Pipe Flow – A Tutorial for Electrical Engineers

by

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Abstract Energy is transported over long distances primarily by electric power transmission lines and by pipelines carrying gas or liquid fuel. Electric power engineers need to compare these two energy transmission methods, but often lack knowledge of fluid mechanics pre-requisite to understanding pipeline transmission. This article provides a focused tutorial to provide the necessary background in the fluid mechanics of pipe flow.

1. INTRODUCTION

When you approach a gas-fueled electric power plant you immediately notice large power transmission lines dominating the landscape, carrying the generated electric power away from the plant. But where is the pipeline carrying natural gas into the plant? This pipeline carries three to four times more power than the electric transmission lines leaving. Yet it is completely inconspicuous. Often it is buried underground; always it is very much smaller than the massive structures required to support high-voltage electric power lines. Distributed generation is likely to become more prominent in the energy systems of the future, thus enabling significant gains in energy efficiency through the use for combined cooling heat and power of heat rejected during electric power generation and otherwise wasted. Much of this distributed generation will be derived from fuel supplied in pipelines. One of the factors to be considered in finding the right balance between centralized and distributed generation is the transport of energy to the site where it will be utilized. When is energy best carried as fuel in pipelines and when as electric power on transmission lines? To appreciate the trade-offs involved in answering this question, an understanding of the fluid mechanics of pipe flow is required. This tutorial is intended to guide the reader, assumed to be an electrical engineer without prior knowledge of fluid mechanics, to that understanding.

2. SUMMARY

Electric power engineers know that losses are reduced by operating transmission lines at high voltage. To understand this one needs to know that transported power is $P_t = VI$ and loss power is $P_l = I^2R$. The two are connected through Ohm's law, $V = IR$. Power can also be transmitted by a fuel flowing in a pipeline. The equivalent of Ohm's law for pipe flow is a relation between pressure gradient and fluid velocity (or volume or mass flow rate). Pressure gradient is like voltage drop; fluid velocity is like current. Impressing a pressure gradient across a pipe causes fluid to flow, just as impressing a voltage drop across a resistor causes current to flow. The resistor can be at high voltage with respect to ground, but it is only the voltage drop (difference across its terminals) that determines current flow. The pipe can be at high pressure with respect to the atmosphere, but flow is determined only by the pressure gradient (difference between inlet and outlet). The pressure gradient is expressed in dimensionless form as the friction factor. The velocity is expressed in dimensionless form as the Reynolds number. The relation between the two is the resistance law. It is the Ohm's law of pipe flow, but it lacks the simplicity and fundamental character of Ohm's law. The resistance law of pipe flow is:

$Re < Re_c$: $\lambda = 64/Re$

$Re_c < Re < Re_t$: $\dfrac{1}{\sqrt{\lambda/8}} = \dfrac{1}{\kappa}\ln\left(\dfrac{Re\sqrt{\lambda/8}}{2\alpha}\right)$

$Re_t < Re$: $\lambda = 8\kappa^2\left[\ln\left(\dfrac{1}{2\beta\varepsilon}\right)\right]^{-2}$.

This law is essentially empirical in character. The constants κ, α and β appearing in the law are empirical curve-fitting parameters. They are universal in that they are the same for a wide variety of flow conditions (different combinations of pressure gradient, flow velocity, pipe diameter and wall roughness). That there are such constants is a consequence of a remarkable fact: widely disparate flows obey the same law when scaled to dynamically similar non-dimensional quantities. The non-dimensional quantities that correspond to variable flow conditions are the friction factor (λ) representing the pressure gradient; the Reynolds number (Re) representing the average flow velocity; and the relative wall roughness (ε). The pipe diameter does not appear explicitly in the dimensionless resistance law; it re-appears when the dimensionless law is scaled to a dimensioned form. A purpose of the subsequent exposition is to explain what the dimensionless constants (κ, α and β) and variables (λ, Re and ε) are, and how they relate to the dimensioned characteristics of a specific fuel-carrying pipeline. Fuel-carrying pipelines operate at high flow rate for which $Re > Re_t$ where λ is independent of Re. Under these conditions fluid viscosity does not affect pressure drop in the pipeline which is determined by the wall roughness of the pipe. The relation between power lost due to friction and power transmitted is

$P_l = \dfrac{1}{G}\cdot\dfrac{1}{F}P_t^3$

where

$$G = A\sqrt{\frac{2D}{\lambda_t L}}$$

and

$$F = \rho h\sqrt{h}.$$

G depends only on geometric properties of the pipeline: diameter (D), length (L) and friction factor (λ_t, which in turn depends only on pipe rms wall roughness e). A is the cross-sectional area of the pipe, $\pi D^2/4$. F has units W m⁻² and may be called the fuel power density. It depends only on the fluid properties of the fuel carried in the pipeline: density (ρ) and specific enthalpy of combustion (h, energy per unit mass). The viscosity (μ) does not appear because friction losses are dominated by turbulence which in turn is controlled by the pipe wall roughness rather than fluid viscosity. The fuel power density is a figure of merit for ranking the suitability of different fuels as energy vectors for low-loss long-distance power transmission. The larger is this figure of merit, the more suitable is the fuel. Transmission losses are minimized by using a high-density fluid. Liquid fuels are therefore preferred to gas, other factors being equal. For an incompressible liquid the fuel power density is an intrinsic property. For a gas, it can be increased by compression, but then means to recover the power required for compression must be provided and losses associated with the compression/decompression cycle must be accounted for. For gases the fuel power density is best expressed

$$F = \frac{p_{op}M}{RT}h\sqrt{h}$$

where p_{op} is the operating (or optimum) pressure of the pipeline. A typical value of p_{op} for existing gas pipelines is 50 bar.

3. VISCOUS FLUID FLOW

To understand the results summarized in the previous section it is necessary to appreciate viscosity and the elements of viscous fluid flow. Viscosity is friction, the friction of fluid layers sliding over each other. It is perceived both as the resistance one feels when stirring a viscous fluid, and as the heat such stirring generates. It is the equivalent of resistance in electrical circuits, and is the reason why there are losses associated with power transmission by fuels flowing in pipelines.

3.1 Shear stress Lay a plank down on the ground and place on it a brick to which you have tied a string. Stand at one end of the plank and drag the brick toward you by pulling the string. You apply force to the string and momentum is transferred in the same direction you pull: the brick moves toward you. If the force with which you pull is F and the cross-sectional area of the string is A, the *normal stress* in the string is F/A. Another name for normal stress is pressure; both have units N m⁻² or Pa (Pascal). Normal stress transmits momentum in the same direction as the stress. As you pull the brick the plank is likely to move toward you too, perhaps at a slower speed if the brick slips over the surface of the plank. Momentum is transmitted from the brick to the plank. The direction of this momentum transfer is *perpendicular* to the direction of the momentum; it is transmitted by *shear force*. The shear force is in the direction of the pulled string, but the momentum transfer is perpendicular to that, from brick to plank. The shear force is proportional to the area A of the face of the brick in contact with the plank. The *shear stress* is this force divided by A. Like normal stress or pressure, the units of shear stress are N m⁻².

3.2 Couette flow Now imagine a trough filled with water on which a plank floats, the plank and the water all being at rest. Let the plank be set into motion, pulled along down the trough which is assumed long, like a canal. The water in the trough begins to move. At the bottom of the trough the water remains stationary (as is the trough itself, assumed fixed to the earth). At the top of the trough the water moves at the same speed as the plank, thus remaining stationary with respect to the surface of the (now moving) plank. Idealize this picture to parallel planes infinite in the y and z directions, separated by distance D in the x direction. Let one plane be located at $x = 0$ and the other at $x = D$. These planes represent solid walls, and the space between them is filled with fluid. Now let the bottom wall be stationary but the top wall be moving at speed u_w in the $+z$ direction. This situation, depicted in Figure 1, is Couette flow.
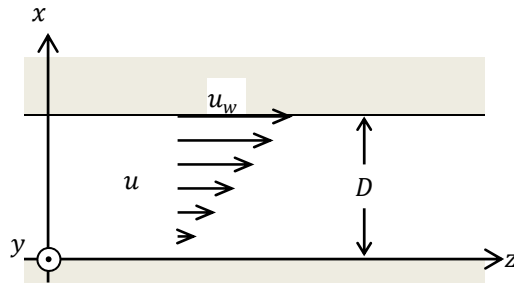
Figure 1: Couette flow

3.3 Viscosity In Couette flow each layer of fluid pulls along the layer beneath it, just as the brick resting on a plank and pulled by a string pulls the plank along with it. The friction force increases with the difference in velocity between that which is pulling and that which is pulled. If the speed of the top wall (u_w) is not too great, the fluid can be imagined to consist of layers that slide over each other without mixing, like a deck of cards resting on a table when the top of the deck is pushed. This is laminar flow; it is illustrated in Figure 2.
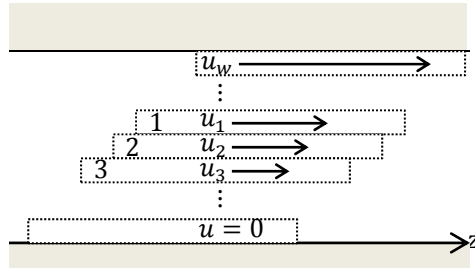

Figure 2: Laminar flow

A layer (card) experiences a force only if its velocity differs from its neighbors. For small velocities shear stress is proportional to velocity gradient:

$$\tau = \mu \frac{du}{dx}. \tag{1}$$

The proportionality constant μ is the viscosity. The units of viscosity are Pa s cm^{-1} or N s m^{-3} or kg s^{-1} m^{-2}. The shear stress is in the z direction, the same as the fluid velocity. The gradient of fluid velocity is perpendicular to that, in the x direction. Think of the fluid as like a deck of cards. The viscosity represents the friction force between two cards. For a new deck it is low; the cards slide easily. For an old deck it is high; the cards stick to each other.

