Existence and uniqueness of v-asymptotic expansions and Colombeau's generalized numbers

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Abstract

We define a type of generalized asymptotic series called v-asymptotic. We show that every function with moderate growth at infinity has a v-asymptotic expansion. We also describe the set of v-asymptotic series, where a given function with moderate growth has a unique v-asymptotic expansion. As an application to random matrix theory we calculate the coefficients and establish the uniqueness of the v-asymptotic expansion of an integral with a large parameter. As another application (with significance in the non-linear theory of generalized functions) we show that every Colombeau's generalized number has a v-asymptotic expansion. A similar result follows for Colombeau's generalized functions, in particular, for all Schwartz distributions.

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1. Introduction

Our framework is the ring $\mathcal{M}$ of functions with moderate growth at infinity, supplied with a pseudovaluation $v : \mathcal{M} \to \mathbb{R} \cup \{\infty\}$ and pseudometric $d_v : \mathcal{M}^2 \to \mathbb{R}$ (Section 2). We denote by $\mathcal{F}_v$ the ring of functions with the non-negative pseudovaluation and by $\mathcal{C}_v$ the set

of functions with zero pseudovaluation. Here are some typical examples: $r + 1/x^n \in \mathcal{F}_v$, $r \in \mathbb{R}$, $n = 0, 1, 2, \ldots$ and $\sin x, \cos x, \ln x \in \mathcal{C}_v$.

In Section 4 we prove the existence of a linear homomorphism (a linear operator) $\widehat{\mathrm{st}}$, called the "pseudostandard part mapping," from $\mathcal{F}_v$ into $\mathcal{F}_v$ with range within $\mathcal{C}_v \cup \{0\}$. The mapping $\widehat{\mathrm{st}}$ is an extension of the limit $\lim_{x \to \infty}$, considered as a functional (part (iii) of Theorem 6).

In Section 5 we define the concept of v-asymptotic series; these are series of the form $\sum_{n=0}^{\infty} \frac{\varphi_n(x)}{x^{r_n}}$, where (r_n) is a strictly increasing (bounded or unbounded) sequence in $\mathbb{R}$ and the set $\{\varphi_n(x) \mid n = 0, 1, \ldots\}$ is a linearly v-independent subset of $\mathcal{C}_v$ (Section 3). In that sense, the functions in $\mathcal{C}_v$ play the role of "our generalized numbers." There are many examples in this section: some series are both v-asymptotic and asymptotic (in the usual sense), some series are v-asymptotic, but not asymptotic and others are asymptotic, but not v-asymptotic.

The main result is in Theorem 8 (Section 7): Every function with moderate growth at infinity has a v-asymptotic expansion. We also describe the sets of v-asymptotic series, where a given function in $\mathcal{M}$ has a unique v-asymptotic expansion. We shall try to explain the idea of the proof by modifying a familiar theorem in asymptotic analysis: let $f(x)$ be a function, (r_n) be a strictly increasing sequence in $\mathbb{R}$ and (c_n) be a sequence in $\mathbb{C}$. The asymptotic expansion $f(x) \sim \sum_{n=0}^{\infty} \frac{c_n}{x^{r_n}}$, $x \to \infty$, holds iff

$$c_n = \lim_{x \to \infty} \left(x^{r_n} \left[f(x) - \sum_{k=0}^{n-1} \frac{c_k}{x^{r_k}} \right] \right), \tag{1}$$

for $n = 0, 1, 2, \ldots$. This result can be rephrased as follows: if all limits (for $n = 0, 1, 2, \ldots$) on the RHS of (1) exist, then $f(x)$ has a unique asymptotic expansion of the form $\sum_{n=0}^{\infty} \frac{c_n}{x^{r_n}}$. It is clear that most of the functions in $\mathcal{M}$ do not have asymptotic expansions of this form. Actually, there are very few general existence results in asymptotic analysis (N.G. De Bruijn [1], R. Estrada and R.P. Kanwal [5]). This is, perhaps, the reason why many consider the asymptotic analysis as a "kind of art" or "collection of different methods," rather than a mature mathematical theory. Our article is an attempt to improve this situation. Here is the way we modify the above asymptotic theorem:

(a) Instead of series of the form $\sum_{n=0}^{\infty} \frac{c_n}{x^{r_n}}$, we consider v-asymptotic series $\sum_{n=0}^{\infty} \frac{\varphi_n(x)}{x^{r_n}}$, where $v(\varphi_n) = 0$ and v is the pseudovaluation mentioned earlier.

