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ADDITIVE COMPARISONS OF STOP RULE AND SUPREMUM EXPECTATIONS OF UNIFORMLY BOUNDED INDEPENDENT RANDOM VARIABLES

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ABSTRACT. Let $X_1, X_2, \dots$ be independent random variables taking values in $[a, b]$, and let T denote the stop rules for $X_1, X_2, \dots$. Then $E(\sup_{n \geq 1} X_n) - \sup\{EX_t: t \in T\} \leq (1/4)(b - a)$, and this bound is best possible. Probabilistically, this says that if a prophet (player with complete foresight) makes a side payment of $(b - a)/8$ to a gambler (player using nonanticipating stop rules), the game becomes at least fair for the gambler.

1. Introduction. Suppose that $X_1, X_2, \dots$ are independent nonnegative random variables on a probability space $(\Omega, \mathfrak{A}, P)$, and let T_n denote the stop rules for $X_1, \dots, X_n$ and T denote the stop rules for $X_1, X_2, \dots$. The inequality $E(\max\{X_1, \dots, X_n\}) \leq k\sup\{EX_t: t \in T_n\}$ has been studied in the theory of semiamarts where Krengel and Sucheston [3] discovered that k can always be taken ≤ 4. Garling's proof [4] showed that $k = 2$, and that 2 is the best possible universal bound, and Hill and Kertz [2] found that in all nontrivial situations, weak inequality actually holds.

Such comparisons of expectations of the maximum with optimal stop rule expectations have been interpreted in probabilistic terms as comparisons between the optimal expected return of a prophet (a player with complete foresight), and a gambler (player using only nonanticipating stop rules). In this language, the $k = 2$ result says that the odds 2:1 make the game at least favorable for the gambler (versus a prophet playing the same game).

The purpose of this paper is to study the difference $E(\sup_{n \geq 1} X_n) - \sup\{EX_t: t \in T\}$ in the case the $\{X_i\}$ are uniformly bounded. The main result is

THEOREM A. *If $X_1, X_2, \dots$ are independent random variables taking values in $[a, b]$, then $E(\sup_{n \geq 1} X_n) - \sup\{EX_t: t \in T\} \leq (1/4)(b - a)$ (equivalently $\inf\{EX_t: t \in T\} - E(\inf_{n \geq 1} X_n) \leq (1/4)(b - a)$), and the bound is best possible.*

This result may be interpreted probabilistically as saying that in a uniformly bounded situation, a side payment, from a prophet to a gambler, of $(b - a)/8$ makes the game at least fair for the gambler.

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2. Preliminaries. For random variables X and Y, $X \vee Y$ denotes the maximum of X and Y, X^+ is $X \vee 0$, and EX is the expectation of X. The *value* $V(X_1, \ldots, X_n)$ of an ordered collection of independent random variables $X_1, \ldots, X_n$ is defined to be $V(X_1, \ldots, X_n) = \sup\{EX_t\colon t \text{ is a stop rule for } X_1, \ldots, X_n\}$.

For ease of reference, we include the following lemma, a consequence of backward induction.

LEMMA 2.1 ([**1**, p. 50]). *Let $X_1, \ldots, X_n$ be independent random variables. Then*
(a) $V(X_j, \ldots, X_n) = E(X_j \vee V(X_{j+1}, \ldots, X_n))$ *for* $j = 1, \ldots, n-1$; *and*
(b) *if t^* is the stop rule defined by* $t^* = j \Leftrightarrow \{t^* > j-1$ *and* $X_j \geqslant V(X_{j+1}, \ldots, X_n)\}$, *then* $EX_{t^*} = V(X_1, \ldots, X_n)$.

DEFINITION 2.2. For an integrable random variable Y and constants $-\infty < a < b < \infty$, let Y_a^b denote a random variable with $Y_a^b = Y$ if $Y \notin [a, b]$, $= a$ with probability $(b-a)^{-1}\int_{Y\in[a,b]}(b - Y)$, and $= b$ otherwise (i.e., $= b$ with probability $(b-a)^{-1}\int_{Y\in[a,b]}(Y - a)$).