In Figures 1 and 2, a force per unit area is exerted by the top (moving) wall on the topmost layer of fluid. It is directed in the $+z$ direction. If the flow is laminar it has magnitude

$$\tau_w = \mu \left(\frac{du}{dx} \right)_w. \tag{2}$$

The subscript w indicates that the derivative is taken next to the moving wall (at $x = D$). This force is in the $+z$ direction, the same direction as the velocity of the top layer of fluid. Work is done by whatever external agent maintains the speed of the moving wall. This agent expends power to maintain the motion. The power per unit area flowing into the fluid from the top surface is

$$P'' = \tau_w u_w = \mu \left(\frac{du}{dy} \right)_w u_w. \tag{3}$$

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The double-prime superscript means "per unit area." This power heats the fluid. It is proportional to the viscosity; in the absence of viscosity it disappears. Viscosity represents friction: the conversion of relative motion (kinetic energy) into heat.

3.4 Steady flow With both top and bottom walls at rest the fluid between them is also at rest. When the top plane starts moving the fluid also begins to move; all parts of it accelerate. Once the top wall reaches speed u_w and remains at that speed, steady flow is achieved. In steady-state there is no further acceleration and therefore no net force on any part of the fluid. Imagine a block of fluid of width W (in the y direction) and length L (in the z direction) and divided into layers of thickness Δx in the x direction, like a deck of cards. Let A be the area WL. Consider three layers of the fluid (three cards in the deck). This is shown in Figure 2, where the top layer is labeled 1, the middle layer 2, and bottom layer 3. All have volume $A\Delta x$. The middle layer is used as a *control volume*. This volume is shown in Figure 3.
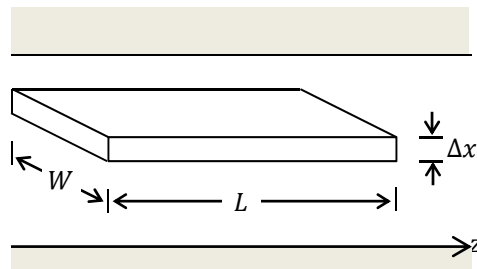

Figure 3: Control volume

The control volume is delineated (in imagination) from the rest of the fluid layer by the six faces of the rectangular prism that is the middle layer. These six faces constitute the *control surface*. In steady state, the net force on this (or any other) control volume is zero. This assertion requires justification. If layer thickness Δx in Figure 3 is small enough then the fluid inside the control volume all moves at the same velocity (u). The control volume itself can then be visualized as moving at that velocity, with the fluid inside stationary. If the velocity is constant the net force on this moving control volume must be zero, else the control volume would accelerate. Subsequently it will be necessary to consider a control volume which does not have uniform fluid velocity throughout, and then there is no single velocity that makes all the fluid inside stationary with respect to the control surface. Then the control volume may as well be fixed in space, with the fluid flowing through it. Steady flow still requires that the net force on the control volume be zero. The force balance must now be interpreted as momentum conservation. Net force equals the time rate of change of momentum. Constant momentum requires zero net force. As fluid flows through a control volume fixed in space, it carries momentum in as it enters and out as it exits. In steady flow the total momentum in the control volume remains constant: the momentum flowing in must equal the momentum flowing out. This is the same as asserting that the net force on the stationary control volume must be zero.

3.5 Force balance We begin by identifying all the forces applied to the control volume. The forces are of two kinds: body forces and surface forces. Body forces are exerted on the interior of the control volume and are due to fields, gravitational or electromagnetic. They are excluded (assumed zero or negligible) in the following. Surface forces are exerted on the control surface by the surrounding fluid and are specified as force per unit area, or stress. These are the forces included in the present analysis. Surface forces are of three kinds: static pressure, dynamic pressure, and shear stress. Static pressure and dynamic pressure are normal stresses: force per unit area acting in a direction perpendicular to the control surface. Shear stress is force per unit area acting in a direction parallel to the control surface. Static pressure exists even in the absence of fluid motion (of course it also exists in a moving fluid). Dynamic pressure and shear stress occur *only* in a moving fluid. To carry out a force balance on the control volume shown in Figure 3, consider each of the six faces of the control surface and ask: what is each of the three surface forces on each of these six faces? Set the vector sum of all these forces to zero to find the force balance. Take each of the three kinds of surface forces in turn.

First, there is static pressure. This is the force per unit area exerted by the pressure of the surrounding fluid on the six faces of the control volume. The static pressure vector points inward, from the outside to the inside of the control volume. In Couette flow the pressure is uniform throughout the fluid, and the six faces occur in pairs for which the pressure forces are equal and opposite, and thus cancel. This is shown in Figure 4. It is not necessary that static pressure forces cancel in steady flow. For example in the next system to be treated, channel flow, they will not all cancel.
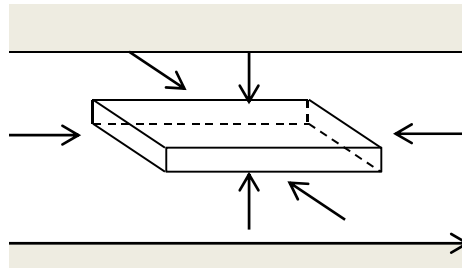

Figure 4: Static pressure force balance

Second, there is dynamic pressure due to the momentum of the flowing fluid. This pressure is $p_d = \frac{1}{2}\rho u^2$, where ρ is the fluid density and u its velocity. There is a force $p_d W \Delta x$ on the left (upstream) face of the control surface. This force is in the direction of flow. It is the force associated with the momentum of the fluid impinging on the surface; it is the force you feel on your hands when you catch a ball. You must exert the opposite force when you throw the ball. Likewise a surface of a control volume from which a fluid is exiting also experiences a force, in the direction opposite to the flow. This force is $-p_d W \Delta x$ on the right (downstream) face of the control surface, pointing in the direction opposite to the flow. It is a reaction to the momentum of the fluid leaving the control volume. This reaction force is equal and opposite to the force on the upstream face, as shown in Figure 5. These two forces cancel each other. There is no dynamic pressure on the other four faces of the control surface because those faces are parallel to the flow direction, so there is no perpendicular momentum.
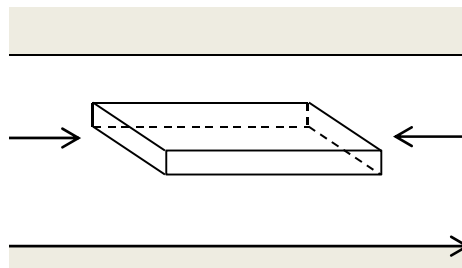

Figure 5: Dynamic pressure force balance

We consider two kinds of fluids: incompressible liquids and ideal gases. The dynamic pressure forces on a control volume always balance in steady flow of an incompressible fluid in a channel of constant cross section, for which the velocity profile at the upstream and downstream faces of the control surface are the same. For a compressible fluid such as an ideal gas, however, these forces do not cancel. When we come to carry out the analysis of ideal gas flow we will find that there is a net force due to dynamic pressure.

Third, there are shear stresses on the layers moving relative to each other. Only the top and bottom faces in Figure 3 are in motion relative to their neighbors; the other four faces of the control surface move at the same speed as the adjacent fluid. Therefore only the top and bottom faces experience shear stress. Consider again Figure 2, which shows three layers of fluid labeled 1, 2 and 3. The three layers move at different velocities,

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fastest at the top, slowest at the bottom. The top layer exerts a force on the middle layer, $F_{12} = A\tau_{12}$. The middle layer in turn exerts a force on the bottom layer, $F_{23} = A\tau_{23}$. The bottom layer exerts a reaction force back on the middle layer that is equal and opposite: $F_{32} = -A\tau_{23}$. The force balance is shown in Figure 6.
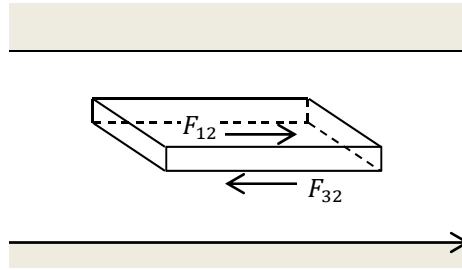

Figure 6: Shear stress force balance

The net force $F_{12} + F_{32}$ must be zero when the flow is in steady-state, thus $\tau_{12} = \tau_{23}$. This reveals that the same shear stress passed from the top layer to the middle layer is also passed from the middle layer to the bottom layer. Thus, in steady flow the shear stress profile is constant:

$$\tau = \tau_w. \tag{4}$$

This is shown in Figure 7. The shear stress profile is a consequence of the steady flow assumption, for which force balance (zero net force) must apply at all points in the fluid.
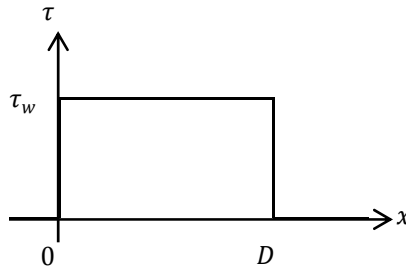

Figure 7: Shear stress distribution in
Couette flow

In the foregoing, a force balance was applied to a layer of fluid of thickness Δx, located somewhere in the middle of the fluid layer. The same argument applies to the total fluid layer. In Figure 1, the force per unit area exerted by the top (moving) wall on the fluid is τ_w, directed in the $+z$ direction. At steady-state, this force must be balanced by an equal and opposite force in the $-z$ direction. That force is supplied by the bottom (stationary) wall, which would be pulled in the $+z$ direction were it not fixed in place. The bottom wall therefore exerts on the fluid layer a force per unit area equal to τ_w and directed in the $-z$ direction. The same argument applies to any layer of fluid with top surface at $x = D$, where the stress is τ_w, and bottom surface at any x between 0 and D (thus thickness $D - x$). This is shown in Figure 8.
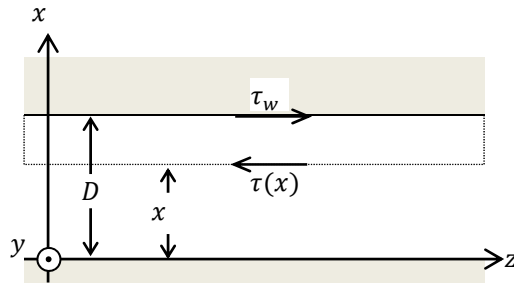
Figure 8: Shear stress in Couette flow is uniform

At the bottom surface of such a layer the stress is $\tau(x)$ and force balance requires that $\tau(x) = \tau_w$. This is true for any value of x; thus the shear stress is constant throughout the fluid layer. The foregoing illustrates that different control volumes can be used to find the shear stress profile in the fluid. It comes from the steady flow condition, which is that there is zero net force on any control volume of the fluid.

