

# Toral Algebraic Sets and Function Theory on Polydisks

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## Abstract

A toral algebraic set  $A$  is an algebraic set in  $\mathbb{C}^n$  whose intersection with  $\mathbb{T}^n$  is sufficiently large to determine the holomorphic functions on  $A$ . We develop the theory of these sets, and give a number of applications to function theory in several variables and operator theoretic model theory. In particular, we show that the uniqueness set for an extremal Pick problem on the bidisk is a toral algebraic set, that rational inner functions have zero sets whose irreducible components are not toral, and that the model theory for a commuting pair of contractions with finite defect lives naturally on a toral algebraic set.

## 0 Introduction

Throughout this paper, we shall let  $\mathbb{D}$  denote the unit disk in the complex plane,  $\mathbb{T}$  be the unit circle, and  $\mathbb{E} = \mathbb{C} \setminus \overline{\mathbb{D}}$  be the complement of the closed

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disk. Let  $A(\mathbb{D}^n)$  denote the polydisk algebra, the algebra of functions that are continuous on the closure of  $\mathbb{D}^n$  and holomorphic on the interior.

When studying function theory on the polydisk  $\mathbb{D}^n$ , it is often useful to focus on the torus  $\mathbb{T}^n$ , which is the distinguished boundary of  $\mathbb{D}^n$ . In several important ways, the behavior of a function in  $A(\mathbb{D}^n)$  is controlled by its behavior on  $\mathbb{T}^n$ : not only is  $\mathbb{T}^n$  a set of uniqueness, but every function in the algebra attains its maximum modulus on  $\mathbb{T}^n$ .

Consider now some algebraic set  $A$  contained in  $\mathbb{C}^n$ . When studying function theory on  $A$ , the intersection of the torus with  $A$  may or may not play an important role. For example, if  $A = \{(z, \dots, z) : z \in \mathbb{C}\}$ , then  $A$  is a plane, and  $A \cap \mathbb{T}^n$  is a unit circle. However, if  $A = \{(z, 0, \dots, 0) : z \in \mathbb{C}\}$ , then  $A \cap \mathbb{T}^n$  is empty.

We shall say that a variety (by which we always mean an irreducible algebraic set)  $V$  is *toral* if its intersection with  $\mathbb{T}^n$  is fat enough to be a determining set for holomorphic functions on  $V$  (see Section 1 for a precise definition). Otherwise we shall call the variety *atoral*. We shall say that a polynomial in  $\mathbb{C}[z_1, \dots, z_n]$  is toral (respectively, atoral) if the zero set of every irreducible factor is toral (respectively, atoral).

It turns out that factoring polynomials into their toral and atoral factors is extremely useful when studying function theory on  $\mathbb{D}^n$ .

Consider first the Pick problem on the bidisk,  $\mathbb{D}^2$ . Let  $H^\infty(\mathbb{D}^2)$  denote the Banach algebra of bounded analytic functions on the bidisk. A *solvable Pick problem on  $\mathbb{D}^2$*  is a set  $\{\lambda_1, \dots, \lambda_N\}$  of points in  $\mathbb{D}^2$  and a set  $\{w_1, \dots, w_N\}$  of complex numbers such that there is some function  $\phi$  of norm less than or equal to one in  $H^\infty(\mathbb{D}^2)$  that interpolates (satisfies  $\phi(\lambda_i) = w_i \forall 1 \leq i \leq N$ ). An *extremal Pick problem* is a solvable Pick problem for which no function of norm less than one interpolates.

Unlike extremal Pick problems on  $\mathbb{D}$ , an extremal Pick problem on  $\mathbb{D}^2$  need not have a unique solution. However there is some subset  $\mathcal{U}$  of  $\mathbb{D}^2$ , called the uniqueness set of the problem, on which all solutions must agree

( $\mathcal{U}$  obviously contains all of the points  $\lambda_i$ ). We prove in Theorem 4.2 that the uniqueness set equals the intersection of  $\mathbb{D}^2$  and the zero set of a toral polynomial.

A second place where the concept of toral/atoral polynomials arises is in the study of rational inner functions on  $\mathbb{D}^n$ . An inner function is a function  $\phi$  that is holomorphic and bounded on  $\mathbb{D}^n$  and whose radial boundary values, which exist almost everywhere [12, Thm. 3.3.5], have modulus one almost everywhere. W. Rudin showed [12, Thm. 5.2.5] that every rational inner function can be represented in the form

$$\phi(z) = z^h \frac{\overline{p(1/\bar{z})}}{p(z)} \quad (0.1)$$

for some polynomial  $p$  that does not vanish on  $\mathbb{D}^n$ , and some monomial  $z^h$  such that  $h \geq \deg p$ . We show in Theorem 3.1 that in the representation (0.1), the atoral factor of  $p$  is uniquely determined, and the toral factor is completely arbitrary. As a consequence of this analysis, we show in Proposition 3.4 that the zero set of a rational inner function is an atoral algebraic set. In Proposition 3.6 we show that the singular set of a rational inner function, namely the set of points on  $\mathbb{T}^n$  to which the function cannot be continuously extended from  $\mathbb{D}^n$ , is always of dimension at most  $n - 2$ .

The Sz.Nagy-Foiaş model theory for a pair of commuting contractions  $(T_1, T_2)$  realizes them as the compression of a pair of commuting isometries  $(S_1, S_2)$  [13]. In the event that one of the isometries has finite defect, it can be represented as multiplication by the independent variable on a vector-valued Hardy space, and the other isometry becomes multiplication by a matrix-valued inner function  $B$ . The set

$$A = \{(z, w) \in \mathbb{C}^2 : \det(B(z) - wI) = 0\}$$

is toral, and the extension spectrum of  $(S_1, S_2)$  is  $A \cap \mathbb{T}^2$ . This means that in addition to the  $\mathbb{D}^2$  functional calculus, one has an  $A$  functional calculus, which is stronger. In other words, for every polynomial  $p$ , instead of Andô's

inequality [3]

$$\|p(T_1, T_2)\| \leq \|p\|_{\mathbb{D}^2},$$

one has

$$\|p(T_1, T_2)\| \leq \|p\|_A.$$

Therefore the study of the function theory of toral algebraic sets is important in understanding the functional calculus for commuting pairs of matrices. See the papers [6] and [7] by J. Ball and V. Vinnikov and [4] by J. Ball, C. Sadosky and V. Vinnikov for another viewpoint.