The random variable Y_a^b is extremal with respect to Y, a, and b in the following sense, which is fundamental to the results in this paper.

LEMMA 2.2. *Let Y be any integrable random variable and $-\infty < a < b < \infty$. Then $EY = EY_a^b$, and if X is any integrable random variable independent of both Y and Y_a^b, then $E(X \vee Y) \leqslant E(X \vee Y_a^b)$.*

PROOF. That $EY = EY_a^b$ is immediate. For the second part of the conclusion, fix any X independent of both Y and Y_a^b, and verify that the function $\psi(y) = E(X \vee y)$ is convex. From the independence of X and Y, and convexity of ψ, it follows that

$$\int_{Y\in[a,b]}(X \vee Y) \leqslant (b-a)^{-1}\left\{[E(X \vee a)]\int_{Y\in[a,b]}(b - Y) + [E(X \vee b)]\int_{Y\in[a,b]}(Y - a)\right\}. \tag{1}$$

Thus

$$\begin{aligned} E(X \vee Y) &= \int_{Y\notin[a,b]}(X \vee Y) + \int_{Y\in[a,b]}(X \vee Y) \\ &\leqslant \int_{Y\notin[a,b]} X \vee Y + (b-a)^{-1}\left\{[E(X \vee a)]\int_{Y\in[a,b]}(b - Y) + [E(X \vee b)]\int_{Y\in[a,b]}(Y - a)\right\} \\ &= E(X \vee Y_a^b), \end{aligned}$$

where the inequality follows from (1). □

An alternative characterization for the extremal random variable Y_a^b is the following: Y_a^b is the random variable with maximum variance which coincides with Y off $[a, b]$, and which has expectation EY. In this respect, the conclusion of Lemma 2.2 becomes rather intuitive.

3. Proof of Theorem A. Without loss of generality (add, or multiply by, suitable constants) it will be assumed throughout the remainder of this paper that all random variables take values in [0, 1].

DEFINITION 3.1. For random variables $X_1, \ldots, X_n$, define $D(X_1, \ldots, X_n)$, the additive advantage of the prophet over the gambler, by $D(X_1, \ldots, X_n) = E(X_1 \vee \cdots \vee X_n) - V(X_1, \ldots, X_n)$.

As in [2], the main step in the proof will be to show constructively that for any sequence of $n > 2$ random variables there is a sequence of $n - 1$ random variables offering at least as large an additive advantage to the prophet.

LEMMA 3.1. *Given $n > 2$ and independent r.v.'s $X_1, \ldots, X_n$, there exists a zero-one valued random variable W independent of $X_2, \ldots, X_{n-2}$, and satisfying $D(X_1, \ldots, X_n) \leqslant D(\mu, X_2, \ldots, X_{n-2}, W)$, where $\mu = V(X_2, \ldots, X_n)$.*

PROOF. By Lemma 2.1, $V(X_1, \ldots, X_n) = V(\mu, X_2, \ldots, X_n) + E(X_1 - \mu)^+$. Since $E(X_1 \vee \cdots \vee X_n) \leqslant E(\mu \vee X_2 \vee \cdots \vee X_n) + E(X_1 - \mu)^+$, it follows that

$$D(X_1, \ldots, X_n) \leqslant D(\mu, X_2, \ldots, X_n). \tag{2}$$

Let $Z = (X_n)_0^1$ and $Y = (X_{n-1})_{EX_n}^1$ be independent of each other and of $X_1, \ldots, X_{n-2}$. By Lemmas 2.1 and 2.2, $V(Y, Z) = V(X_{n-1}, X_n)$, and therefore $V(\mu, X_2, \ldots, X_n) = V(\mu, X_2, \ldots, X_{n-2}, Y, Z)$. By Lemma 2.2,

$$E(\mu \vee X_2 \vee \cdots \vee X_n) \leqslant E(\mu \vee X_2 \vee \ldots \vee X_{n-2} \vee Y \vee Z).$$

