Calculating and visualizing the density of states for simple quantum mechanical systems

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We present a graphical approach to understanding the degeneracy, density of states, and cumulative state number for some simple quantum systems. By taking advantage of basic computing operations, we define a straightforward procedure for determining the relationship between discrete quantum energy levels and the corresponding density of states and cumulative level number. The density of states for a particle in a rigid box of various shapes and dimensions is examined and graphed. It is seen that the dimension of the box, rather than its shape, is the most important feature. In addition, we look at the density of states for a multi-particle system of identical bosons built on the single-particle spectra of those boxes. A simple model is used to explain how the N-particle density of states arises from the single particle system it is based on. © 2014 American Association of Physics Teachers.

I. INTRODUCTION

The concept of the density of states (DOS) is used in many areas of physics. For example, it is important for reaction rates in nuclear physics, the calculation of specific heat rates in nuclear physics, the calculation of specific heat...
It is natural to associate each quantum number with a number line and each integer value with a point along this line. In two dimensions the pair of quantum numbers defines a plane or two-dimensional space called $n$-space. This space could also be called momentum space by using the identity $p_x = \hbar k_x = n_x \pi \hbar / L$. The energy depends on the sum of the squares of $n_x$ and $n_y$. One way of thinking about the distribution of these energies is to locate them in $n$-space. Using the horizontal axis for $n_x$ and the vertical axis for $n_y$, we can write all energy values at their corresponding grid points. This is done in Fig. 1, where at each pair of quantum numbers $(n_x, n_y)$ the energy is written (in units of $\epsilon_0$); for example, at grid points $(n_x, n_y)=(1, 3)$ and $(3, 1)$ we see a “10.” Suppose $n$ is the radius of a circle in $n$-space such that $n^2 = n_x^2 + n_y^2$. Rewriting the energy as $\epsilon_n = \epsilon_0 n^2$ we see that in $n$-space a given energy (in units of $\epsilon_0$) corresponds to a circle of radius $\sqrt{\epsilon/\epsilon_0}$. The radius $n$ is related to the magnitude of the momentum vector by $|p| = n \pi \hbar / L$. In Fig. 1, the circles corresponding to energies “36” and “65” are shown as dotted curves with radii $\sqrt{36} = 6$ and $\sqrt{65} \approx 8$, respectively.

When energies correspond to more than one independent state we say they are “degenerate.” If we make a list of all energies corresponding to the various quantum numbers $(n_x, n_y)$ and order them by energy, we can make a plot of the “number of states with energy $\epsilon$” vs $\epsilon$. This quantity is $d_n$, the degeneracy of the energy $\epsilon$, and for discrete spectra it is an integer. A plot of $d_n$ is a series of spikes, of height $d_n$, at each allowed $\epsilon$.

To further illustrate this idea, imagine a hypothetical single-particle spectrum where the lowest nine energies are $\{\epsilon_i\} = \{2, 2, 3, 3, 3, 5, 5, 5\}$. We have plotted $d_n$ vs $\epsilon$ for this system in Fig. 2 (upper left panel). At this point, it is helpful to introduce the cumulative state number $N(\epsilon)$, defined as the number of states with energy less than or equal to $\epsilon$; its graph is a staircase where each step has a height $d_n$ and a width determined by the gap to the next energy (lower left panel of Fig. 2). In other words, given an ordered list of energies $\{\epsilon_i\}$ we have $N(\epsilon) = i$ for $\epsilon_i < \epsilon < \epsilon_{i+1}$.

A plot of the spikes $d_n$ gives a visual measure of the degeneracies of the energies. These degeneracies are quite delicate in the sense that most perturbations to the potential will break them and the picture for $d_n$ will change dramatically. Each $d_n$-high spike will turn into a cluster of $d_n$ separate spikes, each one unit high. The spacing of the spikes will be determined by the strength of the perturbation. The corresponding change in $N(\epsilon)$ is that each step in the unperturbed system that was $d_n$ high will now become a series of short steps, each one unit high (upper and lower right panels of Fig. 2). It is hoped that Figs. 1 and 2 will be a useful starting point for student discussion.