3.6 Laminar flow When the fluid flows like a deck of cards, one layer sliding over the other without mixing, momentum is transferred from one layer to the next only by the sliding friction at the interface between the layers. The shear stress is then given by (1). When this is combined with (4), the velocity profile is determined:

$$\frac{du}{dx} = \frac{\tau_w}{\mu} \tag{5}$$

$$u = \frac{\tau_w}{\mu}x + C. \tag{6}$$

There are two conditions imposed: $u = 0$ at $x = 0$ and $u = u_w$ at $x = D$. The first condition determines that $C = 0$. The second condition is that

$$u_w = \frac{D\tau_w}{\mu}. \tag{7}$$

The fluid velocity u is constrained to be u_w at the top wall because the relative velocity must be zero—the no-slip condition. An external agent sets the top wall velocity to u_w. In order to do so it must supply sufficient power to set the shear stress at τ_w. There is perhaps a feedback control mechanism monitoring the wall velocity and governing the power applied to pull the wall. For given velocity, more power is required for fluids of greater viscosity. The velocity of the fluid between the two planes increases linearly from zero at the bottom plane to u_w at the top plane:

$$u = \frac{u_w}{D}x. \tag{8}$$

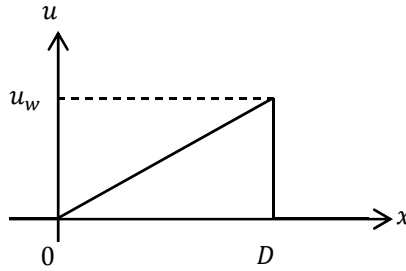This velocity profile is shown in Figure 9.

Figure 9: Velocity profile in Couette flow

3.7 Turbulent flow If the top wall velocity is high enough the flow will cease to be like a deck of cards. Instead the layers of fluid will mix; this is turbulent flow. Mixing means the cards emit pieces of themselves into the neighboring cards, and absorb pieces from the neighboring cards. The mixing length is the distance over which two cards can exchange momentum, and is the thickness of a layer (a card). In steady flow the shear stress profile is still constant (it must be in order that the net force anywhere in the fluid be zero). This requirement is independent of whether the flow is laminar or turbulent. The turbulent velocity profile, however, need no longer be linear. The linear velocity profile came from the laminar shear stress law (1) which stipulates a linear relation between shear stress and velocity gradient. This is valid only at low velocity. At high velocity the non-linearity of the true relationship between shear stress and velocity gradient reveals itself.

The problem of understanding turbulent flow in a fundamental way is a famous one, unsolved at present. Substantial insight is gained, however, from the Prandtl mixing length model. According to this model, the shear stress in turbulent flow is given by

$$\tau = \rho l^2 \left| \frac{du}{dx} \right| \frac{du}{dx}, \tag{9}$$

where ρ is the density of the fluid and l is the mixing length. The notation is chosen so that τ has the same sign as du/dx. The idea is that as a fluid layer of thickness l flows along in the z direction, it mixes with its neighboring layers by emitting particles of fluid (not molecules—supra-molecular aggregates like small droplets or "parcels") in the transverse (x) direction. In doing so, it transfers momentum in the direction perpendicular to the flow; i.e. it exerts shear stress on its neighboring layers. It likewise absorbs fluid particles emitted from its neighboring layers, and thus experiences shear stress exerted by them. The value of this model is that simple and intuitively reasonable assumptions about the mixing length l lead to agreement with experiment. Although not fundamental in a reductive sense, the Prandtl mixing length model provides useful insight.

Take (8) as given and combine it with the shear stress profile (4). If a further assumption about l (for example an assumed functional dependence on distance from the wall) be made, a differential equation results from which the velocity profile can be determined. This method is not here pursued for Couette flow but is applied to pipe flow in subsequent sections.

3.8 Reynolds number As the fluid velocity increases, flow transitions from laminar to turbulent. The velocity at which this transition takes place is determined by a dimensionless number that represents the ratio of inertial to viscous force, the Reynolds number:

$$Re = \frac{\text{Inertial force}}{\text{Viscous force}} = \frac{F_I}{F_V}. \tag{10}$$

The dimensions of inertial force are

$$F_I = ma = mvt^{-1}. \tag{11}$$

The dimensions of viscous force are

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$$F_V = \tau A = \mu \frac{du}{dx} A = \mu l^3 t^{-1}. \tag{12}$$

Thus:

$$Re = \frac{mvl}{l^3 \mu}. \tag{13}$$

Let v be a velocity and D be a length characteristic of the system under study; for example in Couette flow v is the average velocity and D is the distance between the walls. Then

$$Re = \frac{\rho v D}{\mu} = \frac{vD}{v}. \tag{14}$$

The fluid property $v = \mu/\rho$ is the momentum diffusivity, also known as kinematic viscosity. It has units of diffusivity (m² s⁻¹), thus v/v has units of length (m). The momentum diffusivity is a property of the fluid, the same for any geometry or flow conditions, whereas the average velocity depends on the velocity profile which varies with geometry and flow conditions. For Couette flow between infinite walls, an essentially one dimensional geometry, it is

$$v = \frac{1}{D} \int_0^D u(x) dx. \tag{15}$$

For laminar Couette flow, with velocity profile shown in Figure 9, $v = u_w/2$. In turbulent flow the velocity profile, and hence the average velocity, is different. The Reynolds number is a dimensionless version of the average velocity. It is the same for dynamically similar flows, no matter what the fluid, whether liquid or gas. It is known empirically that there is a critical Reynolds number at which the flow transitions from laminar to turbulent, and this critical value is

$$Re_c \approx 2100. \tag{16}$$

Note that for given fluid and geometry Re depends only on velocity. The average velocity corresponding to the critical Reynolds number is

$$v_c = vRe_c/D. \tag{17}$$

3.9 Channel flow A channel is like Figure 1, except that both top and bottom walls remain stationary. It is redrawn in Figure 10, now with the walls placed symmetrically around the z axis, at $x = \pm R$ (where $R = D/2$).
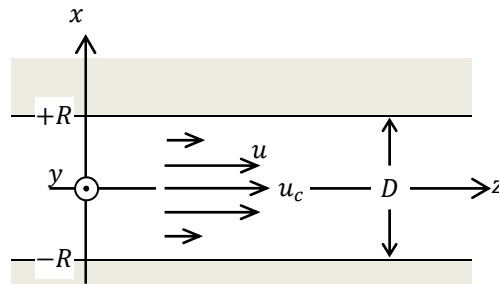
Figure 10: Channel flow

In this system flow results not from a moving wall, but from a pressure gradient. If the pressure gradient is in the z direction there will be flow in that direction. Since flow is from high to low pressure it is in the direction of the negative gradient, $-dp/dz$. At the walls the velocity is zero. At the center of the channel ($x = 0$) the velocity is maximum. The velocity profile in laminar (low velocity) flow is parabolic:

$$u = u_c \left[1 - \left(\frac{x}{R} \right)^2 \right]. \tag{18}$$

To prove that this is the case, first find the shear stress profile in the channel, then combine it with the law of laminar shear (1) to find the velocity profile. Find the shear stress profile from the steady flow condition: zero net force on any control volume. A convenient control volume is a slab of thickness t, centered on the flow (z) axis. The top of the slab is at $x = +t/2$ and the bottom at $x = -t/2$. Let the width be W in the y direction and length L in the z direction. This is shown in Figure 11.
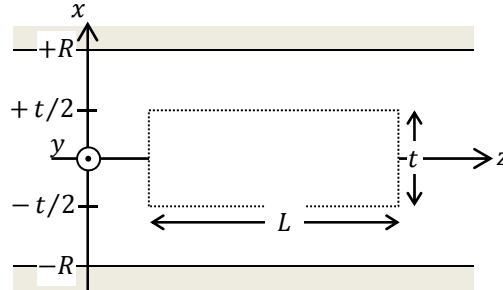

Figure 11: Control volume for channel flow

To perform the necessary force balance on this control volume, note first that both normal and shear stresses contribute. The normal stress is the static pressure. In the case of Couette flow it was assumed constant, but for channel flow it is assumed to have a *constant gradient* in the flow direction. This gradient is denoted by p':

$$p' = -dp/dz. \tag{19}$$

The flow direction is $+z$; dp/dz is a negative number; p' is a positive number. The single-prime superscript signifies "per unit length." In the transverse directions we assume the pressure is uniform: $dp/dx = dp/dy = 0$. As was the case in Couette flow, the force balance on the control volume has components due to static pressure, dynamic pressure, and shear stress. Because the fluid is incompressible the dynamic pressure components on the upstream and downstream faces cancel. There are also static normal components on all six faces. The components on the four faces perpendicular to the x and y axes cancel each other because there is no pressure gradient in those directions. Due to the pressure gradient in the flow direction the normal components on the two faces perpendicular to the z axis do not cancel each other. It is this net force that drives the flow. The static pressure force on the upstream face is $F_u = p_0 Wt$; that on the downstream face is $F_d = -p_L Wt = -(p_0 - p'L)Wt$. The negative sign on F_d signifies that it points in the $-z$ direction, opposing the flow. The net force in the flow direction due to static pressure is $F_u + F_d = p'LWt$. At steady state this has to be balanced by the net shear force on the top and bottom faces (the faces parallel to the flow direction) as shown in Figure 12.