Thus

$$D(\mu, X_2, \ldots, X_n) \leqslant D(\mu, X_2, \ldots, X_{n-2}, Y, Z). \tag{3}$$

Let W be any random variable independent of $X_2, \ldots, X_{n-2}$, and satisfying $P(W = 1) = V(Y, Z) = 1 - P(W = 0)$. Since $EW = V(Y, Z)$, and since $EZ = EX_n \leqslant \mu$, it follows from Lemma 2.1 and the definitions of Y, Z, and W that

$$D(\mu, X_2, \ldots, X_{n-2}, Y, Z) = D(\mu, X_2, \ldots, X_{n-2}, W). \tag{4}$$

Combining (2), (3), and (4) completes the proof. □

PROOF OF THEOREM A. It is clear that it suffices to prove the result for a finite number of random variables (e.g., see [4, p. 237]), and, by Lemma 3.1, the proof is further reduced to showing that $D(X_1, X_2) \leqslant 1/4$, and that the bound is sharp. Letting $EX_2 = \mu$, it follows as in (2) and (3) that $D(X_1, X_2) \leqslant D(\mu, Z)$, where $P(Z = 1) = \mu = 1 - P(Z = 0)$. But $D(\mu, Z) = E(\mu \vee Z) - V(\mu, Z) = (\mu + (1 - \mu)\mu) - \mu = \mu - \mu^2 \leqslant 1/4$ for $\mu \in [0, 1]$, and the bound $D = 1/4$ is attained for $\mu = 1/2$. □

4. Remarks. The parenthetical conclusion in Theorem A that $\inf\{EX_t : t \in T\} - E(\inf_{n \geqslant 1} X_n) \leqslant (1/4)(b - a)$ is immediate by symmetry. In contrast, no corresponding universal constant exists for ratio comparisons of $E(\min\{X_1, \ldots, X_n\})$ and $\inf\{EX_t : t \in T_n\}$, even if the random variables are indentically distributed as well as uniformly bounded, as the following example shows.

EXAMPLE 4.1. Fix $n > 1$ and $0 < p < 1/2$, and let $X_1, X_2, \ldots, X_n$ be i.i.d. each with common distribution given by $X_i = 0$ with probability $1 - p - p^2$, $= p^2/(1 - p)$ with probability p, and $= 1$ otherwise. Then

$$E(\min\{X_1, \ldots, X_n\}) = p^2[(p + p^2)^n - p^{2n} + (1 - p)p^{2n-2}]/(1 - p),$$

and $\inf\{EX_t\colon t \in T_n\} = p^2(p + p^2)^{n-1}/(1 - p)$. For any $M > 0$, and p sufficiently small, the random variables $X_1, \ldots, X_n$ satisfy $\inf\{EX_t\colon t \in T_n\} > ME(\min\{X_1, \ldots, X_n\})$.

If the independence assumption in Theorem A is dropped, the conclusion may fail, even if the sequence $X_1, X_2, \ldots$ is both a martingale and Markovian.

EXAMPLE 4.2. Define X_1, X_2, X_3 jointly distributed as follows: $(X_1, X_2, X_3) = (1/2, 2/3, 1)$ with probability $1/3$, $= (1/2, 2/3, 0)$ w.p. $1/6$, $= (1/2, 1/3, 1)$ w.p. $1/6$, and $= (1/2, 1/3, 0)$ w.p. $1/3$. For $n > 3$, let $X_n = X_3$. Then $E(\sup X_n) = 7/9$, and since $X_1, X_2, \ldots$ is a martingale, $\sup\{EX_t\colon t \in T\} = EX_1 = 1/2$. Thus $D(X_1, X_2, X_3, \ldots) = 5/18 > 1/4$.

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