A smooth DOS function $g(\epsilon)$ is useful because at higher energies one is interested in the number of states in an interval, or the density of spikes along the $\epsilon$-axis. The DOS and cumulative state number are related by $g(\epsilon) d\epsilon = dN(\epsilon)$, so the DOS is recognized as the slope of $N(\epsilon)$. We must be careful here because the slope of a staircase is infinite at each step, so we mean “slope” in an averaging sense.

To get $g(\epsilon)$ from $N(\epsilon)$, we can take a numerical derivative using a finite difference scheme:

$$g(\epsilon) = \frac{dN(\epsilon)}{d\epsilon} = \frac{N(\epsilon + \delta\epsilon/2) - N(\epsilon - \delta\epsilon/2)}{\delta\epsilon}. \quad (2)$$

It is worth saying explicitly that if the energies are put into bins of width $\delta\epsilon$, with center $\epsilon$, then $g(\epsilon) = (\text{number of states in bin})/(\text{width of bin})$.

Now that we have a numerical representation for $g(\epsilon)$, we would like to get an analytic expression for $g(\epsilon)$ that is valid for the statistical region where the DOS is large and well approximated by a smooth function. We will do this for the rigid box potentials where $\epsilon = n^2$ (in units of $\epsilon_0$). In $n$-space, the states with energy in an interval $d\epsilon$ centered on $\epsilon$ correspond to a set of points in a spherical shell of thickness $dn$ with all-positive coordinates. In Fig. 1, the number of states between the quarter circles of radius $n$ and $n+dn$ is proportional to the area of the curved band. This result is an
approximation, because \( n_x \) and \( n_y \) are discrete and there are fluctuations, but we will see that this is a good approximation nevertheless. The number of states is the “volume” of this shell but by definition it is also \( g(\epsilon) \, d\epsilon \), so in 2-D we have 
\[ g(\epsilon) \, d\epsilon = \frac{1}{2} \pi n \, d\epsilon. \]
If \( n(\epsilon) \) is a smooth function we can write 
\[ g(\epsilon) = \frac{1}{2} \pi n(\epsilon) \, d\epsilon, \]
and \( n(\epsilon) = \sqrt{\epsilon} \) and \( d\epsilon/dn = 1/(2\sqrt{\epsilon}) \) we see that the DOS is constant: 
\[ g(\epsilon) = \pi/4. \]
Since the DOS is the slope of \( N(\epsilon) \), the function 
\[ N(\epsilon) \] should be well approximated by the straight line 
\[ N(\epsilon) = \frac{\epsilon}{4}. \]
We remark here that if we have \( N \) non-interacting independent particles in a rigid cube, then the total energy would be the sum of the individual energies, and this sum could be related to the surface of a \( D = 3N \)-dimensional hypersphere. This problem is similar to the case of a single particle in a rigid box in a \( D \)-dimensional space. We would need to discuss the volume of a sphere of radius \( R \) in \( D \)-dimensions, denoted by \( V_D \) and given by 
\[ V_D = \frac{\pi^{D/2}}{\Gamma(D/2 + 1)} \, R_D = C_D R_D^D. \]

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\[ V_D = \frac{\pi^{D/2}}{\Gamma(D/2 + 1)} \, R_D = C_D R_D^D. \]

Here, \( \Gamma(D + 1) = D! \), giving \( C_2 = \pi \) and \( C_3 = 4\pi/3 \) as expected. The surface area of this \( D \)-sphere is given by 
\[ S_D = D C_D R_D^{D-1}. \]
Before we were concerned with the length of a quarter circle of radius \( n \) \((D = 2)\) and the area of a quadrant of a sphere of radius \( n \) \((D = 3)\), we now have the area of the positive portion of the \( D \)-dimensional sphere, given by \( (1/2)^D S_D \) (the need for \( n_x, n_y, \ldots \) to be positive gives a factor of 1/2 for each dimension). The cumulative state number is given by the volume of phase space enclosed by the boundary \( n = \sqrt{\epsilon} \):

\[ N(\epsilon) = \frac{1}{2^D} \frac{\pi^{D/2}}{\Gamma(D/2 + 1)} \epsilon^{D/2}. \]
and differentiation gives
\[
g(\varepsilon/C_15) = \frac{1}{2^{d+1}} D \frac{\pi^{D/2}}{\Gamma(D/2+1)} (\varepsilon/C_0)^{(D/2)-1}. \tag{5}\]

The forms of \(N(\varepsilon)\) and \(g(\varepsilon)\) for some representative \(D\) are given in Table I. Later we will see that the DOS for one particle in \(D\) dimensions is proportional to the DOS of \(D\) particles in one dimension, so long as the particles do not interact.