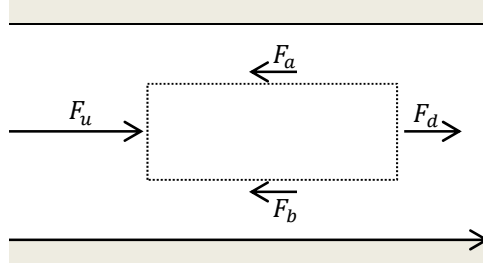
Figure 12: Force balance for channel flow

By symmetry the forces on both top and bottom faces are the same: $F_a = F_b = -\tau WL$. τ is the shear stress at transverse distance x from the flow (z) axis, the same as that at $-x$. F_a is the shear force exerted on the control volume by the fluid layer just above it, and F_b is that by the fluid layer just below it. The net force in

the flow direction due to shear stress is $F_a + F_b = -2\tau WL$. Setting the sum of all four force components to zero yields:

$$\tau = \tfrac{1}{2}p't = p'|x|. \tag{20}$$

The shear stress is zero at the center of the flow and increases linearly toward the walls. At a wall it is

$$\tau_w = p'R, \tag{21}$$

thus the shear stress profile is

$$\tau = \tau_w \frac{|x|}{R}. \tag{22}$$

This is shown in Figure 13.
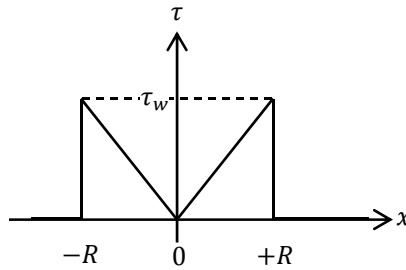
Figure 13: Shear stress distribution in
channel flow

The shear stress profile has been derived on only one assumption, steady state, and is valid for both laminar and turbulent steady flows. If we further assume laminar flow the velocity profile follows by the use of (1):

$$\mu \frac{du}{dx} = \tau_w \frac{|x|}{R}. \tag{23}$$

Integrating (23) and using the condition that the velocity is zero at the wall gives the expected parabolic profile (18). Further, we find that the center velocity is

$$u_c = \frac{\tau_w R}{\mu} = \frac{p'R^2}{\mu}. \tag{24}$$

The average velocity is found by substituting (18) into (15) and carrying out the integration, with the result:

$$v = \frac{1}{2}u_c = \frac{p'R^2}{2\mu}. \tag{25}$$

As expected the average velocity increases with pressure gradient and decreases with viscosity. The R^2 term emphasizes the role of the fixed walls: they are the source of losses in the flow. The farther the fluid can get from the walls, the faster it can flow.

4. PIPE FLOW

Pipe flow is just like channel flow, if cylindrical coordinates be substituted for Cartesian. We consider a pipe of radius R centered on the z axis, as shown in Figure 14.
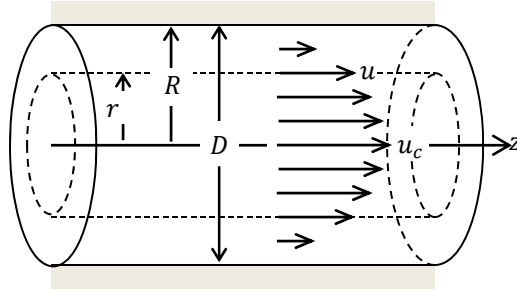
Figure 14: Pipe flow

A force balance on a control volume of radius r and length L is shown in Figure 15.
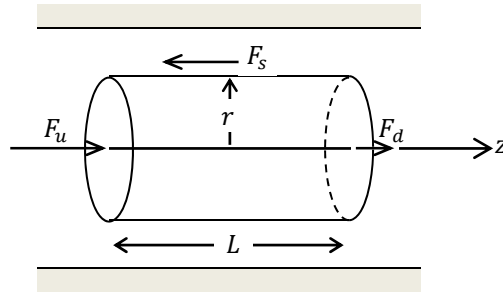
Figure 15: Force balance for pipe flow

The forces are $F_u = p_0 \pi r^2$ on the upstream face and $F_d = -p_L \pi r^2 = -(p_0 - p'L)\pi r^2$ downstream. The negative sign on F_d signifies that it points in the $-z$ direction, opposing the flow. The net force in the flow direction due to normal stress is $F_u + F_d = p'L\pi r^2$. There is also normal force in the radial direction, but its net value is zero. Net dynamic pressure is also zero for incompressible flow.

At steady state, the net longitudinal normal force has to be balanced by the net shear force on the cylindrical surface. This force is $F_s = -\tau 2\pi r L$. τ is the shear stress at radius r from the flow (z) axis. Setting the sum of all three force components to zero yields the shear stress:

$$\tau = \frac{1}{2}p'r. \tag{26}$$

The shear stress is zero at the center of the flow and increases linearly toward the walls. At a wall it is

$$\tau_w = \frac{1}{2}p'R, \tag{27}$$

thus the shear stress profile is

$$\tau = \tau_w \frac{r}{R}. \tag{28}$$

The shear stress profile is shown in Figure 16. It is valid for steady flow of any kind, laminar or turbulent.

Figure 16: Shear stress distribution in pipe flow

4.1 Laminar pipe flow The laminar velocity profile is parabolic as in (18) with x replaced by r. The center velocity is:

$$u_c = \frac{\tau_w R}{\mu} = \frac{p'R^2}{2\mu}. \tag{29}$$

The average velocity is

$$v = \frac{1}{\pi R^2} \int_0^R u(r) \cdot 2\pi r dr. \tag{30}$$

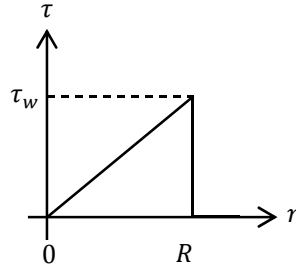Using (18) with x replaced by r yields

$$v = \frac{1}{2} u_c = \frac{p'R^2}{4\mu}. \tag{31}$$

Note that the center and average velocities for pipe flow are half the corresponding values for channel flow, a result of going from a one dimensional to a two dimensional geometry.

4.2 Pipeline power in laminar flow The average velocity connects pipe flow to pressure gradient, leading to the resistance law for pipe flow. This in turn leads to a relation between power lost and power transported in a pipeline. The flow through the pipe is specified either as volume or mass flow. Volume flow is the cross-sectional area of the pipe, $A = \pi R^2$, times the average velocity:

$$\dot{V} = Av. \tag{32}$$

The mass flow is the density times the volume flow:

$$\dot{m} = \rho \dot{V}. \tag{33}$$

The pressure gradient required to establish a given laminar volume flow is:

$$p' = \left(\frac{4\pi\mu}{A^2}\right)\dot{V}. \tag{34}$$

The power per unit length of pipeline to maintain the flow is $p'\dot{V}$. This power is lost: it heats the transported fluid and assuming isothermal conditions that heat is dissipated to the environment surrounding the pipeline. The total power lost in a pipeline of length L is

$$P_l = p'\dot{V}L = \left(\frac{4\pi\mu L}{A^2}\right)\dot{V}^2. \tag{35}$$

The power transported is

$$P_t = h\dot{m} = \rho h \dot{V}, \tag{36}$$

where h is the specific enthalpy of combustion (enthalpy per unit mass) of the fuel carried in the pipeline and ρh is the enthalpy density (enthalpy per unit volume). The relation between power lost and power transported is:

$$P_l = \frac{4\pi L}{A^2}\frac{\mu}{(\rho h)^2}P_t^2 = \frac{64L}{\pi D^4}\frac{\mu}{(\rho h)^2}P_t^2. \tag{37}$$

The group $\pi D^4/64L$ is a figure of merit with units m^3 that characterizes the geometric structure of a given pipeline. The group $(\rho h)^2/\mu$ is a fuel materials figure of merit, with units W m^{-3}. For any given fuel pipeline, transmission losses are minimized by maximizing these two parameters. This result, however, assumes laminar pipe flow, which is never the case in practice. For a given energy transport rate (P_t) and length (L), a pipe diameter (D) must be chosen. Larger diameter results in lower losses but costs more to build and operate. This is an optimization problem driven by the cost of the pipeline. As it turns out, the optimal diameter is always one that results in flow rates that are far beyond the laminar flow limit. Flow in fuel pipelines is always turbulent. We turn next to modifying the foregoing analysis for the case of turbulent flow.

4.3 Friction factor The friction factor is central to the analysis of losses in turbulent pipe flow. It is a dimensionless number proportional to the pressure gradient. It is defined by

$$f = \frac{\tau_w}{p_d} = \frac{\tau_w}{\frac{1}{2}\rho v^2} \tag{38}$$

where τ_w is the shear stress at the wall and $p_d = \frac{1}{2}\rho v^2$ is the dynamic pressure (or kinetic energy density). The wall stress is proportional to pressure gradient according to (27); recall that this equation is valid in all flow regimes whether laminar or turbulent. Therefore the friction factor is also proportional to pressure gradient. As defined above f is the Fanning friction factor. An alternative is the Darcy-Weissbach-Moody friction factor λ. The two are related by

$$\lambda = 4f. \tag{39}$$

Like the Reynolds number the friction factor is a ratio of viscous to inertial forces. The Reynolds number is proportional to average velocity, the friction factor to wall shear and hence pressure gradient.

