Smooth Representation of Functions on Non-Periodic Domains by Means of the Fourier Continuation Method

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Introduction

The Fourier Continuation (FC) method\(^1\) provides a smooth, periodic continuation to a non-smooth function. This allows for a Fourier series to be used in the solution of Partial Differential Equations where periodicity is not applicable. Furthermore, this provides a smooth continuous interpolation of the function throughout the domain. In this investigation, the accuracy of the Fourier series and its derivative was compared to the analytical solution.

Background

Fourier series was initially developed to solve the heat equation in the 1820s. These methods approximate functions, \( f(x) \), in terms of a series of sine and cosine. The Fourier series is defined as:

\[
f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(2\pi nx) + b_n \sin(2\pi nx)
\]

where the coefficients \( a_n \) and \( b_n \) are given by:

\[
a_n = \frac{2}{T} \int_{c}^{c+T} f(x) \cos\left(\frac{2\pi nx}{T}\right) dx
\]

\[
b_n = \frac{2}{T} \int_{c}^{c+T} f(x) \sin\left(\frac{2\pi nx}{T}\right) dx
\]

However, something known as the Gibbs phenomenon occurs at the endpoints or discontinuities of a function, especially non-periodic functions (e.g., linear) when it is expressed as this series of waves. This phenomenon is an oscillation around a “jump” which results in an underestimation or overestimation of the value of this function at these points of interest.

Methods

Using a C++ code, the Fourier Continuation algorithm was applied to various functions on different, non-periodic domains. Python scripts were then used to plot the graphs of three different aspects of the given functions: 1) the newly formed smooth and periodic extension of the original function, 2) and interpolation of the original function using the new continued function \( f^c(x) \), and 3) the derivative of the original function using the Fourier Coefficients. Focusing on the interpolation and the derivative calculated by the program, in particular, the accuracy of the algorithm was determined by means of the absolute error.

Conclusions

The results given by this algorithm are a stepping stone to prove the viability and accuracy of the FC method. Further study should investigate its numerical efficiency (especially in regards to interpolation) and its applications to solving PDEs, in particular, the Vlasov and Poisson’s equations.

References

\(^1\) High-order unconditionally stable FC-AD solvers for general smooth domains I. Basic Elements

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