### III. PROCEDURE FOR CALCULATING AND VISUALIZING THE DENSITY OF STATES

For any quantum-mechanical system we can determine the cumulative state number and the corresponding density of states using the following procedure:

1. Solve the relevant quantum-mechanical problem to get a list of the energies of the system. (This can be an analytical expression or a list of energies obtained numerically.)
2. Sort the list of energies to get a set \(\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots, \varepsilon_n\}\) up to some maximum energy \(\varepsilon_{\text{max}}\).
3. Create the cumulative state number \(N(\varepsilon)\) by making the set \(\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots, \varepsilon_n\}\).
4. Find the DOS by taking the numerical derivative of the cumulative state number, as in Eq. (2). Choose a window size and locate the window so that its center is at \(\varepsilon\). The value of \(g(\varepsilon)\) is the number of energies that are within that window divided by the window size. Evaluate this for all locations of the window, and you now have a list of ordered pairs \(\{\varepsilon, g(\varepsilon)\}\).

As supplementary materials\(^{10}\), we include a MATLAB\(^{11}\) program used to obtain Figs. 5 and 6; with appropriate modifications the program can be used to produce Figs. 3 and 4.

We now apply this procedure to calculate and visualize the DOS for particles in rigid boxes with various geometries.

#### A. Cubical box

We first will consider a cube of side length \(L\).
Step 1: The energies are $\epsilon = (\pi^2 h^2)(n_x^2 + n_y^2 + n_z^2)/(2MV^{2/3})$, where we have used $L^2 = V^{2/3}$ with $V$ the volume of the box. If we set $(\pi^2 h^2)/(2M) = 1$ and $V = 1$, then the energies are all integers.

Step 2: Using Mathematica, we determined the lowest 15,954 energy levels. To calculate the energies we need all the triplets $\{n_x, n_y, n_z\}$ within the sphere of radius $n_{\text{max}}$ in $n$-space. We make a list of $n_x^2 + n_y^2 + n_z^2$ with $1 \leq n_x \leq n_{\text{max}}$, $1 \leq n_y \leq \sqrt{n_{\text{max}}^2 - n_x^2}$, and $1 \leq n_z \leq \sqrt{n_{\text{max}}^2 - n_x^2 - n_y^2}$, this list has all energies with $\epsilon \leq n_{\text{max}}^2$ (we chose $n_{\text{max}} = 32$). This is a very degenerate system—of the 15,954 energy levels only 818 are distinct and the average degeneracy is 19.50. As seen in Fig. 6, for $\epsilon \sim 1000$ we have $d_{\epsilon} \sim 60$. Specifically, the energies from 939–954, along with their degeneracies, are given by: $\{\epsilon, g(\epsilon)\} = \{933, 24\}, \{934, 39\}, \{936, 24\}, \{937, 27\}, \{938, 24\}, \{940, 6\}, \{941, 66\}, \{942, 18\}, \{944, 9\}, \{945, 48\}, \{946, 24\}, \{947, 15\}, \{948, 18\}, \{949, 12\}, \{950, 63\}, \{952, 12\}, \{953, 45\}, \{954, 42\}.

Step 3: Plot $N(\epsilon)$. It is instructive to construct the set of points for the plot explicitly using the list of energies. There are only 818 steps, one for each value of $\epsilon$. The height of the step at some energy $\epsilon$ is $d_{\epsilon}$, so the degeneracy, so this staircase has steps that get higher with energy (e.g., the step at $\epsilon = 950$ has a height of 63).

Step 4: While calculating the DOS, the window size needs to be adjusted to give a reasonable-looking graph. If the window is too small, the graph will look noisy; if it is too big, the graph may appear too coarse. We use trial and error to select an appropriate window size. We have included code in the online supplement in which the reader can select the maximum energy and window size and generate plots like Figs. 5 and 6.

