OPERATOR SPLIT TECHNIQUE APPLIED TO RANDOM VIBRATION OF “SMART” SDOF MECHANICAL SYSTEMS

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Abstract

A significant amount of research in the last few years has greatly advanced the field of structural control through the use of smart materials. Smart materials have the ability to change shape, stiffness, natural frequency, damping and other mechanical characteristics in response to environmental condition changes. One of the new class of materials with promising applications in structural and mechanical systems are Shape Memory Alloys (SMAs). However, there are few studies where the random response of mechanical systems containing shape memory components has been conducted. Such study is important to verify the feasibility on the use of SMAs in structural components.

The present contribution reports on the operator split technique applied to random vibration of single-degree-of-freedom (SDOF) mechanical system with a shape memory helical spring. A constitutive theory originally developed by Flmond with internal variables that must satisfy internal constrains is used to model the restoring force provided by the spring. The operator split technique has been used on the solution of well-posed nonlinear dynamic problem and its basic idea is to promote a partition of state space in sub-spaces that may be solved separately. Results in terms of root-mean-square for both zero and non-zero mean random vibration is presented for SDOF shape memory oscillator. Numerical simulations of the system under a wide range of white noise excitation are presented.

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Introduction

Contrary to other metallic materials, which have a well-defined yield limit, SMAs can undergo large amount of inelastic deformations, without permanent plastic deformation, and recover that upon temperature change and/or loading. The fact behind this extraordinary ability, known as superelasticity, is the phase transformation mechanism. A temperature gradient and/or externally applied load can provide the required energy for the phase transformation. These two types of transformation are referred to as temperature-or stress-induced phase transformation respectively. In most cases for practical applications either temperature gradient or externally applied load can be treated as driving force.

The phenomena associated with martensitic phase transformations are highly nonlinear. As a consequence, when subjected to dynamic inputs, a mechanical system that contains a shape memory element may experience a number of quite complex behaviors.

The authors have previously investigated the random vibration of mechanical systems where a SMA element is incorporated (Duval et. al., 1999). In that article, the restoring force provided by a SMA helical spring has been described using a constitutive model based on Devonshire’s theory for temperature induced first-order phase transitions combined with hysteresis (Duval et. al., 1999). This problem is revised in the present contribution, where the spring is now analytically modeled using a theory with internal constraints that are imposed by the coexistence of distinct metallic phases in the alloy. As a consequence, the SDOF system is mathematically represented as a nonlinear system with four state variables.

The operator split technique has been used on the solution of different nonlinear problems and its basic idea is to promote a partition of state space in sub-spaces that may be solved separately. The solution of each sub-space permits to perform a sequence of updating where the results of the previous part are used as input on the next one. The main advantage of this procedure is the use of a combination of classical algorithms to evaluate the response of each part of the system. As example of this application one could mention the elasto-plastic behavior (Ortiz, et. al., 1983; Chorin, et. al., 1978) and nonlinear dynamical systems (Savi, et. al., 1997).

This paper addresses the random response of a SDOF system with a shape memory element. First, we briefly explain the operator split technique used to analyze the response of the oscillator. A constitutive theory for martensitic transformation induced by pure shear stress states is discussed. Then a formulation of the system which incorporates a constitutive model originally developed by Fland (Flmond, 1987,1996,1999) is presented. The random response of the system is then investigated under a wide range of temperature. Response statistics for zero and non-zero mean analysis of the system is carried out.
The Operator Split Technique

In order to present the operator split technique, one can envision a dynamical system described by the following equation of motion,

\[ \dot{x} = f(x) \quad x \in \mathbb{R}^n \]  

(1)

The technique considers a partition of the \( \mathbb{R}^n \) space in sub-spaces that may be solved separately. Initially, one separates the system in \( N \) parts, which may be written as,

\[ x^i = (x^i_1, x^i_2, \ldots, x^i_m) \quad \forall i = 1, 2, \ldots, N \]  

(2)

where \( x^i \in \mathbb{R}^m \). After this partition, the equations of motions can be written as,

\[ \dot{x}_i = f^i(x, t) \quad \forall i = 1, 2, \ldots, N \]  

(3)

Suppose that for each part, it is possible to define a differentiation scheme for updating the variable \( x \) from time-step \( n \) to time-step \( n+1 \). This scheme is defined by a classical integration procedure and may be associated with some specific algorithm used to treat nonlinearities of the problem. Hence,

\[ x^i_{n+1} = F^i(x^i_n, t_{n+1}) \quad \forall i = 1, 2, \ldots, N \]  