The lay-out of this paper is as follows. In Section 1 we give precise definitions of the concepts of toral and atoral, and make some basic observations. In Section 2, we study how various geometric properties are related to the analytic notion of torality. In Section 3 we give applications to the study of rational inner functions, and in Section 4 we characterize the uniqueness set for a Pick problem on the bidisk.

We have tried to write our paper to be intelligible to both algebraic geometers and to analysts. Consequently we apologize for belaboring points that will be obvious to some readers.

## 1 Toral and Atoral Algebraic sets in $\mathbb{C}^n$

For  $p \in \mathbb{C}[z_1, \dots, z_n]$ , the polynomials in the commuting variables  $z_1, \dots, z_n$  with complex coefficients, we shall denote the zero set of  $p$  by  $Z_p$ . An *algebraic set* in  $\mathbb{C}^n$  is a finite intersection of zero sets of polynomials. If the algebraic set cannot be written as a finite union of strictly smaller algebraic sets, it is called a *variety*. Every algebraic set  $A$  can be written as a union of varieties, no one containing another, called the irreducible components of  $A$ .

The algebra of holomorphic functions on an algebraic set  $A$ , denoted  $Hol(A)$ , consists of all complex-valued functions  $f$  on  $A$  with the property that, for every point  $\lambda$  in  $A$ , there is an open set  $U$  in  $\mathbb{C}^n$  containing  $\lambda$ , and a holomorphic function  $g$  on  $U$ , such that  $g|_{U \cap A} = f|_{U \cap A}$ .

Let  $X \subseteq \mathbb{C}^n$ , and let  $A$  be an algebraic set in  $\mathbb{C}^n$ . We shall say that  $X$  is *determining for  $A$*  if  $f \equiv 0$  whenever  $f \in \text{Hol}(A)$  and  $f|_{X \cap A} = 0$ . Note that  $X$  is not required to be a subset of  $A$ , which is a departure from the usual use of “determining”. However, it is also determining in the following sense.

**Proposition 1.1** *Let  $X \subseteq \mathbb{C}^n$ , and let  $A_1$  and  $A_2$  be algebraic sets for which  $X$  is determining. If  $A_1 \cap X = A_2 \cap X$ , then  $A_1 = A_2$ .*

PROOF: There exist nonzero polynomials  $p_1, \dots, p_m$  such that  $A_1 = Z_{p_1} \cap Z_{p_2} \cap \dots \cap Z_{p_m}$ . For each  $j$ ,  $p_j$  vanishes on  $A_1 \cap X$  (since  $p_j$  vanishes on  $Z_{p_j}$ ) and so  $p_j$  vanishes on  $A_2 \cap X$ . Therefore each  $p_j$  vanishes on  $A_2$ , and so  $A_1 \subseteq A_2$ . By reversing the roles of  $A_1$  and  $A_2$ , we get that  $A_2 \subseteq A_1$ .  $\square$

**Definition 1.2** Let  $A$  be an algebraic set in  $\mathbb{C}^n$ . Let us agree to say  $A$  is *toral* if  $\mathbb{T}^n$  is determining for  $A$  and say  $A$  is *atoral* if  $\mathbb{T}^n$  is not determining for any of the irreducible components of  $A$ . If  $p \in \mathbb{C}[z_1, \dots, z_n]$ , let us agree to say  $p$  is *toral* (resp, *atoral*) if the algebraic set  $Z_p$  is.

Note that if  $V$  is a variety, then  $V$  is either toral or atoral and if, in addition,  $V$  is nonempty, then  $V$  cannot be both toral and atoral. Also, the empty set is both toral and atoral.

The definition of “is determined by” immediately implies that if  $B$  is determining for both  $A_1$  and  $A_2$ , then  $B$  is determining for  $A_1 \cup A_2$ . Therefore, finite unions of toral sets are toral. That finite unions of atoral sets are atoral is also true and follows immediately from the definition. Notice that it also follows immediately from the definition of atoral, that if  $A$  is an atoral set, then each irreducible component of  $A$  is atoral. The corresponding assertion for toral algebraic sets is also true as the following proposition shows.

**Proposition 1.3** *If  $A$  is a toral algebraic set, then each irreducible component of  $A$  is toral.*

This proposition follows from the following lemma.

**Lemma 1.4** *If  $X$  is a set in  $\mathbb{C}^n$ ,  $A$  is an algebraic set in  $\mathbb{C}^n$ , and  $X$  is determining for  $A$ , then  $X$  is determining for each irreducible component of  $A$ .*

PROOF: Suppose that  $X$  is determining for  $A$ , and  $C$  is an irreducible component of  $A$ . Set  $A_0 = A \setminus C$ . Let  $f \in \text{Hol}(C)$  and assume that  $f|_{C \cap X} = 0$ . There exists a function  $\chi \in \text{Hol}(A)$  such that  $\chi|_{A_0} = 0$  and  $\chi|_C \not\equiv 0$ . If  $z \in \overline{A_0} \cap C$ , then  $\chi(z)f(z) = 0$ . Therefore, the function defined by setting  $g(z) = \chi(z)f(z)$  for  $z \in C$  and  $g(z) = 0$  for  $z \in \overline{A_0}$  is well defined.

To show  $g \in \text{Hol}(A)$ , let  $\zeta \in A$ . We seek to show that there is a neighborhood  $U$  of  $\zeta$  and an analytic function defined on  $U$  which equals  $g$  on  $U \cap A$ .