(b) We replace $\lim_{x \to \infty}$ (considered as a functional) in the counterpart of (1) by the linear operator $\widehat{\mathrm{st}}$ mentioned earlier. The advantage of $\widehat{\mathrm{st}}$ over $\lim_{x \to \infty}$ is that $\widehat{\mathrm{st}}$ is defined on the whole $\mathcal{F}_v$ (including functions such as $\sin x, \cos x, \ln x, \ln(\ln x)$, etc.). In contrast, $\lim_{x \to \infty}$ is defined only on a proper subset of $\mathcal{F}_v$ (e.g., $\lim_{x \to \infty} \sin x$ and $\lim_{x \to \infty} \ln x$ do not exist).

(c) Instead of a (fixed) sequence (r_n) (which corresponds to a choice of a fixed asymptotic scale), we allow different sequences (r_n) for the different functions $f(x)$.

Two applications are presented in the article: At the end of Section 7 we prove the uniqueness of the v-asymptotic expansion of an integral with a large parameter with origin in random matrix theory. In Section 8 we show that every Colombeau's generalized number has a v-asymptotic expansion. A similar result follows for Colombeau's generalized functions, in particular, for all Schwartz distributions. This result has importance in the

non-linear theory of generalized functions (J.F. Colombeau [2,3]) and its applications in ordinary and partial differential equations (M. Oberguggenberger [11]).

This article has many features in common with another article by T.D. Todorov and R. Wolf [12] on A. Robinson's asymptotic numbers in the framework of the non-standard asymptotic analysis (A.H. Lightstone and A. Robinson [9]).

In what follows $\mathbb{N} = \{1, 2, 3, \ldots\}$ denotes the set of natural numbers and $\mathbb{R}$ and $\mathbb{C}$ denote the fields of the real and complex numbers, respectively.

2. Our framework: Functions of moderate growth at infinity

We define and study the properties of a pseudovaluation and the corresponding pseudo-metric in the ring of complex valued functions with moderate growth at infinity. Most of the results are elementary and presented without proofs.

Definition 1 (*Functions of moderate growth*).

(i) We denote by $G_\infty(\mathbb{R}, \mathbb{C})$ the ring (under the usual pointwise addition and multiplication) of all complex valued functions f defined on $\mathrm{dom}(f) \subseteq \mathbb{R}$ such that $(n, \infty) \subseteq \mathrm{dom}(f)$ for some $n \in \mathbb{N}$. We shall treat the elements of $G_\infty(\mathbb{R}, \mathbb{C})$ as germs at ∞, i.e., two functions will be identified if they have the same values for all sufficiently large x. We define a partial order in $G_\infty(\mathbb{R}, \mathbb{C})$ by $f > 0$ if $f(x) > 0$ for all sufficiently large x.

(ii) The functions in

$$\mathcal{M} = \{f \in G_\infty(\mathbb{R}, \mathbb{C}) \mid f(x) = O(x^n) \text{ as } x \to \infty \text{ for some } n \in \mathbb{N}\}$$

are called *functions with moderate growth at* ∞ (or, *moderate functions*, for short) and those in

$$\mathcal{N} = \{f \in G_\infty(\mathbb{R}, \mathbb{C}) \mid f(x) = O(1/x^n) \text{ as } x \to \infty \text{ for all } n \in \mathbb{N}\}$$

are called *null-functions*.

(iii) We define a *pseudovaluation* $v : \mathcal{M} \to \mathbb{R} \cup \{\infty\}$ by

$$v(f) = \sup\{r \in \mathbb{R} \mid f(x) = O(1/x^r) \text{ as } x \to \infty\}. \tag{2}$$

The functions in the sets:

$$\mathcal{I}_v = \{f \in \mathcal{M} \mid v(f) > 0\}, \tag{3}$$
$$\mathcal{F}_v = \{f \in \mathcal{M} \mid v(f) \geqslant 0\}, \tag{4}$$
$$\mathcal{C}_v = \{f \in \mathcal{M} \mid v(f) = 0\}, \tag{5}$$

are called v-*infinitesimal*, v-*finite* and v-*constant*, respectively. The functions in $\mathcal{M} \setminus \mathcal{F}_v$ will be called v-*infinitely large*.

