Student understanding of the superposition of perpendicular harmonic oscillators can be enhanced using a spark generator to record position-versus-time data for small-amplitude pendulum motion in two dimensions. Our students have used this arrangement to analyze the motion of a spherical pendulum and a “Y-suspended” (Blackburn) pendulum that has two effective lengths.

Spherical Pendulum

The spherical pendulum is a simple pendulum whose motion is not restricted to a single plane (see Fig. 1a). Our pendulum consists of a metal bob (e.g., a 100-g mass from a standard hanging mass set) suspended by a string ($L \approx 1$ m) over an air table.\(^1\) Fine magnet wire runs from the high voltage terminal of the spark generator\(^2\) to the bob. A sharp point on the bottom of the bob ensures that all the sparks jump from the same point.\(^3\) Carbon paper\(^4\) is placed on the table and connected to the ground terminal of the spark generator to complete the circuit.

For small-amplitude motion, the height of the bob above the table is approximately constant (gap~0.5 cm) and can be adjusted so that sparks jump between the sharp point on the bob and the carbon paper on the table. With ordinary 8.5 in x 11 in white paper placed over the carbon paper, a record of the bob’s position in the horizontal (x-y) plane is obtained on the underside of the paper. If the height of the pendulum bob increases much, sparking to the carbon paper will cease, so recording of small-amplitude motion is guaranteed.

For small-amplitude motion in the x-y plane, the pendulum should approximate simple harmonic motion with period $T = 2\pi(L/g)^{1/2}$ in both the x and y directions. In general, $x(t) = A \cos(\omega t)$ and $y(t) = B \cos(\omega t + \phi)$, provided we let $t = 0$ occur when $x = A$. The angle $\phi$ represents the relative phase difference between the perpendicular oscillations. The position vector $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$ represents the superposition of these sinusoidal motions and describes elliptical trajectories (including linear and circular paths as special cases).\(^5\) For an ellipse with axes aligned along the coordinate axes, $\phi = \pi/2$ and $x^2/A^2 + y^2/B^2 = 1$.

Spark data for a single elliptical orbit are readily obtained for analysis (see Fig. 2a). (Multiple orbits make it difficult to assign times to the data points.) These data are then traced by hand onto graph paper using a light box for illumination. In this way the major and minor axes of the ellipse can be aligned with the coordinate system to simplify the analysis. (Alternatively, one may record the data directly onto graph paper, without controlling the orientation of the
Fig. 2a. Raw data for one elliptical orbit of the spherical pendulum. Spark frequency = 20 Hz.

Fig. 2b. \( x(t) \) data (dots) plotted along with a sinusoidal curve fit (solid line).

Fig. 2c. \( y(t) \) data (dots) plotted along with a sinusoidal curve fit (solid line).

\[ T = \frac{2\pi}{\sqrt{L/g}} \]

Students find that the trajectory of the spherical pendulum is indeed composed of perpendicular sinusoidal motions. They see that within the measurement uncertainties, these oscillations appear to have the same period, \( T = 2\pi(L/g)^{1/2} \), and are \( \pi/2 \) out of phase with each other. In addition, the amplitudes (A and B) are easily identified with the semimajor and semiminor axes on the graph paper.

**Y – Suspended Pendulum**

As an extension of the first exercise, two strings forming a “Y” shape are used to suspend the pendulum bob (see Fig. 1b). This pendulum, sometimes called a Blackburn pendulum, has two effective lengths and thus different periods of oscillation in the two perpendicular planes. For small-amplitude motion, complicated trajectories, known as Lissajous figures, can be observed and recorded. With properly chosen lengths, the ratio of the periods can be made rational (i.e., a ratio of two integers), resulting in closed orbits.