Suppose  $\zeta \in C \setminus \overline{A_0}$ . There exists an open disk  $U$  centered at  $\zeta$  such that  $U \cap A_0 = \emptyset$ , and there exist  $\Gamma, F \in \text{Hol}(U)$  such that  $\Gamma|_{U \cap A} = \chi|_{U \cap A}$  and  $F|_{U \cap A} = f|_{U \cap A}$ . Therefore,  $g|_{U \cap A} = \chi f|_{U \cap A} = \Gamma F|_{U \cap A}$ .

Now suppose  $\zeta \in A_0 \setminus C$ . There exists an open disk  $U$  centered at  $\zeta$  such that  $U \cap C = \emptyset$ . If  $G(z) = 0$  for all  $z \in U$ , then  $g|_{U \cap A} = G|_{U \cap A}$ .

Now suppose  $\zeta \in \overline{A_0} \cap C$ . There exists an open disk  $U$  centered at  $\zeta$  and  $\Gamma, F \in \text{Hol}(U)$  such that  $\Gamma|_{U \cap A} = \chi|_{U \cap A}$  and  $F|_{U \cap C} = f|_{U \cap C}$ . Let  $G(z) = F(z)\Gamma(z)$  for  $z \in U$ . For  $z \in A_0 \cap U$ ,  $G(z) = \Gamma(z)F(z) = \chi(z)F(z) = 0 = g(z)$  by the choice of  $\chi$ . For  $z \in C \cap U$ ,  $g(z) = f(z)\chi(z)$  and  $G(z) = \Gamma(z)F(z) = \chi(z)f(z) = g(z)$ . Therefore,  $g|_{U \cap A} = G|_{U \cap A}$ .

We have shown that  $g \in \text{Hol}(A)$ . Note that  $g|_{A \cap X} = 0$  and therefore,  $g = 0$  since  $X$  is determining for  $A$  and  $g \in \text{Hol}(A)$ . Therefore,  $\chi f|_C \equiv 0$ . Since  $C$  is a variety,  $\text{Hol}(C)$  is an integral domain. Since  $\chi f|_C = 0$  and  $\chi|_C \neq 0$ ,  $f|_C = 0$ . Therefore,  $X$  is determining for  $C$ .  $\square$

If  $B$  is an algebraic set, we define the *toral component* of  $B$  to be the union of the irreducible toral components of  $B$  and define the *atoral component* of  $B$  to be the union of the irreducible atoral components of  $B$ .

Let  $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$  and, for  $\zeta \in (\mathbb{C}^*)^n$ , let  $1/\bar{\zeta} = (1/\bar{\zeta}_1, \dots, 1/\bar{\zeta}_n)$ . For  $\zeta \in \mathbb{C}^n$  and  $d = (d_1, \dots, d_n)$ , an  $n$ -tuple of nonnegative integers, let  $\zeta^d =$

$\zeta_1^{d_1} \zeta_2^{d_2} \cdots \zeta_n^{d_n}$ . Note that  $1/\bar{\zeta} = \zeta$  if and only if  $\zeta \in \mathbb{T}^n$  and that  $\zeta^d \in \mathbb{C}^*$  whenever  $\zeta \in (\mathbb{C}^*)^n$ .

Let us agree to say that an algebraic set  $A \subseteq \mathbb{C}^n$  is  $\mathbb{T}^n$ -symmetric if, for all  $\zeta \in (\mathbb{C}^*)^n$ ,  $\zeta \in A$  implies  $1/\bar{\zeta} \in A$ . Note that, since  $1/\overline{1/\bar{\zeta}} = \zeta$ ,  $A$  is  $\mathbb{T}^n$ -symmetric if and only if, for all  $\zeta \in (\mathbb{C}^*)^n$ ,  $\zeta \in A$  if and only if  $1/\bar{\zeta} \in A$ .

**Proposition 1.5** *If  $A$  is a toral algebraic set in  $\mathbb{C}^n$ , then  $A$  is  $\mathbb{T}^n$ -symmetric.*

PROOF: Let  $A$  be a toral algebraic set in  $\mathbb{C}^n$ . If  $A = \mathbb{C}^n$ , then  $A$  is  $\mathbb{T}^n$ -symmetric. Suppose  $A \neq \mathbb{C}^n$ . There exist nonzero polynomials  $p_1, \dots, p_m$  such that  $A = Z_{p_1} \cap Z_{p_2} \cap \cdots \cap Z_{p_m}$ . Let  $q_j(z) = z^{\deg p_j} \overline{p_j(1/\bar{z})}$ . Each  $q_j$  is a nonzero polynomial. For  $\zeta \in A \cap \mathbb{T}^n$ ,  $q_j(\zeta) = \zeta^{\deg p_j} \overline{p_j(\zeta)} = 0$ . Therefore,  $q_j|_{A \cap \mathbb{T}^n} = 0$ . Since  $A$  is toral,  $q_j|_A \in \text{Hol}(A)$ , and  $q_j|_{A \cap \mathbb{T}^n} = 0$ , we have  $q_j|_A = 0$ .

If  $\zeta \in A \cap (\mathbb{C}^*)^n$ , then, for each  $j$ ,  $q_j(\zeta) = 0$  and  $p_j(1/\bar{\zeta}) = 0$ . Therefore,  $1/\bar{\zeta} \in A$ . Thus,  $A$  is  $\mathbb{T}^n$ -symmetric.  $\square$

In the special case when  $n = 2$ , it is an easy matter to describe the torality (resp, atorality) of any algebraic set in terms of the torality (resp, atorality) of a single polynomial. In  $\mathbb{C}^2$ , varieties are either points,  $\mathbb{C}^2$ , or  $Z_p$  for some irreducible  $p \in \mathbb{C}[z_1, z_2]$ , so any algebraic set  $A$  in  $\mathbb{C}^2$  can be represented in the form  $A = F \cup Z_p$  for some  $p \in \mathbb{C}[z_1, z_2]$  where  $F$  is the finite set of isolated points of  $A$ . (If  $A$  is finite, choose  $p(z_1, z_2) = 1$ .) The following proposition follows from this characterization of irreducible algebraic sets in  $\mathbb{C}^2$ .