(4)

where \( F \) represents the nonlinear function \( f \) associated with the differentiation scheme applied to the vector field \( x \). The operator-split technique conceives a sequence of updating, as follows,

\[
\begin{align*}
 x^1_{n+1} &= F^1(x^1_n, x^2_n, x^3_n, \ldots, x^n_n, t_{n+1}) \\
 x^2_{n+1} &= F^2(x^1_n, x^2_n, x^3_n, \ldots, x^n_n, t_{n+1}) \\
 x^3_{n+1} &= F^3(x^1_n, x^2_n, x^3_n, \ldots, x^n_n, t_{n+1}) \\
 & \vdots \\
 x^n_{n+1} &= F^n(x^1_n, x^2_n, x^3_n, \ldots, x^n_n, t_{n+1})
\end{align*}
\]  

(5)

It should be pointed out that this sequence permits solving each sub-space separately by considering the variables of the other parts as known parameters. The solution of the previous part is used as input on the next one. An iterative procedure may be used to assure the convergence of the process.
Shape Memory Alloy Constitutive Model

Frémond (Frémond, 1987, 1996, 1999), has proposed a three-dimensional model for the thermoelastic response of SMA where martensitic transformations are described with the aid of two internal variables, these variables represent volumetric fractions of two variants of martensite, and most satisfy constraints regarding the co-existence of three distinct phases, the third being the parent austenite phase.

It has been noted (Savi, et. al., 1997) that Frémond’s original model can not describe martensite phase changes induced by pure shear stress states. Nevertheless, experimental torsion test curves presented by Jackson and co-workers (Jackson, et. al., 1972) indicate that these transformations occur in Nickel-Titanium (NiTi) alloys. Qualitatively, these curves are very similar to those obtained in tension tests performed in NiTi and other SMA. Based on this observation, we employ a one-dimensional version of Frémond’s theory that we assume is valid for pure shear stress states.

Considering the three-dimensional version of Frémond’s theory (Savi, et. al. 1997, Duval et. al., 2000) and replacing the stress, strain, and elasticity tensors respectively by the uniaxial shear stress $\sigma$, shear strain $\gamma$, and shear modulus $G$, we obtain the following constitutive law:

\[
\begin{align*}
\sigma &= G\gamma + \alpha(\beta_2 - \beta_1) \\
B_1 &= \alpha\gamma - L(\theta - 1) - \partial_1 I(\beta_1, \beta_2) \\
B_2 &= -\alpha\gamma - L(\theta - 1) - \partial_2 I(\beta_1, \beta_2)
\end{align*}
\]

where $\alpha$ is a constant proportional to the coefficient of thermal expansion, $L$ is the latent heat of the martensite-austenite phase change, and $\theta = \frac{T}{T_M}$, $T_M$ being the temperature below which austenite becomes unstable. Further, $\beta_i$ ($i = 1, 2$) represents the volumetric fraction of the $i$ martensite variant, $B_i$ a thermodynamical stress, and $\partial_i$ sub-differential with respect to $\beta_i$ (Rockafellar, 1970), while $I(\beta_1, \beta_2)$ is the indicator function associated with the following constraints:

\[
\begin{align*}
h_1 &= -\beta_1 \leq 0; \quad h_2 = -\beta_2 \leq 0; \quad h_3 = \beta_1 + \beta_2 - 1 \leq 0
\end{align*}
\]

The use of Lagrange multipliers offers an alternative approach to represent the indicator function and the sub-differentials. Rewriting the indicator function as

\[
I(\beta_1, \beta_2) = \lambda_1 h_1 + \lambda_2 h_2 + \lambda_3 h_3
\]

It follows that
Further, we consider a pseudo-potential of dissipation $\Phi(\beta_1, \beta_2)$, which is quadratic. Thus, the evolution equations for the internal variables are written as:

$$\eta_i \beta_i = B_i$$

where $\eta$ is a coefficient associated with internal losses occurring during the phase changes (Frémond, 1987, 1996, 1999).

**SDOF System with Shape Memory**

We consider an SDOF system where the nonlinear restoring force is provided by a helical spring with shape memory and the damping is assumed to be linear. By considering a constitutive equation proposed by Frémond (Fremond, 1987, 1996, 1999) it is possible to present the following equations of motion, (Savi et al., 1997; Duval et al., 2000):

$$\begin{align*}
\dot{y}_1 &= y_2 \\
\dot{y}_2 &= -\xi v_2 - y_1 - \alpha'(y_4 - y_3) + f(t) \\
\dot{y}_3 &= \frac{1}{\eta} \left[ y_1 - L'(\theta - 1) + \lambda_2 - \lambda_3 \right] \\
\dot{y}_4 &= \frac{1}{\eta} \left[ y_1 - L'(\theta - 1) + \lambda_2 - \lambda_3 \right]
\end{align*}$$

