OPERATIONS WITH DISTRIBUTION VECTORS

The space of distributions $D'$ is isomorphically embedded in the space of distribution vectors $D'_1$ (1) and this larger space $D'$ is equipped with operations of multiplication and integration. Several formulae for $\delta^2(x)$, $\delta(p)\delta(q)$, $x^p\delta^q(x)$, etc., are derived which, as we know, are significant for some applications, in particular, in quantum field theory but they do not make sense in $D'$ itself. The paper is a continuation of a previous work (1) but it could be read independently.

In the following we let $C$ be the field of complex numbers, $D$ be the space of infinitely differentiable functions defined in the real line with compact support and $D'$ be the space of all distributions on $D$.

Definition 1. Let $\alpha_r$ be in $C$ for $r = 0, 1, \ldots$. We say that $\alpha = [\alpha_0, \alpha_1, \ldots]$ is a number vector (2). We denote the vector space of all number vectors, with the usual definition of the sum and product by a scalar, by $C$.

Definition 2. Let $\eta_r$ be in $D'$ for $r = 0, 1, \ldots$. We say that $\eta = [\eta_0, \eta_1, \ldots, \eta_r, \ldots]$ is a distribution vector.
If $h_{n+1} = 0$ for $i = 1, 2, \ldots$, we write

$$h = [h_0, h_1, \ldots, h_r, 0, 0, \ldots] = [h_0, h_1, \ldots, h_r]$$

and if $h_1 = 0$ for $i = 1, 2, \ldots$ we write

$$h = [h_1, 0, \ldots, 0].$$

We denote the vector space of all distribution vectors, with the usual definition of sum and product by a scalar, by $\mathbb{D}'$.

**Definition 3.** Let $\mathbf{h} = [h_0, h_1, \ldots, h_r, \ldots]$ be in $\mathbb{D}'$ and let $\varphi$ be in $\mathbb{D}$. We define $(\mathbf{h}, \varphi)$ to be the number vector

$$(h, \varphi) = [(h_0, \varphi), (h_1, \varphi), \ldots, (h_r, \varphi), \ldots].$$

**Definition 4.** Let $\mathbf{h} = [h_0, h_1, \ldots, h_r, \ldots]$ be in $\mathbb{D}'$. We define the derivative $\mathbf{h}'$ of $\mathbf{h}$ by

$$\mathbf{h}' = [h_0', h_1', \ldots, h_r', \ldots].$$

**Theorem 1.** Let $\mathbf{h} = [h_0, h_1, \ldots, h_r, \ldots]$ be in $\mathbb{D}'$ and let $\varphi$ be in $\mathbb{D}$. Then

$$(h, \varphi) = -(h_0, \varphi).$$

The proof of the theorem follows easily.

**Definition 5.** Let $\omega$ be a fixed function in $\mathbb{D}$ having the properties:

(i) $\omega(x) = 0$ for $|x| \leq 1$,

(ii) $\omega(x) \geq 0$,

(iii) $\omega(x) = \omega(-x)$,

(iv) $\int_{-1}^{1} \omega(x) \, dx = 1$.

We define the function $\delta_\varphi$ by $\delta_\varphi(x) = \varphi(\omega(x))$ for all $\varphi > 0$.
Proof. Suppose
\[ (f_0g_0)^\varphi = \sum_{i=0}^{\varphi} (h_{i, \varphi}) \varphi^i + \Delta(\varphi), \]
\[ (f'_{0'}g'_{0'}) = \sum_{i=0}^{\varphi} (k_{i, \varphi}) \varphi^i + \Delta_1(\varphi), \]
for arbitrary \( \varphi \) in \( D \), so that
\[ f \circ g = [h_{0,0}, h_{1,0}, \ldots, h_{r,0}] , \]
\[ f' \circ g' = [k_{0,0}, k_{1,0}, \ldots, k_{r,0}] . \]
Then
\[ ((f_0g_0)', \varphi) = -(f_0g_0, \varphi') = (f_0' + f_0g_0, \varphi) \]
and so
\[ (f_0g_0, \varphi') = -(f_0g_0, \varphi') - (f'_{0'}g'_{0'}) = \]
\[ = - \sum_{i=0}^{\varphi} (h_{i, \varphi'}) \varphi^i - \sum_{i=0}^{\varphi} (h_{i, \varphi}) \varphi^i + \sum_{i=0}^{\varphi} (h_{i, \varphi'}) \varphi^i - \Delta_2(\varphi) = \]
\[ = - \sum_{i=0}^{\varphi} (h_{i, \varphi'}) \varphi^i + (h_{i, \varphi}) \varphi^i + \sum_{i=0}^{\varphi} (h_{i, \varphi'}) \varphi^i - \Delta_1(\varphi) = \]
\[ = \Delta_1(\varphi) + \Delta_2(\varphi) \]
for some function \( \Delta_2 \), where
\[ \lim_{\varphi \to \infty} \Delta_2(\varphi) = \lim_{\varphi \to \infty} \Delta_1(\varphi) = 0 . \]
It follows that the product \( f \circ g' \) is in \( D' \) and
\[ f \circ g' = [h_{0,0}, h_{1,0}, \ldots, h_{r,0}] = (f \circ g)' - f' \circ g . \]
The results of the theorem follows.