4.4 Resistance law The relation between pressure gradient and average velocity is a relation between friction factor and Reynolds number. This is the fundamental relation, the resistance law that we need to analyze losses in pipe flow. It is the equivalent of Ohm's law for the flow of electric current. Graphically the law is most often presented as a log-log plot of λ vs. Re; that is, as a plot of $\log \lambda$ vs. $\log Re$. This is shown in Figure 17, a schematic version of a Blasius-Stanton-Moody diagram.
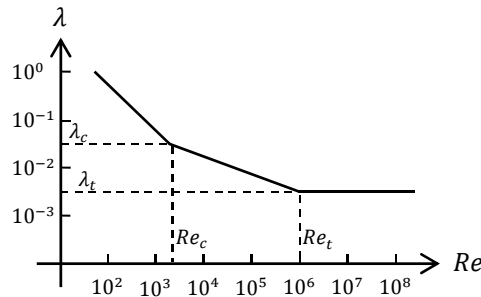
Figure 17: Blasius-Stanton-Moody
diagram

There are two breakpoints in the curve with corresponding Reynolds numbers Re_c and Re_t. Re_c, the critical Reynolds number, divides the laminar and turbulent flow regimes. Its value is 2100. Re_t, the transition Reynolds number, divides the turbulent flow regime into smooth wall and rough wall domains. All flows are dominated by wall roughness at sufficiently high Reynolds number. The wall roughness is characterized by a length parameter e, the rms value of the deviation of the wall surface from its average level. The roughness is characterized by the ratio of this parameter to the pipe diameter: $\varepsilon = e/D$. The transition Reynolds number is determined by this parameter; a formula will be subsequently derived.

For power engineers the most interesting case is $Re > Re_t$, the typical range for fuel pipelines. In developing the resistance law, however, it is easiest and also instructive to begin with the laminar range, $Re < Re_c$. Reviewing (28) to (31), one observes that the average velocity is directly related to wall shear stress:

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$$\tau_w = \frac{4\mu v}{D}. \tag{40}$$

Substitute this in (38) and use the definitions $\lambda = f/4$ and $Re = \rho v D/\mu$ to obtain

$$\lambda = \frac{64}{Re}. \tag{41}$$

This is the Hagen-Poiseuille law that relates pressure gradient (λ) to fluid velocity (Re) for laminar flow. In a Blasius-Stanton-Moody diagram such as Figure 17 it appears as a straight line with slope -1. Compare the same law written in dimensional form:

$$p' = \left(\frac{16\mu}{D^2}\right)v. \tag{42}$$

It is striking that, whereas in dimensional form the direct proportionality between pressure gradient and average velocity is evident, their dimensionless surrogates λ and Re stand in inverse relationship. It is because the Reynolds number is directly proportional to the velocity, whereas the friction factor is *inversely* proportional to the *square* of the velocity. This gives Figure 17 a non-intuitive character: the pressure drop parameter (λ) *decreases* as the flow rate parameter (Re) *increases*.

4.5 Universal velocity profile Review the foregoing procedure; note that the resistance law was obtained by finding the relation between average velocity (v) and wall shear stress (τ_w) which in turn came from the velocity profile (u). The velocity profile is fundamental to an analysis of flow phenomena. Therefore turn attention to deriving a velocity profile valid in all pipe flow regimes: a universal velocity profile.

Before deriving it we state the result:

$$\frac{u}{u_*} = \begin{cases} \dfrac{1}{\kappa}\ln\left(\dfrac{R-r}{\gamma y_*}\right) & \text{for } 0 \leq r < R - t \\ \dfrac{R}{y_*}\left[1 - \left(\dfrac{r}{R}\right)^2\right] & \text{for } R - t \leq r < R. \end{cases} \tag{43}$$

The cross-section of the pipe is divided into two regions, as shown in Figure 18.
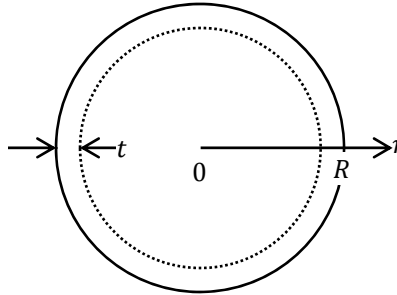
Figure 18: Turbulent core with annular
laminar layer

The pipe radius is R. An inner circle of radius $R - t$ is the turbulent core, and an outer annulus bounded by the wall is the laminar layer. The laminar layer thickness is t. In laminar flow $t = R$; the laminar layer extends over the entire cross-section of the pipe and there is no turbulent core. At an average velocity corresponding to the critical Reynolds number (Re_c) the turbulent core begins to develop. The laminar layer thickness becomes smaller as the average fluid velocity increases. At high velocity, it shrinks to a thin layer next to the wall. Eventually it approaches the size of the wall rms roughness (e). The average velocity then corresponds to the transition Reynolds number (Re_t). From that point on wall roughness dominates the flow. The laminar layer ceases to shrink as the velocity increases; it remains fixed at a value on the order of the wall roughness. The universal velocity profile consists of two pieces: one in the laminar layer and one in the turbulent core. The piece in the laminar layer is just the parabolic profile we already know. The piece in the turbulent core is Prandtl's logarithmic profile.

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4.6 Logarithmic velocity profile This is derived from Prandtl's mixing length model of turbulent shear stress (9) combined with the following assumption about the mixing length:

$$l = \kappa y \sqrt{r/R} = \kappa y \sqrt{1 - y/R} \tag{44}$$

where $y = R - r$ is the radial wall distance (radial distance measured from the wall rather than the center of the pipe). In developing the mechanics of turbulent flow y is more suitable than the radial center distance (r) because turbulence is strongly influenced by the wall. Prandtl assumes that near the wall, the mixing length increases linearly going away from the wall (toward the center). The proportionality factor is κ, the Kármán constant, a dimensionless number. The assumption $l = \kappa y$ gives a logarithmic velocity profile near the wall (where $r/R \approx 1$), but not too near (the log function has a singularity at zero). Prandtl then extrapolates this profile all the way to the pipe center. The Prandtl extrapolation is equivalent to augmenting the linear mixing length κy with the factor $\sqrt{r/R}$ (thus creating a non-linear mixing length formula) as in (44) above. This is an ad hoc assumption: there is no justification for it except that it results in good agreement with experiment. For present purposes that is justification enough. There are many attempts in the literature (starting with von Kármán) to replace this assumption with one that has a more satisfactory theoretical justification. But no theory of the Prandtl mixing length that is both simple and in agreement with experimental observations has yet emerged.

Combining (9), (28), and (44) the result is:

$$\tau_w = \rho \kappa^2 y^2 \left(\frac{du}{dy}\right)^2. \tag{45}$$

It is natural to normalize the velocity u in this equation using the *friction velocity* u_* defined by $\tau_w = \rho u_*^2$, that is

$$u_* = \sqrt{\tau_w/\rho}. \tag{46}$$

A corresponding natural length measure for normalizing distances is the *friction length* defined by

$$y_* = \nu/u_*. \tag{47}$$

Recall that ν is the momentum diffusivity (also known as kinematic viscosity) with units m^2 s^{-1}. Use (14), (38) and (39) (the definitions of Re, f, and λ) to eliminate τ_w and thus express the friction velocity and length in the alternative forms:

$$\frac{u_*}{\nu} = \sqrt{f/2} = \sqrt{\lambda/8}, \tag{48}$$

$$\frac{y_*}{D} = \frac{1}{Re\sqrt{f/2}} = \frac{1}{Re\sqrt{\lambda/8}}. \tag{49}$$

With these normalizing factors the dimensionless velocity and radial wall distance are:

$$u_+ = u/u_* \tag{50}$$

$$y_+ = y/y_*. \tag{51}$$

In terms of these dimensionless variables (45) is

$$\frac{du_+}{dy_+} = \frac{1}{\kappa}\frac{1}{y_+}. \tag{52}$$

The solution is

$$u_+ = \frac{1}{\kappa}\ln y_+ + C. \tag{53}$$

Let y_0 be the distance from the wall at which u would be zero if the logarithmic profile continued that far. Let γ be the normalized zero-distance:

$$\gamma = y_0/y_*. \tag{54}$$

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In terms of this parameter, the turbulent piece of the velocity profile is

$$u_+ = \frac{1}{\kappa} \ln \left(\frac{y_+}{\gamma} \right).$$

(55)

This is valid in the turbulent core, $\alpha < y_+ < R_+$, where

$$\alpha = t/y_*$$

(56)

$$R_+ = R/y_*.$$

(57)

The centerline (maximum) velocity is

$$u_{c+} = \frac{1}{\kappa} \ln \left(\frac{R_+}{\gamma} \right).$$

(58)

The velocity at the boundary of the laminar layer is

$$u_{t+} = \frac{1}{\kappa} \ln \left(\frac{\alpha}{\gamma} \right).$$

(59)

The logarithmic velocity profile is shown in Figure 19.
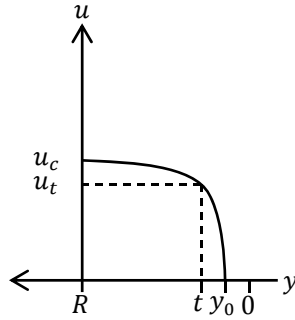
Figure 19: Logarithmic velocity profile

4.7 Parabolic velocity profile As is already known, the laminar piece of the velocity profile is parabolic. It is

$$u = u_l \left[1 - \left(\frac{r}{R} \right)^2 \right] = u_l \left[\frac{y}{R} \left(2 - \frac{y}{R} \right) \right]$$

(60)

where u_l is the laminar center velocity

$$u_l = \frac{\tau_w R}{\mu} = \frac{\tau_w}{\rho} \frac{\rho R}{\mu} = u_*^2 \frac{R}{\nu} = u_* \frac{R}{y_*}.$$

(61)

In dimensionless form the parabolic piece of the velocity profile is

$$u_+ = y_+ \left(2 - \frac{y}{R} \right) = y_+ \left(2 - \frac{y_+}{R_+} \right).$$

(62)

This is valid in the laminar layer, $0 < y_+ < \alpha$. The laminar centerline velocity (an imaginary construct in turbulent flow) is the value u_+ would achieve if the laminar layer reached to the center ($y_+ = R_+$):

$$u_{l+} = R_+.$$

(63)

The velocity at the boundary of the laminar layer ($y_+ = \alpha$) is

$$u_{t+} = \alpha \left(2 - \frac{\alpha}{R_+} \right).$$

(64)

The parabolic velocity profile is shown in Figure 20.