Figures 5 and 6 display $d(\epsilon)$, $N(\epsilon)$, and $g(\epsilon)$ for this system. Again, it is evident that the agreement between the analytic result and the explicit counting of energies improves as the energy interval increases.

### B. Rectangular box

The rigid rectangular box is a straightforward adjustment to the cubical case and gives us a look at a non-degenerate spectrum. In order to remove the degeneracies from the single-particle spectrum that exist in the cubical case, we will make the sides of the box have incommensurate lengths. The choice $L_x \neq L_y \neq L_z$ with $L_x = 1$, $L_y = 2\epsilon$, and $L_z = e/2$ gives us a box of unit volume and a non-degenerate spectrum.

Step 1: The energies are $\epsilon = (\pi^2 h^2)|n_x/L_x|^2 + |n_y/L_y|^2 + |n_z/L_z|^2/(2M)$. Again, the obvious choice for our energy scale is to set $\pi^2 h^2/(2M) = 1$.

Step 2: This time we use a different method than in the cubical case. The sphere in $n$-space corresponds to an ellipse in phase space. This is a nice illustration of the difference between these abstract spaces. The method used was an exhaustive calculation of all energies corresponding to the points $1 \leq n_x \leq n_{\text{max}}, 1 \leq n_y \leq n_{\text{max}}$, and $1 \leq n_z \leq n_{\text{max}}$, after which we select $\epsilon \leq (2n_{\text{max}}^2/e^2)$. The fact that $n_x$ and $n_y$ are multiplied by a factor of 0.74 and 1.36, respectively, ensures that the sphere radius of $\sqrt{2/e}n_{\text{max}}$ is enclosed in the ellipse in phase space, and we had a complete set of energies. Although the method is inefficient, it has the advantage of transparency.

All the energies in this system are unique. In order to have degeneracies, we would need two sets of integers, $\{n_x, n_y, n_z\}$ and $\{m_x, m_y, m_z\}$, such that $n_x^2 + e^2n_y^2/4 + 4n_z^2/e^2 = m_x^2 + e^2m_y^2/4 + 4m_z^2/e^2$, which is impossible as $\epsilon$ is transcendental. The lowest 62,440 energies of the system are calculated.

Step 3: The cumulative state number $N(\epsilon)$ consists of steps of unit height, though the widths of the steps (level spacings) vary. For our rectangular box there are 41,435 steps, one for each energy. A plot of $d_{\epsilon}$ vs $\epsilon$ is a series of horizontal spikes of height 1.

Step 4: In Fig. 7, we show $g(\epsilon)$ and $N(\epsilon)$ for this system.

### C. Spherical box

The energies for a rigid sphere of radius $R$ are $\epsilon_{nlm} = h^2 k_{lm}^2/(2MR^2)$, where $k_{lm}$ is the $n$th zero of $j_l(r)$, the $l$th spherical Bessel function. Each $\epsilon_{nlm}$ has a degeneracy of $2l + 1$. It is easy to confuse $j_l(r)$ with $J_l(r)$, the ordinary Bessel function, also known as the Bessel function of the first kind; the two are related by $j_l(r) = \sqrt{\pi/2r}J_{l+1/2}(r)$. Note that the $n$th zero of $J_{l+1/2}(r)$ is written $K_{lm}$; MATHEMATICA has a useful add-on called NumericalMath‘BesselZeros’ that lists the zeros of the various Bessel functions. In Sec. VI we define the problem of the rigid cylindrical box, where the energies are proportional to $K_{lm}^2$, so to avoid confusion we provide the following check: the first three zeros of $J_{l+1/2}(r)$ are $[2.4048, 5.5201, 8.6537]$, and the first three zeros of $j_{l+1/2}(r)$ are $[3.9374, 7.8748, 11.8122]$. Actually, the distinction between $K_{lm}$ and $k_{lm}$ is not important as far as $g(\epsilon)$ goes for the sphere and the cylinder, as the two sets of zeros are interspersed.