**Proposition 1.6** *Let  $A$  be an algebraic set in  $\mathbb{C}^2$ . If  $A$  is a finite set, then  $A$  is toral if and only if  $A \subset \mathbb{T}^2$  and  $A$  is atoral if and only if  $A \cap \mathbb{T}^2 = \emptyset$ . If  $A$  is not a finite set and we let  $A = F \cup Z_p$  where  $F$  is the set of isolated points of  $A$  and  $p \in \mathbb{C}[z_1, z_2]$ , then the following statements hold.*

(a)  *$A$  is toral if and only if  $p$  is toral and  $F \subset \mathbb{T}^2$ .*

(b)  *$A$  is atoral if and only if  $p$  is atoral and  $F \cap \mathbb{T}^2 = \emptyset$ .*

## 2 Toral and Atoral polynomials in $\mathbb{C}^n$

Let us agree to say that two irreducible polynomials  $p$  and  $q$  are *essentially equal* if  $p = cq$  for some nonzero  $c \in \mathbb{C}$ . The zero set  $Z_p$  is a variety if and only if  $p$  is irreducible. Since the zero set of a nonzero polynomial  $p$  is equal to the union of the zero sets of its irreducible factors, the results of Section 1 imply the following corollaries.

**Corollary 2.1** *Let  $p$  be a nonzero polynomial in  $\mathbb{C}[z_1, \dots, z_n]$ . The following are equivalent.*

- i)  $p$  is toral (respectively, atoral)
- ii) each irreducible factor of  $p$  is toral (respectively, atoral)
- iii) every divisor of  $p$  is toral (respectively, atoral).

**Corollary 2.2** *Let  $p$  be a nonzero polynomial in  $\mathbb{C}[z_1, \dots, z_n]$ . There exist  $q, r \in \mathbb{C}[z_1, \dots, z_n]$  such that  $p = qr$ ,  $q$  is toral and  $r$  is atoral. Moreover, if  $p = q'r'$  is another factorization with  $q'$  toral and  $r'$  atoral, then  $q'$  is essentially equal to  $q$  and  $r'$  is essentially equal to  $r$ .*

PROOF: The existence of  $q$  and  $r$  follows from factoring  $p$  into irreducible factors and then grouping the toral and atoral factors. Suppose that  $p = q'r'$  is another factorization with  $q'$  toral and  $r'$  atoral. By Corollary 2.1, every irreducible factor of  $q'$  is toral and every irreducible factor of  $r'$  is atoral. Thus,  $q'$  divides  $q$  and  $r'$  divides  $r$ . Since  $qr = q'r'$ ,  $q'$  is essentially equal to  $q$  and  $r'$  is essentially equal to  $r$ .  $\square$

Any nonzero polynomial in  $\mathbb{C}[z_1, \dots, z_n]$  can be reflected in  $\mathbb{T}^n$  in the following way. Let  $p$  be a polynomial of degree  $d = (d_1, \dots, d_n)$ . We define the polynomial  $p^\sim$  by

$$p^\sim(z) = z^d \overline{p(1/\bar{z})}. \quad (2.3)$$

Notice that  $(pq)^\sim = p^\sim q^\sim$  and that  $p^{\sim\sim} = p$  if and only if none of the coordinate functions divide  $p$ .



We shall say that a polynomial  $p$  is  $\mathbb{T}^n$ -*symmetric* if  $p^\sim = p$  or  $p$  is the zero polynomial, and *essentially*  $\mathbb{T}^n$ -*symmetric* if there is a unimodular constant  $\tau$  such that  $\tau p$  is  $\mathbb{T}^n$ -symmetric.

If  $p$  is an irreducible atoral polynomial, then there is some non-zero  $f$  in  $Hol(Z_p)$  that vanishes on  $Z_p \cap \mathbb{T}^n$ . Therefore  $Z_p \cap \mathbb{T}^n$  is contained in the  $(n - 2)$ -dimensional analytic set  $Z_p \cap Z_f$ . In Theorem 2.4 we show that  $Z_p \cap \mathbb{T}^n$  is contained in an  $(n - 2)$ -dimensional algebraic set (in other words,  $f$  can be chosen to be a polynomial). This can be thought of as an algebraic characterization of whether a polynomial is toral: measuring directly how fat  $Z_p \cap \mathbb{T}^n$  is.

**Theorem 2.4** *If  $p \in \mathbb{C}[z_1, \dots, z_n]$ , then  $p$  is atoral if and only if  $Z_p \cap \mathbb{T}^n$  is contained in an algebraic set  $A$  of dimension  $n - 2$ .*

PROOF: By Corollary 2.1, it is sufficient to consider polynomials  $p$  that are irreducible.

(Sufficiency) Let

$$A = \bigcup_{j=1}^m (Z_{q_{1j}} \cap Z_{q_{2j}}),$$

where  $q_{1j}$  and  $q_{2j}$  are relatively prime polynomials, and suppose  $A$  contains  $Z_p \cap \mathbb{T}^n$ . Relabelling, if necessary, we can assume that  $p$  does not divide any  $q_{1j}$ . If  $q = \prod_j q_{1j}$ , then  $q$  vanishes on  $Z_p \cap \mathbb{T}^n$ , but not on all  $Z_p$ . Thus  $p$  is atoral.

