also often requires that we draw the axes of an inertial co-ordinate system K' momentarily co-moving with a particle whose worldline is a non-inertial path. We will use the following standard construction and adopt the notation introduced therein (figure 1(*b*)). Let W be an arbitrary non-inertial worldline and p an event on that worldline. Let $\mathbf{v}_p$ be the tangent vector to W at p. $\mathbf{v}_p$ is the four-velocity of the particle at p. The t'-axis of K' lies on $\mathbf{v}_p$. The x'-axis of K' lies on the plane of simultaneity $\Sigma_W(p)$ for W at p. To construct $\Sigma_W(p)$, we draw the light-cone L at p, measure the angle θ between $\mathbf{v}_p$ and L, and draw a line through p that makes an angle θ on the other side of L. $\Sigma_W(p)$ is the collection of events that are simultaneous with p for an observer moving with four-velocity $\mathbf{v}_p$.

We show that statement (**1**) is true by drawing space–time diagrams to illustrate statements (**1a**) and (**1b**). To illustrate (**1a**), we begin with our base diagram (figure 1(*a*)). We select an arbitrary point p on C and draw the tangent vector $\mathbf{v}_p$ (figure 2(*a*)). We then draw the plane of simultaneity $\Sigma_C(p)$ and the tangent vector $\mathbf{v}_q$ at the point q where $\Sigma_C(p)$ intersects B. We note that $\mathbf{v}_q$ is *not* parallel to $\mathbf{v}_p$. Consequently, O_C judges that at this 'instant', B's four-velocity is less than C's. As one moves along C, i.e., as O_C's proper time elapses, the planes of simultaneity for the co-moving inertial reference frames tilt, which results in an increasing difference between C's velocity and B's. Thus, C judges that B is lagging further and further behind. If we start by selecting an arbitrary point s on B, construct the plane of simultaneity $\Sigma_B(s)$ on that point, find the point of intersection r with C, and compare $\mathbf{v}_s$ and $\mathbf{v}_r$, we can show that (**1b**) is true (figure 2(*b*)).

To show that statement (**2**) is true, we begin by superimposing O_A's co-ordinate system K on our base diagram. For convenience, we shift the origin away from the centre of the spaceship A, and align the co-ordinate system so that the events b and c lie on the x-axis (figure 3). We select an arbitrary point j on B and construct the plane of simultaneity $\Sigma_A(j)$ for O_A, which is a line parallel to O_A's x-axis. We then draw the tangent vectors $\mathbf{v}_j$ and $\mathbf{v}_k$ at the events j and k where $\Sigma_A(j)$ intersects B and C respectively. We note that $\mathbf{v}_j$ and $\mathbf{v}_k$ are parallel. Consequently, O_A judges that the two spaceships have the same velocity at every instant, and hence they remain the same distance apart as measured in K. So why does O_A judge that the string will break?

O_A reasons as follows: 'The taught string spans

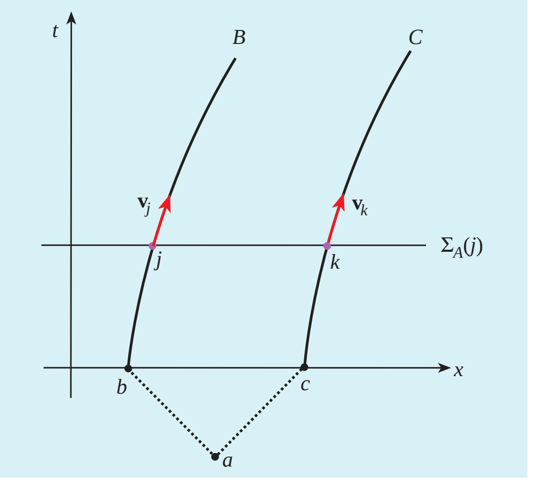

Figure 3. *Space–time diagram showing that* O_A *judges that the spaceships* B *and* C *remain a constant distance apart in her reference frame, since* $\mathbf{v}_j$ *and* $\mathbf{v}_k$ *are parallel.*

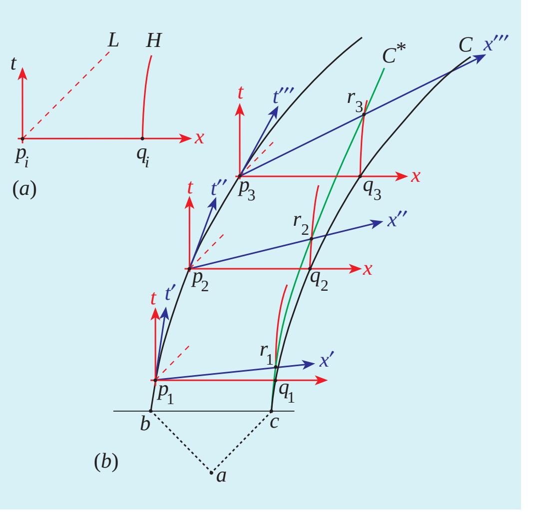

Figure 4. *Space–time diagram showing the path* C* *that the spaceship* C *would have to follow to keep the string taught without breaking.* **Figure 4(*a*)**. *The calibrator diagram.* **Figure 4(*b*)**. *The construction of* C* *from the events* r_i *a constant distance from events* p_i *on* B *at successive co-moving inertial frames.*