Students pick a simple integer period ratio (e.g., 2:1, 3:2, etc.) to investigate and prepare the string lengths based on the relation \( T_x/T_y = (L_x/L_y)^{1/2} \). For this exercise, rather than extracting the position-versus-time data, we ask the students to investigate the shape (\( y \) versus \( x \)) of the trajectory (they may record more than one cycle in this case). This record is then compared visually to the corresponding case displayed on an oscilloscope (in “X-Y” mode) whose inputs are sine waves of the appropriate frequency ratio. Unless care is taken, however, the phases of the two frequency generators tend to drift, so that the Lissajous pattern on
the oscilloscope is constantly changing. This leads to a discussion of the role of the relative phase between the two oscillators. Students also notice that different pendulum records taken using the spark apparatus may produce patterns that look quite different, depending on how the motion is started. Thus, they are led to think about how releasing the pendulum in different ways determines the initial phase difference.

To sort out these issues, the students prepare a spreadsheet to produce \( y \)-versus-\( x \) graphs using \( x(t) = A \cos(\omega_x t) \) and \( y(t) = B \cos(\omega_y t + \phi) \), with \( \omega_x \) and \( \omega_y \) corresponding to the period ratio used in the experiment. The resulting plot represents the superposition of perpendicular harmonic oscillators whose frequencies are different. The angle \( \phi \) fixes the relative phase of the two oscillators at \( t = 0 \). (This does not necessarily represent the release time \( t_r \), however.) This phase angle can be adjusted on the spreadsheet and the resulting patterns observed. In order to compare these patterns to experimental data, students must use techniques for releasing the pendulum bob such that this phase angle is known. Descriptions of the two release methods used by our students follow:

1) The first method involves displacing the bob and releasing it from rest. In this case the phase angle is \( \phi = 0 \), as both \( x(t) \) and \( y(t) \) take on their maximum values (\( x = A \), \( y = B \)) at \( t = 0 \). (In this case \( t = 0 \) does correspond to the moment of release, \( t_r \).)

2) The second release method requires the bob to be released while moving through the origin. With \( x = 0 \) and \( y = 0 \) at the same time \( t_r \), the resulting relative phase angle at \( t = 0 \) can be shown to be \( \phi = (\pi/2)(m - n\omega_y/\omega_x); m, n = \text{odd integers} \).

Note that with both release methods, the spark generator is turned on after the bob is safely released; with closed orbits, it does not matter when the spark record is started.

The resulting experimental trajectories match the corresponding spreadsheet graphs quite well, as shown in Fig. 3, for both release techniques. This confirms, at least qualitatively, that the complex motion of the \( Y \)-suspended pendulum can be represented by perpendicular harmonic oscillators. In this experiment, students produce, manipulate, and analyze the superposition of perpendicular sinusoidal oscillations with integer ratio frequencies in a way that complements the usual “gee whiz” visualization provided by Lissajous figures on the oscilloscope. In addition, students gain insight into the physical significance of the phase angle (\( \phi \)) in terms of how and where the pendulum is released.

**Conclusion**

Using the spark generator to collect data provides the opportunity to do quantitative investigations of two-dimensional, small-amplitude pendulum motion. The two exercises presented here provide nice additions to the standard simple pendulum labs often used to introduce simple harmonic motion (e.g., period-versus-length measurements). As an added benefit, our students seem to enjoy the noise and sense of danger that are characteristic of the spark apparatus!
Acknowledgment

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References

1. For example, the Cenco Precision Air Table, which is typically used to investigate two-dimensional collisions between air-supported pucks. The air supply is not used in our experiments. Although the air table is convenient, any horizontal insulating surface may be used.

2. For example, the Cenco Spark Generator sold for use with the air table. Students should be encouraged to use caution when working with the spark generator.

3. If necessary, a small piece of wire may be attached to the bottom of the bob.

4. The special carbon paper sold by Cenco for use with the air table works well.


9. The values of $m$ and $n$ depend on which $x = A$ point is chosen to correspond to $t = 0$, along with the direction of the orbit. The shape of the trajectory does not depend on the specific choices of $m$ and $n$. When labeling our spreadsheet trajectories, we choose to specify the smallest value of $\phi$. 