(Necessity) If  $p$  is not essentially  $\mathbb{T}^n$ -symmetric, then

$$Z_p \cap \mathbb{T}^n \subseteq Z_p \cap Z_{p^\sim},$$

and we are done. So, assume that  $p$  is  $\mathbb{T}^n$ -symmetric. We shall show that the gradient of  $p$  must vanish on  $Z_p \cap \mathbb{T}^n$ . Indeed, let  $(z'_0, w_0)$  be a point in  $Z_p \cap \mathbb{T}^n$  with  $z'_0 \in \mathbb{C}^{n-1}$ , and assume that

$$\frac{\partial p}{\partial z_n} \big|_{(z'_0, w_0)} \neq 0.$$

By the implicit function theorem, there is an open neighborhood  $U$  of  $z'_0$ , an open neighborhood  $W$  of  $w_0$ , and a holomorphic function  $h$  on  $U$  such that if  $(z', w)$  is in  $U \times W$ , then

$$p(z', w) = 0 \iff w = h(z'). \quad (2.5)$$

By (2.5) for every  $z' \in U \cap \mathbb{T}^{n-1}$ ,  $p$  has only one root  $w$  in  $W$ . Since  $p$  is  $\mathbb{T}^n$ -symmetric, this root must be unimodular. Therefore,

$$Z_p \cap \mathbb{T}^n \cap (U \times W) = \{(z', h(z')) : z' \in U \cap \mathbb{T}^{n-1}\}.$$

Now suppose  $f \in \text{Hol}(Z_p)$  vanishes on  $Z_p \cap \mathbb{T}^n$ . The function  $z' \mapsto f(z', h(z'))$  on  $U$  vanishes on  $U \cap \mathbb{T}^{n-1}$ , and therefore vanishes identically on  $U$ . Therefore, if we let  $S$  be the set of singular points of  $Z_p$ , then  $f$  vanishes on an open subset of  $Z_p \setminus S$ , and as  $Z_p \setminus S$  is connected [10],  $f$  is identically zero. Thus  $p$  must be toral, a contradiction.

Therefore we can let  $A = Z_p \cap Z_{\frac{\partial p}{\partial z_n}}$ . □

**Proposition 2.6** *Every toral polynomial is essentially  $\mathbb{T}^n$ -symmetric.*

PROOF: By Corollary 2.1, it is sufficient to show that every irreducible toral polynomial is  $\mathbb{T}^n$ -symmetric. Since  $Z_p \cap \mathbb{T}^n = Z_{p^\sim} \cap \mathbb{T}^n$ , Theorem 2.4 implies  $p^\sim$  is toral. Since  $p^\sim$  vanished on  $Z_p \cap \mathbb{T}^n$  and  $p$  is toral, we must have  $Z_p \subseteq Z_{p^\sim}$ . Since  $p$  vanished on  $Z_{p^\sim} \cap \mathbb{T}^n$  and  $p^\sim$  is toral, we must have  $Z_{p^\sim} \subseteq Z_p$ . Thus  $Z_p = Z_{p^\sim}$  and since both  $p$  and  $p^\sim$  are irreducible, we must have that one is a unimodular constant times the other. □

Now we give a geometric condition which is sufficient to guarantee that a polynomial is toral.

**Theorem 2.7** *Let  $p \in \mathbb{C}[z_1, \dots, z_n]$ , and suppose  $Z_p$  is disjoint from  $\mathbb{D}^n \cup \mathbb{E}^n$ . Then  $p$  is toral.*

PROOF: Without loss of generality, assume that  $p$  is square free. Suppose  $z' \in \mathbb{T}^{n-1}$ , and  $p(z', w) = 0$ . Then  $|w| = 1$ , or else some small perturbation of  $(z', w)$  would yield a zero of  $p$  in either  $\mathbb{D}^n$  or  $\mathbb{E}^n$ .

Write

$$p(z', w) = \sum_{j=0}^k a_j(z') w^j.$$

Let  $D(z') \in \mathbb{C}[z_1, \dots, z_{n-1}]$  be the discriminant of  $p(z', w)$ ; then  $D$  will vanish precisely at those points  $z'$  such that the function  $w \mapsto p(z', w)$  has a root of multiplicity higher than one for some  $w$ .

Let  $B = Z_D \cup Z_{a_k}$ . This is an algebraic set in  $\mathbb{C}^{n-1}$  that does not disconnect  $\mathbb{C}^{n-1}$ . Off  $B$ , one can choose  $k$  holomorphic functions  $\{w_j(z')\}_{j=1}^k$  that take values in the  $k$  sheets of  $Z_p$  over  $\mathbb{C}^{n-1} \setminus B$ .

Now let  $f \in \text{Hol}(Z_p)$  vanish on  $Z_p \cap \mathbb{T}^n$ . Locally, around any point  $(z', w)$  in  $Z_p$  with  $z' \notin B$ ,  $f$  can be written as a function of  $z'$ , and this function vanishes on  $\mathbb{T}^{n-1} \setminus B$ . By analytic continuation,  $f$  must vanish at any point  $(z', w)$  with  $z' \in \mathbb{C} \setminus B$ , and by continuity of  $f$ , it must vanish on all of  $Z_p$ . Therefore  $Z_p$  is toral, as required.  $\square$

### 3 Inner functions

W. Rudin showed that any rational inner function can be represented as

$$\phi(z) = z^h \frac{p^\sim(z)}{p(z)}$$

for some polynomial  $p$  that does not vanish on  $\mathbb{D}^n$ , and some monomial  $z^h$ , [12, Thm. 5.2.5]. We show that the atoral factor of  $p$  uniquely determines  $\phi$ . Note that in the case  $n = 2$ , B. Cole and J. Wermer have obtained additional information about the relation between  $p$  and  $p^\sim$  [8].

**Theorem 3.1** *Let*

$$\phi(z) = z^h \frac{p^\sim(z)}{p(z)} \quad \text{and} \quad \psi(z) = z^e \frac{q^\sim(z)}{q(z)} \quad (3.2)$$

be two rational inner functions, with  $p$  and  $q$  polynomials that do not vanish on  $\mathbb{D}^n$ . Then  $\phi$  and  $\psi$  are essentially equal if and only if  $p$  and  $q$  have the same atoral factor and  $h = e$ .