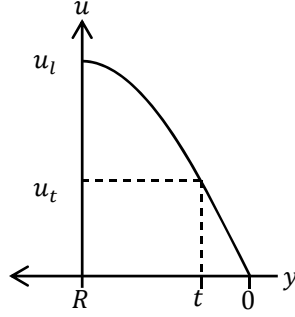Figure 20: Parabolic velocity profile

4.8 Turbulent smooth wall resistance law The complete universal velocity profile is

$$u_+ = \begin{cases} y_+\left(2 - \dfrac{y_+}{R_+}\right) & \text{for } 0 \le y_+ < \alpha \\ \dfrac{1}{\kappa}\ln\left(\dfrac{y_+}{\gamma}\right) & \text{for } \alpha \le y_+ < R_+. \end{cases} \tag{65}$$

It is shown in Figure 21. An un-normalized version in terms of the center radial distance r was presented in (43).
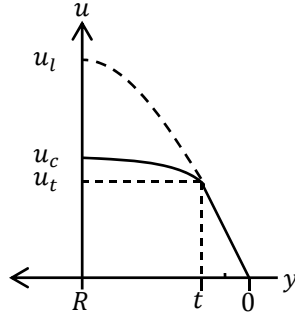
Figure 21: Complete velocity profile

The average velocity of the universal profile is well-approximated by averaging from $y = 0$ to R with the additional assumption that the velocity is zero from $y = 0$ to t. That is, the velocity in the laminar layer is assumed zero for purposes of computing the average. Since the laminar layer is thin in cases of interest this simplification leads to little error. Including the parabolic velocity profile in the average is not hard, but doing so leads to a more complicated formula and the extra complexity adds no value. The approximate average is

$$v_+ = \frac{1}{\pi R_+^2}\int_\alpha^{R_+} u_+ \cdot 2\pi y_+ dy_+ = \frac{2}{\kappa R_+^2}\int_\alpha^{R_+} y_+\ln\left(\frac{y_+}{\gamma}\right)dy_+, \tag{66}$$

thus

$$v_+ = \frac{1}{\kappa}\ln\left(\frac{R_+}{\alpha}\right). \tag{67}$$

The resistance law follows directly from this expression by using (48), (49), (56) and (57) to un-normalize it:

$$\frac{v}{u_*} = \frac{1}{\kappa}\ln\left(\frac{R}{t}\right) = \frac{1}{\kappa}\ln\left(\frac{D}{2\alpha y_*}\right), \tag{68}$$

$$\frac{1}{\sqrt{\lambda/8}} = \frac{1}{\kappa}\ln\left(\frac{Re\sqrt{\lambda/8}}{2\alpha}\right). \tag{69}$$

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This is the resistance law for turbulent smooth wall flow. The experimental fact is that κ and α are universal constants, independent of flow conditions (as specified by the variables Re and λ). Their values are

$$\kappa = 0.40, \tag{70}$$

$$\alpha = 0.51. \tag{71}$$

These values and the value for β to be presented subsequently are derived from the chart of L. F. Moody [1].

The fact that the values of of κ and α are on the order of one gives the Prandtl model the ring of truth. It suggests that the normalizing quantities $u_* = \sqrt{\tau_w/\rho}$ and $y_* = \nu/u_*$, which are intuitively appealing, are indeed the right ones with which to measure turbulent flow; and that the notion of a mixing length on the order of magnitude of y_* is a valid way to understand turbulent momentum transport.

4.9 Plug flow The laminar layer becomes thinner as the average velocity increases:

$$t/R = e^{-\kappa v_+}. \tag{72}$$

For high Reynolds number flows, viz. $Re \gtrsim 2Re_c$ (the case for all flows of interest in fuel transport by pipeline) the laminar layer thickness is much less than the pipe radius: $t/R \ll 1$. Then the parabolic velocity profile (62) is approximately linear:

$$u_+ \cong 2y_+. \tag{73}$$

The velocity at the laminar layer boundary (64) is

$$u_{t+} \cong 2\alpha. \tag{74}$$

Using (58), (59), (67) and (74) the average velocity can expressed

$$v_+ = u_{c+} - u_{t+} \cong u_{c+} - 2\alpha. \tag{75}$$

This reveals that if the laminar layer is thin ($\alpha \ll 1$), the average velocity is only a little less than the center velocity—the flow has a nearly flat velocity profile. A flat velocity profile is called plug flow; turbulent pipe flow approximates closely to plug flow. The turbulent core occupies most of the pipe, and moves like a solid plug—except that the churning fluid of which it is composed dissipates energy due to friction losses as it goes. The laminar layer is thin, like a layer of lubricating oil between the turbulent core and the wall. Within this layer there is a kind of Couette flow, the edge of the turbulent core "plug" being the moving wall, the pipe wall the stationary one.

Un-normalizing (69) gives

$$v \cong u_c - \frac{2u_* t}{y_*} = u_c - \frac{u_*^2 d}{\nu} = u_c - \frac{\tau_w d}{\mu}, \tag{76}$$

where $d = 2t$ is the laminar double thickness. If the velocity profile is approximated in piecewise-linear fashion, as a flat piece in the turbulent core and a linear constant-slope piece in the laminar layer, the double thickness is a better estimate of the laminar layer extent than is the thickness. Compare (76) and (7); note that the velocity of the "plug wall," $u_d = \tau_w d/\mu$, is like Couette flow between walls spaced by distance d. Turbulent pipe flow at high Reynolds number can be visualized as a turbulent plug of radius $R - d$ travelling at velocity v with a laminar layer of effective thickness d between it and the wall. Within the plug the velocity is constant in the radial direction. Within the laminar layer of effective thickness d it decreases linearly from v next to the moving core to zero at the wall. The effective thickness of the laminar layer (d) decreases as the velocity of the plug (v) increases.

4.10 Rough wall resistance law Figure 17 shows that as the Reynolds number increases in the turbulent range it reaches a transition at which the friction factor becomes constant. Referring to (49) note that the friction length (y_*) decreases with Reynolds number (Re). At high flow velocity the friction length y_* must be replaced with the wall roughness e. In place of (56), $t = \alpha y_*$, we have

$$t = \beta e. \tag{77}$$

The dimensionless average velocity then becomes

$$\frac{v}{u_*} = \frac{1}{\kappa} \ln\left(\frac{R}{t}\right) = \frac{1}{\kappa} \ln\left(\frac{R}{\beta e}\right) = \frac{1}{\kappa} \ln\left(\frac{1}{2\beta\varepsilon}\right) \tag{78}$$

where the relative wall roughness ε based on pipe diameter is

$$\varepsilon = \frac{e}{D} = \frac{e}{2R}. \tag{79}$$

Converting this to a resistance law using (48) gives

$$\frac{1}{\sqrt{\lambda/8}} = \frac{1}{\kappa} \ln\left(\frac{1}{2\beta\varepsilon}\right) \tag{80}$$

or

$$\lambda_t = 8\kappa^2 \left[\ln\left(\frac{1}{2\beta\varepsilon}\right)\right]^{-2}. \tag{81}$$

In turbulent rough wall flow, the friction factor is constant (independent of Reynolds number). To emphasize this it has subscript t (for transition) in the foregoing equation. The transition friction factor λ_t depends only on a pipe property (wall relative roughness ε) and not on the fluid properties. The transition Reynolds number at which (81) becomes valid is that at which the laminar layer thickness for smooth wall flow equals that for rough wall flow:

$$\alpha y_{t*} = \beta e, \tag{82}$$

$$\alpha \frac{y_{t*}}{D} = \frac{\beta e}{D}. \tag{83}$$

Using (49):

$$\alpha \frac{1}{Re_t\sqrt{\lambda_t/8}} = \beta\varepsilon \tag{84}$$

$$Re_t = \frac{2\alpha}{\kappa}\left[\frac{1}{2\beta\varepsilon} \ln\left(\frac{1}{2\beta\varepsilon}\right)\right]. \tag{85}$$

β, like α, is a universal constant. A value for it can be derived from the Moody chart [1]. It is

$$\beta = 0.15. \tag{86}$$

This value is far enough from unity to call into question its interpretation as the laminar layer thickness normalized by the wall roughness. Its correct interpretation is left for others; we accept it as an empirical fitting parameter.

Rms surface roughness lengths for the materials of which pipes are made range from 0.1 μm for drawn tubing, to 5 μm for commercial steel or wrought iron, to 20 μm for cast iron, to 200 μm (or more) for concrete and riveted steel. A roughness length of 10 μm and pipe diameter of 0.1 m results in a roughness parameter of $\varepsilon = 1 \times 10^{-4}$. The corresponding limiting friction factor for high Reynolds number turbulent flow is

$$\lambda_t \approx 0.012 \tag{87}$$

or

$$f_t \approx 0.003. \tag{88}$$

These are convenient numbers to use as typical of flow in fuel-carrying pipelines. They are valid for $Re \gtrsim 10^6$.

4.11 Complete resistance law The resistance law in all three flow regimes is summarized as follows:

$$Re < Re_c: \qquad \lambda = 64/Re \tag{89a}$$

$$Re_c < Re < Re_t: \quad \frac{1}{\sqrt{\lambda/8}} = \frac{1}{\kappa} \ln\left(\frac{Re\sqrt{\lambda/8}}{2\alpha}\right) \tag{89b}$$

$Re_t < Re$: $\lambda = 8\kappa^2 \left[\ln\left(\dfrac{1}{2\beta\varepsilon}\right)\right]^{-2}.$ (89c)

For high Reynolds number flow, with Re on the order of 10^6 and greater, the friction factor is a constant determined by the wall roughness of the pipe and independent of the fluid properties. For such flows a typical Fanning friction factor for the kind of pipe expected in long distance fuel pipelines is $f_t \approx 0.003$.

5.