the distance between B and C at $t = 0$. Since the string moves non-inertially, it is as if it is moving from one Lorentz-boosted frame to another, and each subsequent reference frame has a higher value of the Lorentz factor $\gamma(v)$. Consequently, to keep the string taught, without changing the tension on the string, the distance between the spaceships as measured in K should *decrease* continuously. Since the distance between the spaceships in K remains *constant*, the spaceships are exerting a force on the string. Therefore, the string will break.' This reasoning shows that (3) is true. However, we can do more.

We can draw the worldline C^* (approximately) that spaceship C would have to follow to keep the string taught if we leave the worldline B unchanged. To keep the string taught, the distance between spaceships C and B has to remain constant in successive co-moving inertial reference frames. To generate C^*, at arbitrarily selected events p_i on B, we draw the co-moving inertial co-ordinate system K' (adding primes as necessary to distinguish different co-ordinate systems), calibrate the x'-axis, and find the event r_i where the front of the string would be if its length did not change, i.e., if it remained taught. We then trace a smooth curve through the events r_i to get a sense of C^*. Comparing C^* and C, it is clear that if the front spaceship follows path C, it pulls on the string. Consequently, the string must break.

The details of the construction are as follows (figure 4). We begin with our base diagram (figure 1(*a*)). We draw a copy of K off to the side. We draw the light cone L at the origin of K and the hyperbola H defined by $x^2 - t^2 = s^2$, where s is the length of the taught string. We label the origin p_i and the intersection of H and the x-axis q_i to anticipate that we will 'stamp' this entire diagram, complete with light cone and hyperbola, so that the p_i fall on B. The points q_i will fall on C. This is our calibrator diagram (figure 4(*a*)).

To approximate C^*, we 'stamp' the calibrator diagram several times on our base diagram so that the p_i fall on B. At each p_i, we draw the t'-axis and x'-axis of K' (again adding primes as necessary). The intersection of H and x' we label r_i. Since H is an invariant, we have $x'^2 - t'^2 = s^2$. Consequently, the intersection of H and x' designates where the front end of the string would be if it was being kept taught. Thus, the path the front ship would have to follow to keep the string taught is approximated by the smooth curve C^* through the events r_i (figure 4(*b*)). Finally, we can draw planes of simultaneity $\Sigma_A(r_i)$ to illustrate that to keep the string taught, the spaceships must move so that the distance between them, as measured in K, must decrease continuously (figure 5).

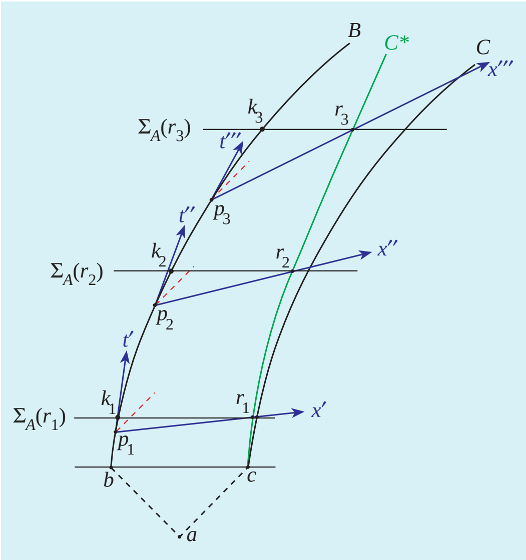

Figure 5. *Space–time diagram showing the path* C* *without the calibrator diagrams. The spatial separation* $r_i k_i$ *between the spaceships in* K *is continuously decreasing (as* O_A*'s proper time increases) along successive planes of simultaneity* $\Sigma_A(r_i)$ *for* O_A.

Bell's spaceships: pedagogy

The pedagogical virtues of dissolving the paradox using space–time diagrams seem clear. First, students in introductory relativity courses can be expected to dissolve the paradox using diagrams. Second, our approach has the virtue that one can dissolve the paradox without introducing the notions of proper acceleration, hyperbolic motion and rigid motion. In more advanced courses, one can introduce these notions and obtain an analytical solution for *C** if one additionally assumes that the ships *B* and *C* move with a constant proper acceleration. Finally, our approach shares the main virtue Bell claims for his approach, namely it can be used to emphasize that in relativity the laws of physics are the same in all inertial frames. All inertial observers agree that if the string is stretched to a certain tension, it will break. All inertial observers agree that in its rest-frame the string is being stretched beyond its breaking point.

References

[1] Bell J S 1993 How to teach special relativity *Speakable and Unspeakable in Quantum Mechanics* (Cambridge: Cambridge University Press) pp 67–80

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