(iv) If $S \subseteq G_\infty(\mathbb{R}, \mathbb{C})$, then the set

$$\mu_v(S) = \{\varphi + d\varphi \mid \varphi \in S, \ d\varphi \in \mathcal{I}_v\} \tag{6}$$

is called the *v-monad* of S. In the particular case $S = \{f\}$, we shall write $\mu_v(f)$ instead of the more precise $\mu_v(\{f\})$.

Example 1. x^r, e^{ix}, $\sin x$, $\cos x$, $\ln x$, $\ln(\ln x)$ are all in $\mathcal{M}$ (where $r \in \mathbb{R}$). Also, 0 and e^{-x} are in $\mathcal{N}$. In contrast, $e^x \notin \mathcal{M}$. Notice that $\mathcal{M}$ is subring of $G_\infty(\mathbb{R}, \mathbb{C})$ with zero divisors. In particular, the functions x^r, e^{ix}, $\ln x$, $\ln(\ln x)$ are all multiplication invertible in $\mathcal{M}$, while $\sin x, \cos x$ are not.

Example 2. If $r \in \mathbb{R}$, then $v(1/x^r) = r$. More generally, $v(P) = -\deg(P)$ for any polynomial $P \in \mathbb{C}[x]$, where $\deg(P)$ denotes the degree of P. We have $v(c) = 0$ for all $c \in \mathbb{C}, c \neq 0$. In other words, $\mathbb{C} \setminus \{0\} \subset \mathcal{C}_v$. Also, $\sin x$, $\cos x$, e^{ix}, $\ln x$, $\ln^n x$ and $\ln(\ln x)$ are all in $\mathcal{C}_v$, because $v(e^{ix}) = v(\sin x) = v(\cos x) = v(\ln x) = v(\ln^n x) = v(\ln(\ln x)) = 0$ ($n \in \mathbb{Z}$). Finally, $v(0) = v(e^{-x}) = \infty$.

The results of the next three lemmas follow immediately from the definition of the pseudovaluation:

Lemma 1 (A characterization of v). *Let* $f \in G_\infty(\mathbb{R}, \mathbb{C})$ *and* $r \in \mathbb{R}$. *Then the following are equivalent*:

(i) $v(f) = r$.
(ii) $\frac{1}{x^r \sqrt[n]{x}} \leqslant |f(x)| \leqslant \frac{\sqrt[n]{x}}{x^r}$ *for all* $n \in \mathbb{N}$.
(iii) $f(x) = \frac{\varphi(x)}{x^r}$ *for some* $\varphi \in \mathcal{C}_v$.

Lemma 2 (A characterization of $\mathcal{N}$). *Let* $f \in G_\infty(\mathbb{R}, \mathbb{C})$. *The following are equivalent*:

(i) $f \in \mathcal{N}$.
(ii) $v(f) = \infty$.
(iii) $\lim_{x \to \infty} x^n f(x) = 0$ *for all* $n \in \mathbb{N}$.

Lemma 3 (Connection with O's symbols).

(i) $\mathcal{I}_v \subset o(1) \subset \mathcal{F}_v$ *in the sense that* $f \in \mathcal{I}_v$ *implies* $f(x) = o(1)$, *as* $x \to \infty$, *which implies* $f \in \mathcal{F}_v$.
(ii) $O(1) \setminus o(1) \subset \mathcal{C}_v$ *in the sense that if* $f(x) = O(1)$ *and* $f(x) \neq o(1)$, *as* $x \to \infty$, *then* $f \in \mathcal{C}_v$. *More generally,* $f \in O(1/x^r) \setminus o(1/x^r)$ *implies* $v(f) = r$.
(iii) *The functions in* $G_\infty(\mathbb{R}, \mathbb{C}) \setminus \mathcal{F}_v$ *are unbounded on every interval of the form* $(n, \infty), n \in \mathbb{N}$.

Part (i) shows that the v-infinitesimals are proper infinitesimals (with respect to the order relation in $\mathcal{M}$). Part (ii) shows that the finite but non-infinitesimal functions (with respect to the order in $\mathcal{M}$) are v-finite. Part (iii) shows that if f is outside $\mathcal{F}_v$ and if $|f|$ happens to be in order with all $n \in \mathbb{N}$, then f is an infinitely large element of $G_\infty(\mathbb{R}, \mathbb{C})$. We should notice that $\mathcal{C}_v$ contains also infinitely large functions (along with finite and infinitesimal).