Step 1: The role of $\pi$ in the energies of the rectangular wells is taken over by the $k_{lm}$ in the rectangular box, the

![Fig. 7. The upper panel shows the cumulative state number for the spectra for a single particle in cubical, rectangular, and spherical boxes. The spectra have been normalized so that $h^2\pi^2/(2M) = 1$ and the volume of each box is 1. The three curves are essentially indistinguishable. The lower panel is a histogram of the energies for each box. The bins are 40 units wide.](image-url)
zeros of the sine function are multiples of π. This allowed us to factor it out and choose the energy scale to be in units of \( \hbar^2 \pi^2 / (2M) \) for the rectangular boxes. Now, we have \( \epsilon = \hbar^2 (k_{ln}/R)^2 / (2M) \). To compare with the square wells we use an energy scale with \( \hbar^2 \pi^2 / (2M) = 1 \); in these units \( \epsilon_{ln} = [k_{ln}/(\pi R)]^2 \). Also, if the volume of the box is to be unity, then \( R = [3/(4\pi)]^{1/3} \), so finally we have \( \epsilon_{ln} = k_{ln}^2 / (3\pi^2 / 4)^{2/3} \). This is the spectrum we will compare to those of the rectangular and cubical boxes.

**Step 2:** We calculate all energies with \( k_{ln} < 95 \). To ensure a complete set, we use the brute force method and calculate all the triplets \( \{k_{ln}, l, n\} \) with \( 0 \leq l \leq 95 \) and \( 1 \leq n \leq 95 \), sort these 9,120 numbers, and select the first 95. Sorting triplets as opposed to just the list of \( k_{ln} \) makes it easy to accommodate the \( 2l + 1 \) degeneracy when making the final spectrum.

**Steps 3 and 4:** See Fig. 7.

### D. Weyl’s theorem

It is clear from Sec. II and Fig. 7 that the general trend for the three boxes we examined is \( N(\epsilon) = a \epsilon^b \). The energy exponent does not vary much with the shape of the box. A plot of \( \ln N(\epsilon) \) vs \( \ln \epsilon \) illustrates this nicely, as shown in Fig. 8. The deviation from the straight line occurs only at low energies, where the shape of the box matters. Using \textsc{Mathematica}, we perform linear fits to the logarithmic plots and obtain the results shown in Table II.

If the box is sufficiently large, a particle in a rigid box should not be aware of the particular shape of the box, so long as \( \lambda^D \ll V \), where \( V \) is the volume of the box. Here, \( \lambda = 2\pi/|\vec{k}| \) is the deBroglie wavelength (\( |\vec{p}| = \hbar |\vec{k}| \) and \( D \) is the dimension of the space. A slow particle will know about the edge, as it has a long wavelength, and a fast particle or high-energy state will not be sensitive to the shape because its wavelength is small. This is the content of Weyl’s theorem,\(^{15}\) which can be paraphrased as “high energy eigenvalues of the wave equation are insensitive to the shape of the boundary.” A nice account of the origin of the theorem is given by Kac\(^{16}\) and an explicit proof at the level of this paper for the case of the cylinder and sphere is given by Lambert.\(^{17}\)

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**Table II. Parameters for \( N(\epsilon) = a \epsilon^b \) for the three different 3-D “boxes.”**

<table>
<thead>
<tr>
<th>Shape</th>
<th>( a )</th>
<th>( b )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cube (theory)</td>
<td>( \pi/6 \approx 0.524 )</td>
<td>3/2 = 1.5</td>
</tr>
<tr>
<td>Cube (numerical)</td>
<td>0.29</td>
<td>1.58</td>
</tr>
<tr>
<td>Rectangular box</td>
<td>0.32</td>
<td>1.56</td>
</tr>
<tr>
<td>Spherical box</td>
<td>0.46</td>
<td>1.51</td>
</tr>
</tbody>
</table>

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### IV. N PARTICLES: STATISTICAL MECHANICS

Statistical mechanics can be introduced by analyzing a model system of \( N \) non-interacting particles on a single-particle spectrum, often chosen as a set of equally spaced levels.\(^7\) This is a valuable pedagogical tool for exploring the statistical distributions for classical particles, identical bosons, and identical fermions. This type of model is rich with subtleties—the particles not only don’t see each other beyond obeying the Pauli exclusion principle if they are identical fermions, but their presence has no influence on the energies of the levels they fill.\(^{18}\) Real particles, on the other hand, have interactions, so how does this model give useful results? One can think of the effect of all the other particles as being a smooth mean field that can be modeled as a potential for that single particle. Another subtlety is the robustness of the spectra of these \( N \)-body systems—they are very insensitive to the details of the single-particle spectrum upon which they are built, depending only on its DOS.