5.1 Friction factor for gas vs. liquid It is striking that gases and liquids of low and high density and viscosity all have the same limiting friction factor for pipe flow at high Reynolds number. The effect of density and viscosity is to determine how high the average velocity must be to achieve a given Reynolds number; this is what differentiates the different fluids. The momentum diffusivity (ν) is much larger in gases than it is liquids, because $\nu = \mu/\rho$ and although viscosity (μ) and density (ρ) are both smaller for gases than for liquids, the density is *very much* smaller. Since $Re = vD/\nu$, the Reynolds number for gases will be larger than for liquids in pipelines of given diameter and flow velocity. In pipelines carrying a gas the velocity will typically be higher than in pipelines carrying a liquid, but not enough to compensate for the very much larger momentum diffusivity. Therefore the Reynolds number for gas-carrying pipelines is typically higher than for liquid-carrying pipelines. Nevertheless, both typically exceed the transition Reynolds number, which is determined only by the pipe properties and not by the fluid properties.

5.2 Pipeline power in turbulent flow Equations (27) and (38) are valid for any flow condition: laminar or turbulent, smooth wall or rough wall. Combining them yields the general relation between pressure gradient and average velocity for any pipe flow:

$$p' = \frac{f\rho}{R}v^2.$$ (90)

In general, the friction factor (f) depends on the velocity (v), so (90) could be misleading if one concluded from it that the pressure gradient goes as the square of the velocity. In laminar flow $f = 64/Re$; f goes as $1/v$ and the pressure gradient is just proportional to the velocity. In turbulent smooth wall flow f and v have a complicated relationship. But in turbulent rough wall flow (high flow rates in which we are most interested) the relation becomes simple: $f = f_t$, a constant that is a property of the pipe only and not of the fluid or its flow rate. In turbulent rough wall flow the pressure gradient does indeed go as the square of the velocity.

Using (32), the pressure gradient required to establish a given volume flow for turbulent rough wall conditions is

$$p' = \frac{f_t\rho}{RA^2}\dot{V}^2.$$ (91)

The power per length required to maintain this flow is $p'\dot{V}$. For a pipeline of length L the power lost is

$$P_l = \frac{f_t L}{RA^2}\rho\dot{V}^3.$$ (92)

Using (36) the relation between power lost and power transmitted is

$$P_l = \frac{L f_t}{RA^2}\frac{1}{\rho^2 h^3}P_t^3 = \frac{1}{G^2}\frac{1}{F^2}P_t^3.$$ (93)

The factor relating P_l to P_t^3 is divided into two groups, one dependent only on geometric dimensions of the pipe (including wall roughness), one only on fluid properties. The geometric factor is

$$G = A\sqrt{\frac{R}{f_t L}} = A\sqrt{\frac{2D}{\lambda_t L}}$$ (94)

with units m². The fluid factor is

$$F = \rho h\sqrt{h}$$ (95)

with units W m⁻². Both F and G should be maximized to minimize losses.

5.3 Ideal gas flow Isothermal flow with constant pressure gradient in the direction of flow and uniform pressure in the transverse (radial) direction has been tacitly assumed in the foregoing. The density (ρ) has been assumed constant. Constant density is what defines an incompressible liquid, and is a good first-order model for any liquid. On the other hand a good first-order model for any gas is an ideal gas, for which

$$\rho = \frac{M}{RT}p. \tag{96}$$

This is the ideal gas law; M is the molecular weight of the gas, T its temperature, and R the gas constant. It states that density is proportional to pressure. Under isothermal conditions it is convenient to write this law as

$$\rho = p/e, \tag{97}$$

where $e = RT/M$ is a specific energy (energy per unit mass) with units J kg^{-1}. In this section we are going to show that the fluid flow figure of merit F defined in (95) remains valid for gaseous fuels under conditions encountered in pipelines if we use as density the average or operating density in the pipe corresponding to the operating pressure of the pipe as specified by (97).

The relation between volume and mass flow rates for an ideal gas can be written

$$\dot{V} = \frac{p_s}{p}\dot{V}_s. \tag{98}$$

where p_s is the standard pressure (1 bar) and $\dot{V}_s$ is the standard volume flow rate, which is in fact a mass flow rate: $\dot{V}_s = \dot{m}/\rho_s$ and ρ_s is the density at standard pressure and temperature ($T_s = 298.15$ K). For steady flow the mass flow rate is a constant independent of distance (z) along the pipe. If the pipe has constant cross sectional area (A) then the pressure-velocity product is also constant:

$$pv = \frac{p_s V_s}{A}. \tag{99}$$

Compressible fluids such as ideal gases have an additional term that must be accounted for in the force balance that led to (27). That term is due to the variation in dynamic pressure from upstream to downstream. The dynamic pressure is

$$p_d = \tfrac{1}{2}\rho v^2. \tag{100}$$

To appreciate the difference between compressible and incompressible flow, first review the force balance for incompressible flow shown in Figure 15. The upstream face of the control volume is on the left, the downstream face is on the right. The left-to-right flow is caused by a static pressure gradient: the static pressure on the upstream face is greater than that on the downstream face. Because the fluid is moving, there is also dynamic pressure on these two faces; but if the fluid is in steady flow and is incompressible, the dynamic pressure on the two faces is equal and opposite, and can be ignored. This is the assumption that led to (27). Note that there are two factors in dynamic pressure: density (ρ) and velocity (v). In Figure 15 the density is the same at the two faces (since it is constant for an incompressible fluid) and the velocity is also the same (because the fluid is in steady flow, so as much must leave as enters in any given time). The force on the upstream face of the control volume is in the flow direction; that face is "catching" the impinging fluid. The force on the downstream face is opposite to the flow direction. It is the reaction force to the momentum of the fluid leaving the control volume: the downstream face is "throwing" the exiting fluid. Hence the two force components on the control volume due to dynamic pressure are equal and opposite. There is no gradient of dynamic pressure for steady flow of an incompressible fluid.

Now assume the fluid is *compressible*. Then neither the density nor the velocity is necessarily the same on the upstream and downstream faces of the control volume; consequently there can be a dynamic pressure gradient. The force balance on the control volume is determined not by the static pressure alone, but by the *total* pressure:

$$p_t = p + p_d = p + \tfrac{1}{2}\rho v^2. \tag{101}$$

This is true for the incompressible fluid also, but then the total and static pressure *gradients* are the same, since only the static pressure varies. For compressible fluids the dynamic pressure also varies. Working

through the revised force balance reveals that the modification required is that total pressure (p_t) be used in place of static pressure (p). Then (27) becomes

$$\tau_w = \frac{1}{2}p_t'R, \tag{102}$$

where

$$p_t' = -\frac{d}{dz}(p + p_d) = -\frac{d}{dz}\left(p + \frac{1}{2}\rho v^2\right). \tag{103}$$

Now invoke the assumption that the fluid is an ideal gas in steady flow. Substitute (97) through (99) into (100) to obtain:

$$pp_d = p_s p_{ds} \tag{104}$$

where

$$p_{ds} = \frac{1}{2}\rho_s \left(\frac{\dot{V}_s}{A}\right)^2. \tag{105}$$

(105) tells that for an ideal gas in steady flow the product of the static and dynamic pressures is constant. Then the total pressure gradient is

$$p_t' = -\frac{d}{dz}\left(p + \frac{p_s p_{ds}}{p}\right) = p'\left(1 - \frac{p_s p_{ds}}{p^2}\right). \tag{106}$$

Combine (102) and (38) to eliminate τ_w and obtain the relation between total pressure gradient and dynamic pressure:

$$p_t' = \frac{4f}{D}p_d. \tag{107}$$

Reduce this to an equation in p and p': multiply (107) by p and then substitute (104) and (106). This yields a differential equation that can be solved for the static pressure (p):

$$p'\left(p - \frac{p_s p_{ds}}{p}\right) = \frac{p_s^2}{Z_l} \tag{108}$$

where

$$Z_l = \frac{Dp_s}{\lambda p_{ds}}. \tag{109}$$

Recall that $p' = -dp/dx$ and integrate (108) over the line length from the high pressure side at $z = 0$ to the low pressure side at $z = L$. The result is

$$\frac{1}{2}\frac{p_s}{p_{ds}}\left(\frac{p_0^2 - p_L^2}{p_s^2}\right) - \ln\left(\frac{p_0}{p_L}\right) = \frac{\lambda L}{D} \qquad (110)$$

$$2\bar{p}\Delta p = \left(\frac{\dot{m}}{A}\right)^2 e\left[\frac{\lambda L}{D} + \ln\left(\frac{p_0}{p_L}\right)\right] \qquad (111)$$

In (111) $\bar{p}$ is the average pressure in the pipe and Δp is the pressure drop from inlet to outlet:

$$\bar{p} = \frac{1}{2}(p_0 + p_L) \tag{112}$$

$$\Delta p = p_0 - p_L. \tag{113}$$

5.4 Approximations for fuel pipelines (110) or (111) is a fairly complicated expression relating mass flow to inlet and outlet pressures for steady flow of an ideal gas through a pipe. The complexity comes from the compressibility of the fluid. Natural gas pipelines are operated at high pressure to increase the fluid density, thus maximizing the fluid factor F. If a length L of gas pipeline between two compressor stations is operated at high average pressure (p) and the pressure drop between the stations is small compared to that average

pressure ($\Delta p < \bar{p}$) then the inlet and outlet pressures are nearly the same ($p_0 \approx p_L$) and the logarithmic term [$\ln(p_0/p_L)$] is negligible. Let us denote the average pressure, which is what we mean by operating pressure, by p_{op}. Then

$$\Delta p \cong \left(\frac{\dot{V}_s}{A}\right)^2 \frac{p_s^2 \lambda L}{2 p_{\mathrm{op}} e D} = \left(\frac{\dot{m}}{A}\right)^2 \frac{e \lambda L}{2 p_{\mathrm{op}} D}. \tag{114}$$

As a further approximation valid for the low-loss conditions of interest to power engineers it may be assumed that $\Delta p \ll p_{\mathrm{op}}$ and thus the pressure gradient is approximately constant along the line. Then

$$p' \cong \frac{\Delta p}{L} \cong \left(\frac{\dot{m}}{A}\right)^2 \frac{\lambda}{2 p_{\mathrm{op}} D}. \tag{115}$$