For example, $\ln x$ is infinitely large in the sense that $n < \ln x$ for all $n \in \mathbb{N}$ and $1/\ln x$ is a positive infinitesimal in the sense that $0 < 1/\ln x < 1/n$ for all $n \in \mathbb{N}$.

Theorem 1 (Properties of v). *The function v is a pseudovaluation on $\mathcal{M}$ in the sense that*:

(i) $v(f) = \infty$ *iff* $f \in \mathcal{N}$.
(ii) $v(fg) \quad v(f) + v(g)$. *In particular, we have* $v(cg) = v(f)$ *for all* $c \in \mathbb{C}, c = 0$ *and also* $v(f/x^r) = v(f) + r$ *for all* $r \in \mathbb{R}$.
(iii) $v(f \pm g) \quad \min\{v(f), v(g)\}$. *Moreover,* $v(f) = v(g)$ *implies* $v(f \pm g) = \min\{v(f), v(g)\}$.

In addition, v has the following properties:

(iv) $v(-f) = v(f) = v(|f|)$;
(v) $v(1/f) = -v(f)$ *whenever f is invertible in $\mathcal{M}$*;
(vi) $|f| < |g|$ *implies* $v(f) \quad v(g)$.

Proof. The above properties follow directly from the properties of the "sup" in the definition of v and we leave the verification to the reader. □

Theorem 2 (Rings and ideals).

(i) $\mathcal{M}$ *is convex subring (with zero divisors) of* $G_\infty(\mathbb{R}, \mathbb{C})$, *where "convex" means that if* $f \in G_\infty(\mathbb{R}, \mathbb{C})$ *and* $g \in \mathcal{M}$, *then* $|f| \quad |g|$ *implies* $f \in \mathcal{M}$. *Also $\mathcal{N}$ is a convex ideal in $\mathcal{M}$*.
(ii) *We have* $\mathcal{F}_v = \mathcal{C}_v \cup \mathcal{I}_v$ *and* $\mathcal{C}_v \cap \mathcal{I}_v = \emptyset$. *Besides, $\mathcal{F}_v$ is convex subring of $\mathcal{M}$ and $\mathcal{I}_v$ is a convex ideal in $\mathcal{F}_v$*.
(iii) *The ring $\mathcal{F}_v$ contains a copy of the field of complex numbers $\mathbb{C}$ presented by the constant-functions. Also we have* $\mathbb{C} \subset \mathcal{C}_v \cup \{0\}$.

Proof. (i) follows immediately from the definitions of $\mathcal{M}$ and $\mathcal{N}$.

(ii) Suppose $f, g \in \mathcal{F}_v$, i.e., $v(f) \quad 0, v(g) \quad 0$. It follows $v(f + g) \quad \min\{v(f), v(g)\} \quad 0$ and $v(fg) \quad v(f) + v(g) \quad 0$, by Theorem 1, i.e., $f + g, fg \in \mathcal{F}_v$. Similarly it follows that $\mathcal{I}_v$ is closed under the addition and multiplication. Suppose also that $h \in \mathcal{I}_v$, i.e., $v(h) > 0$. We have $v(fh) \quad v(f) + v(h) > 0$, i.e., $fh \in \mathcal{I}_v$, as required.

(iii) follows from the fact that $v(c) = 0$ for all $c \in \mathbb{C}, c = 0$. □

Definition 2 (*Pseudometric*). We define a pseudometric $d_v : \mathcal{M}^2 \to \mathbb{R}$ by $d_v(f, g) = e^{-v(f-g)}$ (with the understanding that $e^{-\infty} = 0$). We denote by $(\mathcal{M}, d_v)$ the corresponding pseudometric space.

Theorem 3 (Properties of the pseudometric). *For any $f, g, h \in \mathcal{M}$*:

(i) $d_v(f, g) = 0$ *iff* $f - g \in \mathcal{N}$.
(ii) $d_v(f, g) = d_v(g, f)$.

(iii) $d_v(f, g) \quad \max\{d_v(f, h), d_v(h, g)\} \quad d_v(f, h) + d_v(h, g)$.

