Hands-On Phasors and Multiple-Slit Interference

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Multiple-slit interference is often covered in optics courses to bridge the gap between Young’s double-slit experiment and the diffraction grating. In addition to the standard algebraic analysis, the technique of phasor addition can be used both as a visual aid and a quantitative tool for determining the intensity distribution of light resulting from an arbitrary number (N) of identical, equally spaced slits. We have designed and built mechanical phasor models to give our students a hands-on appreciation of the phasor method and a demonstration of its application, both qualitatively and quantitatively, to the case of multiple-slit interference for small numbers of slits (N=2,3,4). Use of these models in our sophomore level Waves and Optics course for physics majors is described here.

In preparation for a laboratory exercise involving direct observation of multiple-slit interference patterns, we consider a physical situation in which linearly polarized plane waves of wavelength \( \lambda \) and angular frequency \( \omega \) are incident on \( N \) narrow slits of separation \( d \) as shown in Fig. 1. The resulting intensity distribution is proportional to the time average of the net electric field squared, \( I \propto \langle E_T^2 \rangle \), where \( E_T = E_1 + E_2 + \ldots + E_N \). Since light from each successive slit travels an extra distance, \( d \sin \theta \), the phase of each successive field differs by an additional amount, \( \delta = 2\pi d \sin \theta \lambda \), relative to the preceding one. It can be shown that the net electric field is given by

\[
E_T(t) = E_0 \frac{\sin(N\delta/2)}{\sin(\delta/2)} \sin(\omega t + \alpha) = E_{T0} \sin(\omega t + \alpha) \tag{1}
\]

where \( E_{T0} = E_0 \frac{\sin(N\delta/2)}{\sin(\delta/2)} \) is the amplitude and \( \alpha = (N-1)\delta/2 \) is the initial phase of the net electric field. The corresponding intensity is given by

\[
I = I_0 \frac{\sin^2(N\delta/2)}{\sin^2(\delta/2)} \tag{2}
\]

where \( I_0 \) is the uniform intensity from a single slit. Since \( \delta \) is a function of the viewing angle \( \theta \), we can use Eq. (2) to predict the locations of the intensity maxima and minima, including the \( N \)-2 secondary maxima between each pair of principal maxima. However, the algebraic derivation of Eq. (2) did not make the physical origin of the various intensity peaks particularly clear to our students.

Therefore, an alternative derivation using phasors was also presented.

In the phasor approach, light from the \( n \)th slit is represented as a rotating vector of amplitude \( E_0 \) and angle \( (\omega t + n\delta) \) with respect to the horizontal (see Fig. 2). The vertical projection of the resultant vector yields the instantaneous value \( E_T(t) \) of the net electric field. However, since the observed intensity is proportional to the time average of the square of the net electric field, we are concerned only with the amplitude, \( E_{T0} \), of the resultant vector, which depends on the relative phase angle, \( \delta \), and the number of slits, \( N \).

Visualizing the appearance of the phasor diagram for various values of the phase difference \( \delta \) is difficult, and redrawing the diagram for a sequence of values proves tedious. Some texts\(^8\) do show sequences of such drawings, but we thought it would be more instructive if the students could work with a hands-on model. With this in mind, we designed and built a set of simple phasor models that can be easily manipulated to demonstrate the behavior of the interference pattern. The design described here can be assembled quickly using readily available, inexpensive materials.

For our models, each phasor has the same length because the electric field from each slit has the same amplitude, \( E_0 \). To represent the phasors we used plastic rulers,\(^9\)
with holes drilled near the ends, connected end-to-end by round-head fasteners. The resulting phasors have an effective length of 28 cm. The rulers we selected are transparent and have a printed line down the middle which proved useful when setting the angle between adjacent rulers. A circle with cross hairs and 30° tick marks was printed on paper and glued to one end of each ruler (see Fig. 3). The first and last rulers of each set had an additional circle with cross hairs to represent the endpoints of the resultant vector. We built sets of $N=3$ and $N=4$ rulers for student use (a two-ruler model was also built for demonstration purposes).

Working in groups of two or three, students used the mechanical phasor models for the following in-class exercise. With metersticks, they measured resultant vector lengths corresponding to the net electric field amplitudes for phase angles ranging from 0° to 360° in 30° increments. Figure 4 shows the first half of this sequence (δ=0° to 180°) for the $N=3$ case. Students observed how the phasors combined to create the various maxima and minima as they varied the phase angle. Note for example the primary maximum at $\delta=0°$ (Fig. 4a), the zero-intensity minimum at $\delta=120°$ (Fig. 4e), and the secondary maximum at $\delta=180°$ (Fig. 4g).

To analyze the data we noted that the maximum electric field in each case is $N\mathbf{E}_0$ (e.g. when $\delta=0°$); consequently, the maximum intensity is proportional to $N^2\mathbf{E}_0^2$. It is therefore instructive to consider normalized intensities, $I_{\text{norm}} = (I/I_0)/N^2 = (E/E_0)^2/N^2$, where $E_0$ corresponds to the length of one of our phasors. Table I shows typical student data along with the corresponding values of $I_{\text{norm}}$. These $I_{\text{norm}}$ vs $\delta$ data were then plotted along with the appropriate theoretical curves given by

$$\frac{\sin^2(N\delta/2)}{N^2 \sin^2(\delta/2)}$$

As shown in Fig. 5, students found good agreement between their data and the theory for both the $N=3$ and $N=4$ cases. Specifically, they saw that the heights and locations of the various maxima and minima were faithfully reproduced by the phasor models. Our students also recognized the qualitative similarity between these results and the intensity patterns which they observed visually in the laboratory.

Our hands-on exercise using these simple, inexpensive phasor models provides a straightforward means of visualizing the way in which contributions from individual slits combine to produce the complex intensity patterns observed in multiple-slit interference.

References


(continued on page 488)
6. See Ref. 4, p. 214.
9. For example: 12-in ruler, model W-25, The C-Thru Ruler Company, Bloomfield, CT.
10. For example: No. 4 round-head fasteners (1-in length); Acco International, Inc. Chicago, IL.
11. Note that for small viewing angles, as observed in lab, phase angle \( \delta = 2\pi d \sin \theta / \lambda \) is proportional to viewing angle \( \theta \). In addition, the intensities observed in the lab were modulated by the diffraction envelope due to the finite width of the slits; this effect was discussed in class but was not modeled in our exercise.

Fig. 5. \( I_{\text{norm}} \) vs \( \delta \) showing student data (open circles) and theory (solid line): a) \( N=3 \) and b) \( N=4 \).

Table I. Measured phase angle and length, with corresponding normalized intensity obtained using \( N=3 \) phasor model (see Fig. 4).

<table>
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<th>( \delta ) (°)</th>
<th>Length (cm)</th>
<th>( I_{\text{norm}} )</th>
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<tr>
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<tr>
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