Now $\dot{m} = \rho \dot{V} = p\dot{V}/e$, and $p \approx p_{\mathrm{op}}$, thus

$$p' \cong \frac{\lambda_t}{2 D A^2} \cdot \frac{p_{\mathrm{op}}}{e} \cdot \dot{V}^2. \tag{116}$$

Note that $p_{\mathrm{op}}/e = \rho_{\mathrm{op}}$ is the average gas density in the pipe. It is constant, just like the density of an incompressible fluid. Indeed, (91) and (116) are the same. An ideal gas pipeline operated at high pressure and with relatively small pressure drop between compressor stations behaves just like an incompressible liquid pipeline. In both cases the fluid factor is $F = \rho h \sqrt{h}$. For gases the pressure at which the line is operated should be used in computing this number. This may not be the maximum possible pressure, but will presumably be an optimum pressure. The optimum may be lower than the maximum possible to minimize losses incurred in compressing and then expanding the gas. Probably more significant is the effect of construction and operating costs. Like pipe diameter (D), operating pressure (p_{op}) is limited primarily by cost: costs rise steeply as pressure increases. A typical value for existing gas pipelines is $p_{\mathrm{op}} = 50$ bar. Replacing average density with operating pressure gives the appropriate formula for the fluid factor of a gas-carrying pipeline:

$$F = \frac{p_{\mathrm{op}}}{e} h\sqrt{h} = \frac{p_{\mathrm{op}} M}{RT} h\sqrt{h}. \tag{117}$$

It should be born in mind that gaseous fuels suffer an additional loss penalty since power is required to compress them, unless provision is made to recover this power at the receiving end of the pipeline. Even with such provision, some additional loss in inevitably incurred in the compression/expansion cycle.

5.5 Fuel power density and fuel energy velocity The fluid factor F may be called the *fuel power density*. It is a figure of merit for ranking the suitability of different fuels as energy vectors for low-loss long-distance power transmission. The larger is this figure of merit, the more suitable is the fuel. It is composed of two sub-factors: density and specific enthalpy. For an incompressible liquid the fuel power density is an intrinsic property. For a gas it can be increased by compression; but then means to recover the power required for compression should be provided, and losses associated with the compression/decompression cycle must be accounted for. The fuel power density clearly reveals the effect of fuel properties on transmission losses. Transmission losses are minimized by using a high-density fluid. Liquid fuels are therefore preferred to gas, other things being equal.

The factor $\sqrt{h}$ has units m s^{-1}, and may be called the *fuel energy velocity*. The specific enthalpy of combustion for fuels is on the order of 100 MJ kg^{-1}, thus the fuel energy velocity ($\sqrt{h}$) is on the order of 10 km s^{-1}, a thousand times higher than a typical average fluid velocity (v) of 10 m s^{-1}. The transported power density is

$$P_t'' = P_t/A = \rho h v, \tag{118}$$

where A is the pipe cross sectional area and ρh is the fuel energy density [GJ m^{-3}]. Just as combining the energy density with the fluid velocity (v) yields the transported power density, combining the energy density with the energy velocity ($\sqrt{h}$) yields $F = \rho h\sqrt{h}$, the fuel power density. Whereas v determines power actually carried by a fuel in a pipeline, $\sqrt{h}$ is a figure of merit characterizing the *capability* of a fuel to carry power.

The fuel power density F relates power loss due to friction (viscous loss) to transported power. The larger it is, the less the losses for a given amount of power transmitted. It is striking that this figure of merit is independent of the viscosity of the fuel, despite the fact that viscous loss is the cost function from which F derives. It is because F applies to a flow regime in which viscous loss is determined by the pipe (wall roughness) and not the fluid, and the factor ε characterizing pipe roughness is incorporated (via λ_t) in the companion factor to F, the geometric factor G.

5.6 Pipeline design for energy transport Design of pipelines for energy transport requires selecting optimal values for F and G. Optimizing G amounts to choosing the optimal value of pipe diameter (D) and rms surface roughness (e). D should be as large as possible and e as small as possible, but both of these goals increase the cost of the pipeline. The optimum value is determined as a tradeoff between initial cost and the energy savings due to reduced losses during the life of the pipeline.

The possibility of optimizing F is not often considered. Usually a pipeline is built to transport a given fluid and consideration is not given to choice of fluid. If the goal, however, is to transport a given energy per unit time, the choice of fuel to carry that energy may remain open. When that is the case, the fuel factor F is an important figure of merit. It tells us that density and specific enthalpy are all-important, and that viscosity is irrelevant. Given similar specific enthalpies, fuel density is the crucial characteristic. Liquid fuels have a very large advantage over gas fuels since their density is orders of magnitude higher. This is a conclusion one might come to intuitively without the foregoing analysis. The irrelevance of viscosity, however is not so obvious. Further, the analysis enables one to check whether the flow conditions (Reynolds number) do in fact enable viscosity to be ignored. Most important, the analysis enables the liquid vs. gas advantage to be quantified. This enables one to assess the economic viability of energy projects such as gas-to-liquid processes for monetizing stranded natural gas.

5.7 Fuel as a chemical energy vector As an example we compare several renewable and fossil fuels for their suitability as chemical energy vectors in pipelines. Table 1 exhibits the properties of the selected fuels, all at 25°C. The gases hydrogen and methane are at a pressure of 50 bar; the high vapor pressure liquids ammonia and propane are at a pressure of 10 bar; the low vapor pressure liquids methanol and octane are at a pressure of 1 bar. The corresponding densities are listed, as are higher and lower heating values at STP. The lower heating value (h_L) is used to calculate fuel energy density (ρh_L), fuel power velocity ($\sqrt{h_L}$) and fuel power density (F).

Table 1: Figures of merit for representative fuels

Name	Formula	M	p	ρ	h_H	h_L	ρh_L	$\sqrt{h_L}$	F
		g/mol	bar	kg/m^3	MJ/kg	MJ/kg	GJ/m^3	km/s	TW/m^2
Hydrogen	$H_2(g)$	2.02	50	4.06	141.80	120.97	0.49	11.00	5.41
Methane	$CH_4(g)$	16.04	50	35.85	55.50	50.01	1.79	7.07	12.68
Ammonia	$NH_3(l)$	17.03	10	681.90	22.50	18.65	12.71	4.32	54.90
Propane	$C_3H_8(l)$	44.10	10	507.70	50.35	46.36	23.54	6.81	160.24
Methanol	$CH_3OH(l)$	32.04	1	791.80	22.66	19.92	15.77	4.46	70.40
Octane	$C_8H_{18}(l)$	114.23	1	688.00	47.89	44.43	30.57	6.67	203.73

Compare the fuels based on three different figures of merit: specific energy (h_L), energy density (ρh_L), and fuel power density (F). Specific energy is the least useful, over-estimating the value of the two gaseous fuels. Energy density is more useful, but *under*-estimates the value of the two gases. The fuel power density gets it right: it gives the most balanced and useful relative ranking. From it one concludes, in accordance with common knowledge, that liquid hydrocarbons are the preferred fuels, and hydrogen is the least preferred. The fuel power density shows that among the renewable fuels (hydrogen, ammonia and methanol), the liquid fuels ammonia and methanol are better than the gas hydrogen by more than an order of magnitude. The relative rankings of methane, ammonia and methanol are also noteworthy. Ammonia is four and methanol six times better than methane as fuels for transport in pipelines. Gas-to-liquid conversion of natural gas to ammonia and methanol carries with it significant transport and storage benefits, as is known. The fuel power

density helps to quantify these benefits. It suggests that natural gas should be used as a fossil energy *source*, but not used directly as an energy *vector* (fuel). Instead, it should be converted to ammonia and methanol to facilitate trade. When natural gas is converted to ammonia, an additional advantage accrues from the possibility to implement carbon capture sequestration and sale in large-scale concentrated operations conducted at the point of fuel production rather than at the distributed points of fuel use. The fuel power density will prove useful in analyzing under what conditions this conversion reduces greenhouse gas emissions, lowers the cost of energy and stabilizes supply to consumers, and creates competitive advantage for natural gas producers.

6. SUGGESTIONS FOR FURTHER STUDY

Anyone interested in pipe flow should read Moody's classic paper on the friction factor [1]. This paper is widely cited and often reproduced in textbooks. It is the principal source of the Blasius-Stanton-Moody diagram illustrated in Figure 17. Moody provides a data benchmark against which any theoretical model can be compared. He does not, however, offer much in the way of insight into the physics of turbulent pipe flow. This is a topic that has attracted a great deal of sophisticated investigation, resulting in a research record that is quite daunting to the non-specialist. There seems to be no review article, monograph, or textbook that has summarized this body of knowledge in a form that is accessible to those outside the field. The present article does not fill that gap. Instead, I have sought only to present the Prandtl model, developed almost a century ago. This model is simple and insightful, and sufficient to the electric power engineer seeking an understanding of flow phenomena relevant to energy transport by pipeline. Even the very modest goal of presenting the Prandtl model in a concise and accessible way required considerable development beyond what I could find in the literature. To those wishing to pursue this topic further I recommend as a starting point references [2-4]. Prandtl's textbook [2] is a classic, and still the best source for his view of turbulent flow. Schlichting [3] provides in Chapters 19 and 20 a summary of and literature references to the classic experiments and theories of turbulent pipe flow, most notably the experiments of Nikuradse and the theory of von Kármán. Bakhmeteff's book [4] is based on lectures given at Princeton in 1935. His view of the subject is dated; for example the seminal paper of Millikan [5] presented in 1938 was unknown to him. Nevertheless, Bakhmeteff's book is highly recommended, especially for non-specialist readers new to the subject. He provides a view of turbulent pipe flow accessible to those with no prior knowledge of fluid mechanics, and emphasizing physical insight over mathematical formalism. Books devoted to pipeline engineering can be consulted for further development. The textbook by Benedict [6] is recommended for pipelines in general. For fuel pipelines in particular see the textbook by Vincent-Genod [7] (especially section 2.3, "Calculation of absorbed power") and the monograph by Mohitpour [8]. Transport of renewable fuels in pipelines as an alternative to electric power transmission lines is discussed in [9].

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