PROOF: By Proposition 2.6, any toral factor  $r$  of either  $p$  or  $q$  is essentially  $\mathbb{T}^n$ -symmetric, so  $r^\sim/r$  is constant. Thus we can assume that both  $p$  and  $q$  are atoral. Moreover, if  $p$  had any nonconstant  $\mathbb{T}^n$ -symmetric divisor  $r$ , then  $Z_r$  would be disjoint from  $\mathbb{D}^n$  (since  $Z_p$  is) and from  $\mathbb{E}^n$  (by  $\mathbb{T}^n$ -symmetry). Therefore by Theorem 2.7,  $r$  would be toral. So we can assume that neither  $p$  nor  $q$  has any nonconstant  $\mathbb{T}^n$ -symmetric divisors.

To show that  $p$  and  $q$  must then be essentially equal, cross-multiply to get a scalar  $\tau$  such that

$$\tau z^h p^\sim(z) q(z) = z^e q^\sim(z) p(z). \quad (3.3)$$

Since  $p(0) \neq 0 \neq q(0)$ , both  $p^\sim$  and  $q^\sim$  have  $\overline{p(0)}z^{\deg p}$  and  $\overline{q(0)}z^{\deg q}$  respectively as their highest order terms. Therefore the degree of the left-hand side of (3.3) is  $h + \deg p + \deg q$ , and the degree of the right-hand side is  $e + \deg q + \deg p$ . Therefore  $h = e$ .

Now since  $p$  has no nonconstant  $\mathbb{T}^n$ -symmetric divisors,  $p$  is relatively prime to  $p^\sim$  (for if  $r$  were an irreducible polynomial that divided both, either  $r$  would be essentially  $\mathbb{T}^n$ -symmetric, or  $rr^\sim$  would be a  $\mathbb{T}^n$ -symmetric polynomial that divided  $p$ ). Therefore,  $p^\sim$  divides  $q^\sim$ , and, since  $p = p^{\sim\sim}$  and  $q = q^{\sim\sim}$ ,  $p$  divides  $q$ . Interchanging the roles of  $p$  and  $q$ ,  $p$  and  $q$  must be essentially equal.  $\square$

The following proposition shows that the zero set of a rational inner function is atoral, and the level set for any unimodular number is toral. Part (iii) is due to W. Rudin [12, Thm. 5.2.6].

**Proposition 3.4** *Let  $\phi$  be a nonconstant rational inner function, and let  $\alpha \in \mathbb{C}$ . Then*

- (i) If  $\alpha \in \mathbb{D} \cup \mathbb{E}$ , then  $Z_{\phi-\alpha}$  is atoral.
- (ii) If  $\alpha \in \mathbb{T}$ , then  $Z_{\phi-\alpha}$  is toral.
- (iii) If  $\alpha \in \mathbb{D}$ , then  $Z_{\phi-\alpha} \cap \mathbb{E}^n = \emptyset$ .
- (iv) If  $\alpha \in \mathbb{E}$ , then  $Z_{\phi-\alpha} \cap \mathbb{D}^n = \emptyset$ .
- (v) If  $\alpha \in \mathbb{T}$ , then  $Z_{\phi-\alpha} \cap \mathbb{D}^n = Z_{\phi-\alpha} \cap \mathbb{E}^n = \emptyset$ .

PROOF: (i,iii,iv) Write

$$\phi(z) = z^h \frac{p^\sim(z)}{p(z)}, \quad (3.5)$$

where  $p$  (and therefore  $p^\sim$ ) is atoral. Suppose first that  $\alpha = 0$ . Then  $Z_\phi = Z_{z^h} \cup Z_{p^\sim}$ . The set  $Z_{z^h}$  is disjoint from  $\mathbb{T}^n$ , so is atoral, and  $Z_{p^\sim}$  is atoral by the choice of  $p$ . Moreover, since  $Z_p$  is disjoint from  $\mathbb{D}^n$ ,  $Z_{p^\sim}$  is disjoint from  $\mathbb{E}^n$ . Thus,  $Z_\phi \cap \mathbb{E}^n = \emptyset$ .

Now, if  $\alpha \in \mathbb{D}$ , consider

$$\psi(z) = \frac{\phi(z) - \alpha}{1 - \overline{\alpha}\phi(z)}.$$

Then  $\psi$  is rational and inner,  $Z_\psi = Z_{\phi-\alpha}$ , and  $Z_\psi \cap \mathbb{E}^n = \emptyset$ .

Finally, let  $\alpha \in \mathbb{E}$ . Since  $\phi$  is an inner function, by the maximum principle,  $Z_{\phi-\alpha} \cap \mathbb{D}^n = \emptyset$ . Also, since  $\alpha \in \mathbb{E}$ ,  $\phi(\zeta) = \alpha$  if and only if  $\phi(1/\overline{\zeta}) = 1/\overline{\alpha}$ . So the zero set of  $\phi - \alpha$  is the reflection of the zero set of  $\phi - 1/\overline{\alpha}$ , and therefore is atoral.

(ii,v) Suppose  $|\alpha| = 1$ . Off  $Z_p$ , we have  $\phi(\zeta) = \alpha$  if and only if  $\phi(1/\overline{\zeta}) = 1/\overline{\alpha} = \alpha$ . So  $Z_{\phi-\alpha}$  is  $\mathbb{T}^n$ -symmetric.  $Z_{\phi-\alpha}$  is disjoint from  $\mathbb{D}^n$  by the maximum principle, so  $Z_{\phi-\alpha}$  must also be disjoint from  $\mathbb{E}^n$ . Therefore, by Theorem 2.7, the set is toral.  $\square$

The singular set  $S_\phi$  of a rational inner function is the set of points on  $\mathbb{T}^n$  to which the function cannot be continuously extended from  $\mathbb{D}^n$ . If the function has the form (3.5), it is the set  $Z_p \cap \mathbb{T}^n$ . If  $\zeta$  is in this singular set, then  $p^\sim(\zeta) = \zeta^{h+d}\overline{p(\zeta)} = 0$ , so  $S_\phi \subseteq Z_p \cap Z_{p^\sim}$ . Therefore we have:

**Proposition 3.6** *The singular set of a rational inner function is always contained in an algebraic set of dimension  $n - 2$ .*

## 4 Application to Interpolation

Let  $H^\infty(\mathbb{D}^2)$  denote the Banach algebra of bounded analytic functions on the bidisk. A *solvable Pick problem on  $\mathbb{D}^2$*  is a set  $\{\lambda_1, \dots, \lambda_N\}$  of points in  $\mathbb{D}^2$  and a set  $\{w_1, \dots, w_N\}$  of complex numbers such that there is some function  $\phi$  of norm less than or equal to one in  $H^\infty(\mathbb{D}^2)$  that interpolates (satisfies  $\phi(\lambda_i) = w_i \ \forall \ 1 \leq i \leq N$ ). An *extremal Pick problem* is a solvable Pick problem for which no function of norm less than one interpolates. The points  $\lambda_i$  are called the *nodes*, and  $w_i$  are called the *values*. By *interpolating function* we mean any function in the closed unit ball of  $H^\infty(\mathbb{D}^2)$  that interpolates.

Consider the two following examples, in the case  $N = 2$ .

Example 1. Let  $\lambda_1 = (0, 0)$ ,  $\lambda_2 = (1/2, 0)$ ,  $w_1 = 0$ ,  $w_2 = 1/2$ . Then a moment's thought reveals that the interpolating function is unique, and is given by  $\phi(z, w) = z$ .

Example 2. Let  $\lambda_1 = (0, 0)$ ,  $\lambda_2 = (1/2, 1/2)$ ,  $w_1 = 0$ ,  $w_2 = 1/2$ . Then the interpolating function is far from unique — either coordinate function will do, as will any convex combination of them. (A complete description of all solutions is given by J. Ball and T. Trent in [5]). But on the algebraic set  $\{(z, z) : z \in \mathbb{D}\}$ , all solutions coincide by Schwarz's lemma.

For an arbitrary solvable Pick problem, let  $\mathcal{U}$  be the set of points in  $\mathbb{D}^2$  on which all the interpolating functions in the closed unit ball of  $H^\infty(\mathbb{D}^2)$  have the same value. The preceding examples show that  $\mathcal{U}$  may be either the whole bidisk or a proper subset. In the event that  $\mathcal{U}$  is not the whole bidisk, it is an algebraic set intersected with  $\mathbb{D}^2$ . Indeed, for any  $\lambda_{N+1}$  not in  $\mathcal{U}$ , there are two distinct values  $w_{N+1}$  and  $w'_{N+1}$  so that the corresponding  $N + 1$  point Pick problem has a solution. By [5, 1] these problems have interpolating

functions that are rational, of degree bounded by  $2(N + 1)$ . The set  $\mathcal{U}$  must lie in the zero set of the difference of these rational functions. Taking the intersection over all  $\lambda_{N+1}$  not in  $\mathcal{U}$ , one gets that  $\mathcal{U}$  is the intersection of the zero sets of polynomials. Since  $\mathbb{C}[z_1, \dots, z_n]$  is Noetherian [9],  $\mathcal{U}$  is the intersection of the zero sets of a finite number of polynomials. Therefore  $\mathcal{U}$  is an algebraic set, and indeed, by factoring these polynomials into their irreducible factors, we see that  $\mathcal{U}$  is the intersection with the bidisk of the zero set of one polynomial, together with possibly a finite number of isolated points. We shall call  $\mathcal{U}$  the *uniqueness set*. (If the problem is not extremal,  $\mathcal{U}$  is just the original set of nodes).

We shall say that an  $N$ -point extremal Pick problem is *minimal* if none of the  $(N - 1)$  point subproblems is extremal. In [2], a set  $W$  was called a *distinguished variety* if  $W$  was the non-empty intersection of the zero set of a polynomial with the bidisk, and moreover it satisfied the property

$$\overline{W} \cap \partial(\mathbb{D}^2) = \overline{W} \cap \mathbb{T}^2.$$

The following theorem was proved in [2]:

**Theorem 4.1** *The uniqueness variety of a minimal extremal Pick problem on  $\mathbb{D}^2$  contains a distinguished variety that contains all the nodes.*

This theorem left open the possibility that  $\mathcal{U}$  might still have some isolated points in  $\mathbb{D}^2$ . We show that this cannot happen. Indeed,  $\mathcal{U}$  must be a toral algebraic set intersected with  $\mathbb{D}^2$ .

**Theorem 4.2** *The uniqueness set of a minimal extremal Pick problem on  $\mathbb{D}^2$  has the form  $\mathbb{D}^2 \cap Z_p$  where  $p$  is a toral polynomial.*

We shall need to use Lojasiewicz's Vanishing theorem in the proof. See [11, Thm. 6.3.4] for a proof of this form of the theorem:

**Theorem 4.3** [Lojasiewicz] *Let  $f$  be a non-zero real analytic function on an open set  $U$  in  $\mathbb{R}^n$ . Assume that the zero set of  $f$  in  $U$  is non-empty. Let*

$E$  be a compact subset of  $U$ . Then there are constants  $C$  and  $k$  such that  $|f(x)| \geq C \operatorname{dist}(x, Z)^k$  for every  $x \in E$ .