We made a simple model of the \( N \)-particle system to serve as a starting point for students to explore what happens to the DOS as \( N \) increases, and to illustrate that the DOS of many-body systems like this are insensitive to the details of the single-particle spectra they are based on. In this section, \( E \) is used for the total energy of the system, reserving \( \epsilon \) for the single-particle energy. Subscripts will be used on \( g \) and \( N \) to indicate the number of particles in the system, so \( g_N(E) \) is the DOS for a system of seven particles. We compare \( N \)-particle systems built on four distinct single-particle spectra: the square, rectangular, and spherical 3-D boxes, and a spectrum with a specified DOS but random levels. A simple analytical model for the DOS is presented and compared with the numerical results. The focus is on the relationship between the original single-particle energies that the \( N \) particles populate and the resulting \( g_N(E) \). The model demonstrates that \( g_N(E) \) depends on the density of the single-particle spectrum, as opposed to the details of the energies themselves; this is why we can replace the exact single-particle energies (zeros of Bessel functions, etc.) with any set of numbers as long as they have the same range and density.

Using a simple algorithm, the spectrum of \( N \) identical bosons on a set of single-particle energies \( \{\epsilon_i\} \) is calculated. The range of energies is \( 0 \rightarrow E_{\text{max}} \), with \( E_{\text{max}} \) limited only by computational power and user patience. The choice of bosons avoids the subtleties of Fermi statistics. The algorithm is best illustrated with an example. Let’s look at identical bosons on the single-particle energies of the rigid cube in three dimensions. First we make a list of the single-particle-state energies: \( \{\epsilon_i\} = \{3, 6, 9, 12, 14, 17, 20, 23, 26, 29, 32, 35, 38, 41, 44, 47, 50, 53\} \). We choose \( E_{\text{max}} = 53 \) so the list includes the first 140 states. This means that we are guaranteed to have a complete list of energies up to \( E = 53 \). For our purposes an \( N \)-particle state is a list of occupied single-particle states, so
(4, 5, 27) is a four-particle state with a particle in state 4, two particles in state 5, and a particle in state 27. Now list the indices of all possible one-particle states: \(\{(i)\} = \{(1, 2, 3, ..., 140)\}\); the energy of state \(i\) is \(\epsilon_i\). Next list the index pairs of all possible two-particle states: \(\{(i,j)\} = \{(1,1), (1,2), (1,3), ..., (140,140)\}\), sort them and drop duplicates; the energy of the state \((i,j)\) is \(\epsilon_i + \epsilon_j\). Drop all states with energy \(E > 53\) as you are only guaranteed a complete set of energies for \(E \leq 53\). In general, list all possible \(N\)-particle states, sort them, drop duplicates, and sum the corresponding single-particle energies. Once you have these spectra you can use the procedure in Sec. III to get \(g_N(E)\) and \(\mathcal{N}_N(E)\).

Systems based on a random single-particle spectrum are included for comparison with those based on the rigid boxes because this removes the role of geometry. The distribution of the random numbers is such that the DOS is \(g(\epsilon) = N \pi \sqrt{\epsilon}\); this is accomplished by taking a set \(\{x_i\}\) of \(N\) random numbers uniformly distributed on the interval \([0, 1]\), sorting them, raising them to the power of \(2/3\), and multiplying as you are only guaranteed a complete set of energies for \(E \leq 53\). In general, list all possible \(N\)-particle states, sort them, drop duplicates, and sum the corresponding single-particle energies. Once you have these spectra you can use the procedure in Sec. III to get \(g_N(E)\) and \(\mathcal{N}_N(E)\).