We now put for simplicity
\[ \varphi_1 = \varphi^{(1)} = 0 , \]
for \( i = 0,1, \ldots, \) so that in particular
\[ \varphi_1 = 0 \]
for odd \( i . \)

Theorem 4. The product \( g(p) \circ g(q) \) is in \( D' \) and
\[ g(p) \circ g(q) = [h_0(p,q), h_1(p,q), \ldots, h_{p+q}(p,q)] \]
for \( p, q = 1, 2, \ldots, \)
where
\[ h_k(p,q) = \begin{cases} 0, & 0 < k \leq q, \\ (-1)^{1-q-1} \binom{p}{k} \frac{p-k}{q!(p-k)!}, & q < k \leq p+q+1, \end{cases} \]
and \( \binom{p}{q} \) denotes the binomial coefficient
\[ \binom{p}{q} = \frac{p!}{q!(p-q)!} . \]
In particular
\[ \delta^2 = \delta \circ \delta = [0, 0, 0], \]
\[ \delta' \circ \delta = [0, 0, 0], \]
\[ \delta \circ \delta' = 0 . \]

So, we see that the multiplication operation "\( \circ \)" is a non-commutative operation.

Theorem 5. The products \( x_\varphi^{p} \circ g(q) \) and \( g(q) \circ x_\varphi^{p} \) are in \( D' \) and
\[ x_\varphi^{p} \circ g(q) = g(p,q) = [h_0(p,q), h_1(p,q), \ldots, h_{q-p}(p,q)] \]
for \( p = 0, 1, \ldots, q \) and \( q = 0, 1, 2, \ldots, \), where
\[
\begin{align*}
h_1(p,q) &= \begin{cases} 
\frac{1}{2} (-1)^p p! g(q-p), & i = 0, \\
(-1)^{p-1} (q-1) p! g(q-p), & 1 \leq i \leq q-p
\end{cases} \\
g(q) = \begin{cases} 
\frac{1}{2} (-1)^p p! g(q-p), & i = 0, \\
(-1)^{p-1} (q-1) p! g(q-p), & 1 \leq i \leq q-p
\end{cases}
\end{align*}
\]

and
\[
g(q) \circ x^p = g(p,q) = [k_0(p,q), k_1(p,q), \ldots, k_{q-p}(p,q)]
\]

for \( p = 0,1, \ldots, q \) and \( q = 0,1, \ldots \), where
\[
k_1(p,q) = \begin{cases} 
\frac{1}{2} (-1)^p p! g(q-p), & i = 0 \\
(-1)^{p-1} (q-1) p! g(q-p-1), & 1 \leq i \leq q-p
\end{cases}
\]

In particular
\[
x^p \circ g(p) = g(p) \circ x^p = \frac{1}{2} (-1)^p p! g(p)
\]

for \( p = 0,1,2, \ldots \).

These theorems are equivalent to Theorem 3 and 4 proved in [1].