PROOF OF THM. 4.2: If  $\mathcal{U} = \mathbb{D}^2$ , take the polynomial to be 0. Otherwise, let  $\phi_1, \dots, \phi_m$  be rational inner functions that solve the Pick problem and such that

$$\mathcal{U} = \left( \bigcap_{i \neq j} Z_{\phi_i - \phi_j} \right) \cap \mathbb{D}^2.$$

(This can be done because  $\mathbb{C}[z_1, z_2]$  is Noetherian). Let

$$B = \bigcap_{i \neq j} Z_{\phi_i - \phi_j},$$

and let  $V$  be the union of the irreducible toral components  $V_i$  of  $B$ . Notice that by Theorem 4.1,  $V$  contains all the nodes of the interpolation problem. Let

$$\psi = \frac{1}{m}(\phi_1 + \dots + \phi_m) = q/p \quad (4.4)$$

be a rational solution, with  $p$  and  $q$  coprime polynomials, and  $q$  normalized to have modulus less than or equal to 1 on  $\mathbb{D}^2$ . Let

$$S = \bigcup_{j=1}^m S_{\phi_j}$$

be the union of the singular sets; then  $S$  is finite by Proposition 3.6. As  $p$  and  $q$  are coprime, any zero of  $p$  is a singularity of  $\psi$ , so

$$Z_p \cap \mathbb{T}^2 \subseteq S.$$

Notice that  $\psi$  will be unimodular on  $\mathbb{T}^2 \setminus S$  only when all of the  $\phi_j$ 's are equal. Therefore, we have

$$(B \cap \mathbb{T}^2) \cup S = \{\tau \in \mathbb{T}^2 : |\psi(\tau)| = 1\} \cup S.$$

For each  $\tau \in \mathbb{T}^2$ , define

$$l_\tau(z) = 2 - \bar{\tau}_1 z_1 - \bar{\tau}_2 z_2,$$



a linear polynomial whose only zero in  $\overline{\mathbb{D}}^2$  is at  $\tau$ . For each irreducible component  $V_i$  of  $V$ , let  $r_i$  be an irreducible polynomial that vanishes on  $V_i$ . Define

$$g = (\prod r_i) \prod \{l_\tau : \tau \in [(B \setminus V) \cap \mathbb{T}^2] \cup S\}. \quad (4.5)$$

Since we are working in 2 dimensions, Theorem 2.4 implies that atoral varieties intersect  $\mathbb{T}^2$  in a finite set. As  $B \setminus V$  is contained in the atoral component of  $B$ , we see that  $(B \setminus V) \cap \mathbb{T}^2$  is finite. Thus, the second product in (4.5) is over a finite set and  $g$  is a polynomial. Furthermore, we have

$$Z_g \cap \mathbb{T}^2 = (B \cup S) \cap \mathbb{T}^2. \quad (4.6)$$

Now  $|p|^2 - |q|^2$  is strictly greater than 0 on  $\overline{\mathbb{D}}^2 \setminus (B \cup S)$ . So applying Theorem 4.3 to the real analytic function  $|p|^2 - |q|^2$  on  $\mathbb{T}^2$ , this function must grow at least as fast as some power of the distance to its zero set. Since  $g|_{\mathbb{T}^2}$  vanishes on  $(B \cup S) \cap \mathbb{T}^2$ , and  $g$  is a polynomial, we know that  $|g|^2$  can grow no faster than a constant times the distance to  $(B \cup S) \cap \mathbb{T}^2$ . Therefore we conclude that there exist constants  $\varepsilon > 0$  and  $M \in \mathbb{N}$  such that

$$2\varepsilon|g|^M + \varepsilon^2|g|^{2M} \leq |p|^2 - |q|^2 \quad \text{on } \mathbb{T}^2. \quad (4.7)$$

Now, let

$$h = \varepsilon g^M.$$

With this definition of  $h$ , we have that  $h$  is zero on the nodes of the interpolation problem (since  $g$  vanishes on  $V$ , and  $V$  contains these nodes by Theorem 4.1),  $\|\psi + \psi h\| \leq 1$  (by (4.7)), and  $h \neq 0$  on  $\mathbb{D}^2 \setminus V$  (since by (4.5) the zeroes of  $h$  that are not in  $V$  are a union of hyperplanes that just graze the closed bidisk at a single point).

Therefore  $\psi + \psi h$  also solves the interpolation problem, and so

$$\mathcal{U} \subseteq Z_{\psi h} \cap \mathbb{D}^2 = (Z_\psi \cup V) \cap \mathbb{D}^2. \quad (4.8)$$

Suppose now that there is some point  $\lambda$  in  $(Z_\psi \setminus V) \cap \mathbb{D}^2$ . We wish to rule out  $\lambda$  being in  $\mathcal{U}$ . There are two possibilities: either there is some  $\phi_j$  that does not vanish at  $\lambda$ , or they all vanish there.

In the first case, replace  $\psi$  in (4.4) by  $\psi_\lambda$ , some other strict convex combination of the  $\phi_j$ 's that has the additional property that  $\psi_\lambda(\lambda) \neq 0$ . Now repeat the above argument with  $\psi_\lambda$  instead of  $\psi$ , and in place of (4.8) we get

$$\mathcal{U} \subseteq Z_{\psi_\lambda h} \cap \mathbb{D}^2 = (Z_{\psi_\lambda} \cup V) \cap \mathbb{D}^2. \quad (4.9)$$

As  $\psi_\lambda(\lambda) \neq 0$ , this precludes  $\lambda$  being in  $\mathcal{U}$ .

In the second case, let  $f$  be a Möbius map of  $\mathbb{D}$  such that  $f(0) \neq 0$ . Repeat the above analysis for the Pick problem that sends each  $\lambda_i$  to  $f(w_i)$ . It is clear that a function  $\vartheta$  solves the first problem if and only if  $f \circ \vartheta$  solves the second one, so the two problems have the same uniqueness sets. Moreover,  $f \circ \phi_j(\lambda) \neq 0$ , so we can conclude that  $\lambda$  is not in  $\mathcal{U}$ .

Therefore we have

$$\mathcal{U} \subseteq V \cap \mathbb{D}^2.$$

As

$$B \cap \mathbb{D}^2 \subseteq \mathcal{U},$$

we conclude that

$$\mathcal{U} = V \cap \mathbb{D}^2,$$

and so  $\mathcal{U}$  equals the intersection of the bidisk and the zero set of a toral polynomial, as desired.

□

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