where $\alpha$, $L$ and $\theta$ are constants associated with temperature, $\xi$, damping coefficient and $\lambda_1, \lambda_2, \lambda_3$ are Lagrange multipliers. The function $f(t)$ is a forcing function. The variable $x_1$ represents displacement and $x_2$ the velocity. The volumetric fraction of martensitic variant is represented by $x_3 = \beta_1$, referred as positive martensite (M+), while $x_4 = \beta_2$ is the volumetric fraction of other variant referred as negative martensite (M). The system is also subjected to the following Tucker-Kuhn conditions (Luenberger, 1973):

$$\lambda_i x_3 = 0, \quad \lambda_2 x_4 = 0, \quad \lambda_3 (x_3 + x_4 - 1) = 0, \quad \text{and} \quad \lambda_i \geq 0 \quad \forall \ i = 1, 2, 3$$

For this particular system, the problem is split into two parts:

$$x^1 = (x_1, x_2) \quad \text{and} \quad x^2 = (x_3, x_4)$$

The solution of this system considers *Fourth Order Runge-Kutta* to solve part (1). Part (2) is split into two sub-parts using *Euler’s method* in the first sub-part of (2), combined with an *orthonormal projection* in the second (Savi et al., 1997). It should be pointed out that after all variables at the instant $t_{n+1}$ are calculated, the variables...
associated with part (1), \((x_1, x_2)\), have been evaluated using the values of \(x_3\) and \(x_4\) at the instant \(t_n\). Hence, it is necessary to return back to part (1) to re-calculate the actual state for the pair \((x_1, x_2)\) using values of \((x_3, x_4)\) at the instant \(t_{n+1}\). This procedure must be repeated until the value converges to two consecutive iterations.

**Numerical Results**

In the forthcoming analysis, we discuss the random vibration analysis of SDOF system with a shape memory element when it is subjected to a white noise excitation with constant level of Power Spectral Density (PSD). The SDOF system is governed by Eqs. (13) and (14), where \(\alpha = 1, L = 10, \eta = 0.1\) and \(\xi = 0; 0.2\) are considered. We assume \(\theta = 0.95\) which is a typical value for NiTi alloy (Tobushi, et. al., 1990). The behavior of the system when subjected to temperature variation has been obtained by assuming that temperature of the system initially at \(\theta_i\), changes to \(\theta_f\) in the following way (Duval et. al., 1999):

\[
\theta = \begin{cases} 
\theta_i, & t \leq t_i \\
\theta_i + (\theta_f - \theta_i) \sin \left[ \frac{\pi}{2} \left( \frac{t - t_f}{t_f - t_i} \right) \right], & t_i < t < t_f \\
\theta_f, & t \geq t_f
\end{cases}
\]  

(16)

This temperature variation has been induced when the mass of the system is displaced from its equilibrium position at instant \(t = 0\) and after an interval of time \(t_i\) (10 sec.), the temperature begins to change, up to the instant \(t_f\) (20 sec.), when it reaches the final temperature. The chosen temperature variation law obeys a simple assumption that this varies in a sinusoidal fashion in the pre-defined time interval. Good convergence rates are obtained with step size \(\Delta t = \frac{2\pi}{100}\). This is possible, since the orthogonal projections tends to correct small errors which otherwise would propagate in the solution.

Due to space limitation only the displacement response for zero mean and velocity for the non-zero mean case (dissipative system) are presented here. Figure 1 shows the zero mean displacement result for non-dissipative system (\(\xi = 0\)) with PSD = 1.0; 0.5 and \(\mu_f = 0.8\) (mean excitation value) at different temperature conditions. As the level of excitation increases, the system dynamics becomes richer and the effect of temperature reduces the displacement system response. Time history velocity response of the dissipative system (\(\xi = 0.2\)) under the same levels of PSD and mean excitation values is shown in Figure 2. As expected the level of response is reduced due to both damping effect and temperature variation. This behavior is of special interest, since it illustrates the capability of altering the dynamics of the system by changing its temperature.
Figure 1. Zero mean RMS Displacement response of SODF non-dissipative system ($\xi = 0$) under stationary white noise input $S_0 = 1.0; 0.5$ and $\mu_f = 0$

Figure 2. Zero mean RMS Velocity response of SODF dissipative system ($\xi = 0.2$) under stationary white noise input $S_0 = 1.0; 0.5$ and $\mu_f = 0$

Conclusions

In this paper we have focussed on the random response of a SDOF system where the nonlinear restoring force is provided by a spring with shape memory. The spring has been modeled using a constitutive theory with internal variables that are subjected to a set of constraints. The system response is then mathematically described by four state variables. An algorithm based on the operator split technique has been developed for numerical
simulations of a mechanical system under random excitation. Results are qualitatively similar to those obtained by authors in a previous analysis of the same system using a simpler constitutive model for the shape memory spring.

References


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