We now consider the product of two distribution vectors. For convenience we note that \( E' \) is isomorphic to the space of power series in an indeterminate \( \nu \) having distributions as coefficients. Under this natural isomorphism we write
\[
f = [f_0, f_1, \ldots, f_r, \ldots] = \sum_{r=0}^{\infty} f_r \nu^r.
\]

Definition 7. Let
\[
f = [f_0, f_1, \ldots, f_r, \ldots] = \sum_{r=0}^{\infty} f_r \nu^r
\]

and
\[
g = [g_0, g_1, \ldots, g_r, \ldots] = \sum_{r=0}^{\infty} g_r \nu^r
\]

be in \( E' \) and suppose that \( f \circ g \) exist for all \( r,s = 0,1, \ldots \) (in the sense of Definition 6) and
\[
f \circ g = \sum_{m=0}^{\infty} h_{rs} \nu^m
\]

for \( r,s = 0,1, \ldots \) for some distributions \( h_{rs} \) and some integers \( \nu_{rs} \geq 0 \). Let now put
\[
h_n = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} h_{rs} \nu^{r+s}
\]

for \( n = 0,1,2, \ldots \). We define the product \( f \circ g \) in \( E' \) by
\[
f \circ g = \left( \sum_{r=0}^{\infty} f_r \nu^r \right) \left( \sum_{s=0}^{\infty} g_s \nu^s \right) = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} (f_r \circ g) \nu^{r+s}
\]

\[
= \sum_{n=0}^{\infty} h_n \nu^n = [h_0, h_1, \ldots, h_n, \ldots]
\]

and say that \( h_0 \) is the finite part of \( f \circ g \).

Theorem 6. Let \( f \) and \( g \) be in \( E' \) and suppose that the products \( f \circ g \) and \( f' \circ g \) (or \( f \circ g' \)) are in \( E' \). Then the product \( f \circ g \) (or \( f' \circ g \)) is in \( E' \) and
\[
(f \circ g)' = f' \circ g + f \circ g'.
\]

Proof. Suppose
\[
f = \sum_{r=0}^{\infty} f_r \nu^r, \quad g = \sum_{r=0}^{\infty} g_r \nu^r.
\]

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Then
\[ f \circ g = \sum_{r=0}^{\infty} \sum_{s=0}^{r} (f^r \circ g_s) \psi^{r+s} \]
and
\[ f' \circ g = \sum_{r=0}^{\infty} \sum_{s=0}^{r} (f^r \circ g_{s'}) \psi^{r+s} \]
so that the products \( f^r \circ g_s \) and \( f^r \circ g_{s'} \) are in \( D' \). By Theorem 3
the product \( f^r \circ g_s \) is in \( D' \) and
\[ (f^r \circ g_s)' = f^r \circ g_s + f^r \circ g_{s'} . \]
Thus
\[ (f \circ g)' = \sum_{r=0}^{\infty} \sum_{s=0}^{r} (f^r \circ g_s)' \psi^{r+s} = \sum_{r=0}^{\infty} \sum_{s=0}^{r} (f^r \circ g_s) \psi^{r+s} + \sum_{r=0}^{\infty} \sum_{s=0}^{r} (f^r \circ g_{s'}) \psi^{r+s} \]
which implies the existence of \( f \circ g' \) and
\[ (f \circ g)' = f' \circ g + f \circ g' . \]

Example 1. \((\delta, \delta \varphi) = (\delta, \varphi(\varphi(x) \varphi(x)) = \varphi'(0) \varphi(0) = \varphi(0)^2) \) for arbitrary \( \varphi \) in \( D \) and so
\[ \delta^2 \circ \delta + 0 = [0, \delta] \] .

Example 2. \( \delta^3 \circ \delta + 0 = \delta^2 + 0 = [0, \delta] \) .

More general for the \( n \)-th power of the delta-function \( \delta \) we obtain
\[ \delta^n \circ \delta = [0, \delta^n] \] .

Example 3. \((\delta, \delta \varphi) = (\delta, \varphi(\varphi(x) \varphi(x)) = \varphi'(0) \varphi(0) = 0 \) for arbitrary \( \varphi \) in \( D \) and so
\[ \delta \circ \delta' = 0 . \]

Example 4. Using Theorem 3 we see that \( \delta' \circ \delta \) is in \( D' \) and
\[ \delta' \circ \delta = (\delta \circ \delta') - \delta \\
= (\delta \circ \delta' - \delta \circ \delta) = [0, \delta] \] .

Example 5. \((\delta', \delta' \varphi) = (\delta, \varphi'(\varphi(x) \varphi(x))) = \varphi'(0) \varphi(0) = \varphi(0)^2 \) for arbitrary \( \varphi \) in \( D \) and so
\[ \delta' \circ \delta' = [0, \delta] \) .

These four results are, of course, particular cases of
Theorem 4.