Given a single-particle spectrum with a one-particle DOS \(g_1(\epsilon)\), we can obtain a naive expression for the \(N\)-body density of states \(g_N(\epsilon)\) iteratively. The density of \(N\)-body states with energy \(\epsilon\) is a product of the density of \((N-1)\)-body states with energy \(\epsilon'\) and one-body states with energy \(\epsilon - \epsilon'\), or

\[
\ln N(\epsilon) = \ln g_1(\epsilon) + \int_0^\epsilon g_{N-1}(\epsilon') g_1(\epsilon - \epsilon') \, d\epsilon'.
\]

This expression double-counts some states. Suppose we add to the three-particle state \(\{n_1, n_2, n_3\} = \{3, 17, 17\}\) a fourth particle in state 24; then \(E = \epsilon_3 + 2\epsilon_{17} + \epsilon_{24}\), and there is a contribution to the integral in Eq. (6) from \(g_1(\epsilon_{52})g_3(\epsilon_3 + 2\epsilon_{17})\). However, we can get to the same state by adding a particle in state 17 to the three-particle state \(\{n_1, n_2, n_3\} = \{3, 17, 24\}\), and this gives a contribution to the integral in Eq. (6) from \(g_1(\epsilon_{17})g_3(\epsilon_3 + \epsilon_{17} + \epsilon_{24})\). This is the origin of the double counting and its effect is to increase the number of states by a number close to \(N\) at high energies, where the occupancies of the single-particle states are low. We did not fix this problem.

When iterating Eq. (6) we need the following result for integer \(n\) and \(m\):

\[
g_N(E) = \int_0^E g_{N-1}(\epsilon') g_1(\epsilon - \epsilon') \, d\epsilon'.
\]
and saw how the DOS for N-particle systems was simply related to \( g_1(\epsilon) \).

### VI. ADDITIONAL PROBLEMS

The following problems provide additional practice with the procedure used in Sec. III. The programs available in Ref. 10 may be a helpful starting point.

1. Examine changes to the density of states if we continuously change one of the dimensions of the rigid rectangular box. We expect the density of states to exhibit different behavior as we change continuously from a "2-D" system \((L_z \ll L_x, L_y)\) to a "3-D" system \((L_z \gg L_x, L_y)\). Let \(L_x = L_y = L_0 = 1\), and then let \(L_z\) vary over the range 0.01\(L_0 \leq L_z \leq 100L_0\).

2. The energy levels for a particle in a rigid cylindrical box of height \(H\) and radius \(R\) are given by
   \[
   \epsilon_{2n+1} = \frac{\hbar^2}{2m} \frac{n^2 \pi^2}{L_z^2}, \quad n = 0, 1, 2, 3, \ldots
   \]
   \[
   \epsilon_{2n} = \frac{\hbar^2}{2m} \frac{(n + \frac{1}{2})^2 \pi^2}{L_x^2} + \frac{\hbar^2}{2m} \frac{n^2 \pi^2}{L_y^2}, \quad n = 0, 1, 2, 3, \ldots
   \]
   Relation to the boundary conditions is in Appendix IV of Ref. 3, pp. 297–304.

3. Use the procedure of Sec. II for a relativistic particle in the 2-D rigid square box. The important difference is that relativistically we have \(E = pc = h\kappa c\), where \(k\) is the wave number and \(c\) is the speed of light.

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*Modern Statistical Physics (Cambridge, New York, 1999).*

Geissler Tube Set

“Geissler Tube” is a generic name for any discharge tube: an evacuated glass enclosure with two or more metallic electrodes passing through the walls. Some of the tubes are utilitarian, and some are just for enjoyment. Johann Heinrich Wilhelm Geissler (1815-1879) came from a glass-blowing family. About 1852 he became a maker of scientific instruments in Bonn, and in 1855 he constructed a vacuum pump that used droplets of mercury falling through a tube as a method of producing relatively high vacua. This pump was used to evacuate the discharge tubes that he made a few years later for Julius Plücker. These tubes contained rarified gases, and the discharge was obtained in a narrow channel between the electrodes at either end of a straight tube. We still use discharge tubes in undergraduate laboratory work on spectra of gases. This set of fanciful Geissler tubes was shown to me at Dartmouth College in 1979 by Prof. Allen King, who was responsible for the preservation of the historical physics apparatus at his institution. (Notes and photograph by Thomas B. Greenslade, Jr., Kenyon College)