Example 6. \( (\delta, \delta') \circ (\delta', \delta) = (\delta + \delta') \circ (\delta + \delta) = \delta + \delta + \delta \circ \delta + \delta \circ \delta \) = \( 0, \delta \) \) .

We finally consider integration in \( D' \) and \( D' \) .

Definition 8. Let \( f \) be in \( D' \) , let \( \mu \) be a measure in \( R \) and let \( \delta \) be a measurable subset of \( R \) . We say that \( f \) is integrable on \( \delta \) if there exists an integer \( m \geq 0 \) and complex coefficients \( \alpha_0, \alpha_1, \ldots, \alpha_m \) for which
\[ \int_\delta f(x) \, d\mu(x) = \sum_{i=0}^{m} \alpha_i x^i + c(\varphi) \]
where \( f_\delta = f \circ \delta \) and
\[ \lim_{\varphi \to \infty} \Delta(\varphi) = 0 . \]

We then write
\[ \int_\delta f(x) \, d\mu(x) = \sum_{i=0}^{m} \alpha_i x^i + c(\varphi) \]
and say that \( \alpha_0 \) is the finite part of the integral.
Example 7. \( \int_{-\infty}^{\infty} \delta(x) dx = 1. \)

Example 8. \( \int_{0}^{\infty} \delta'(x) dx = \int_{0}^{\infty} \varphi'(\omega) dx = 1/2 \) and so
\( \int_{0}^{\infty} \delta(x) dx = 1/2. \)

Example 9. \( \int_{-\infty}^{\infty} \delta(x) dx = 0. \)

Example 10. \( \int_{0}^{\infty} \delta'(x) dx = \int_{0}^{\infty} \delta'(n) dx = -\varphi(0) \)
and so \( \int_{0}^{\infty} \delta'(x) dx = [0, -\varphi(0)]. \)

Theorem 7. For all \( f \in D' \) and all \( \varphi \in D \) we have:
\[ \int_{-\infty}^{\infty} (f \varphi) dx = \lim_{\varphi \to 0} \int_{-\infty}^{\infty} f(x) \varphi(x) dx. \]

Proof. It is well known that \( (f \varphi) \xrightarrow{\text{topology of } D'(E')} f \varphi \) in the topology of \( \mathcal{E}'(E') \) is the space of all distributions with compact supports) so that
\[ \lim_{\varphi \to 0} \int_{-\infty}^{\infty} f(x) \varphi(x) dx = (f, \varphi). \]

The proof is finished.

Definition 9. Let
\[ \mathcal{F} = [f_0, f_1, \ldots, f_r, \ldots] = \sum_{r=0}^{\infty} \mathbf{f}_r \mathbf{s}^r \]
be in \( D' \) and suppose that \( f_r \) is integrable on \( \delta \) with
\[ \int_{\delta} f_r(x) d\varphi(x) = \sum_{\alpha \geq 0} \alpha \mathbf{f}_r \mathbf{s}^{\alpha+1} \]
for \( r = 0, 1, \ldots, \). We say that \( \mathcal{F} \) is integrable on \( \delta \) and write
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\[ \int_{\delta} f(x)d\varphi(x) = \sum_{r=0}^{\infty} \sum_{i=0}^{d_r} d_r \mathbf{s}^{r+i} \equiv [d_0, d_1, \ldots, d_n, \ldots] \]
where
\[ d_n = \sum_{r=0}^{\infty} \sum_{i=0}^{d_r(s_i)} (r+i = n) \]
for \( n = 0, 1, \ldots \) and say that \( d_0 \) is the finite part of the integral.

Example 11. \( \int_{0}^{\infty} \delta'(x) dx = [1/2, 0, -\varphi(0)]. \)

We see that the integral of a given distribution vector (if exists) is a number vector.

Remark. The reader could remain disappointed at the fact that the multiplication operation introduced in our paper is nonassociative which follows directly from the example
\[ \delta' \circ (\delta \circ \delta') \neq (\delta \circ \delta') \circ \delta'. \]

Recall, however, that according to the well-known interpretation of the Schwartz example
\[ (x^{-1} \cdot x) \delta(x) \neq x^{-1} (x \cdot \delta(x)) \]
it is principally impossible to supply the distribution space or any of its enlargements (in particular, the space of distribution vectors \( \mathcal{D}' \)) with an associative multiplication operation.

REFERENCES


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