(iv) $\mathcal{I}_v, \mathcal{F}_v$ and $\mathcal{C}_v$ coincide with the open unit ball, the closed unit ball and the unit sphere in $(\mathcal{M}, d_v)$, respectively.

Proof. A direct consequence of the property of the pseudovaluation v (Theorem 1). We should mention that $d_v(f, g) \quad \max\{d_v(f, h), d_v(h, g)\}$ is called *ultrametric inequality*. $\square$

Theorem 4 (Convergence in $\mathcal{M}$). *Let* (f_n) *be a sequence in* $\mathcal{M}$.

(i) *If* $f \in \mathcal{N}$ (*in particular,* $f = 0$), *then* $\lim_{n\to\infty} f_n = f$ *in* $(\mathcal{M}, d_v)$ *iff* $\lim_{n\to\infty} v(f_n) = \infty$ *in* $\mathbb{R} \cup \{\infty\}$.

(ii) *If* $f \notin \mathcal{N}$, *then* $\lim_{n\to\infty} f_n = f$ *in* $(\mathcal{M}, d_v)$ *implies that* $v(f_n) = v(f)$ *for all sufficiently large* n.

Proof. (i) We have $\lim_{n\to\infty} f_n = f$ in $(\mathcal{M}, d_v)$ iff $\lim_{n\to\infty} d(f_n, f) = 0$ in $\mathbb{R}$ iff $\lim_{n\to\infty} e^{-v(f_n - f)} = 0$ in $\mathbb{R}$ iff $\lim_{n\to\infty} v(f_n - f) = \infty$ in $\mathbb{R} \cup \{\infty\}$ iff $\lim_{n\to\infty} v(f_n) = \infty$, as required, since $v(f_n - f) = v(f_n)$ for all sufficiently large n, by part (iii) of Theorem 1.

(ii) We have $\lim_{n\to\infty} v(f_n - f) = \infty$, by (i). Suppose (on the contrary) that there exists a unbounded sequence (v_n) in $\mathbb{N}$ such that $v(f_{v_n}) = v(f)$ for all n. It follows $v(f_{v_n} - f) = \min\{v(f_{v_n}), v(f)\}$ for all n, by part (iii) of Theorem 1, which implies $\lim_{n\to\infty} v(f_{v_n}) = \infty$. Finally, it follows $v(f) = \infty$, a contradiction. $\square$

Corollary 1. *Let* (r_n) *be a sequence in* $\mathbb{R} \cup \{\infty\}$ *such that* $\lim_{n\to\infty} r_n = \infty$. *Let* (f_n) *be a sequence in* $\mathcal{M}$ *such that* $f_n = o(1/x^{r_n})$ *or* $f_n = O(1/x^{r_n})$ *as* $x \to \infty$ *for all* n. *Then* $\lim_{n\to\infty} f_n = 0$ *in* $(\mathcal{M}, d_v)$.

Proof. The result follows, by part (i) of the above theorem, since in both cases $v(f_n) \quad r_n$ for all sufficiently large n. $\square$

Example 3. We have $\lim_{n\to\infty} \frac{1}{x^n} = 0$ in $(\mathcal{M}, d_v)$ since $v(\frac{1}{x^n}) = n \to \infty$ as $n \to \infty$.

3. Linearly v-independent sets

We should notice that the set of the v-constants $\mathcal{C}_v$ is not closed under addition and multiplication in $\mathcal{M}$. Our next goal is to look for vector spaces inside $\mathcal{C}_v \cup \{0\}$.

If $\Phi \subset \mathcal{M}$, we denote by $\mathrm{Span}(\Phi)$ the span of Φ over $\mathbb{C}$ (within $\mathcal{M}$). Recall that $\mathrm{Span}(\Phi)$ is the smallest vector subspace of $\mathcal{M}$ containing Φ.

Definition 3 (v-*Independence*). We say that a set $\Phi \subset \mathcal{M}$ is *linearly v-independent over* $\mathbb{C}$ if $\mathrm{Span}(\Phi) \subset \mathcal{C}_v \cup \{0\}$.

Let Φ be an linearly v-independent set over $\mathbb{C}$. It is clear that $\Phi \subset C_v \cup \{0\}$. Also, every subset Ψ of Φ is also linearly v-independent over $\mathbb{C}$.

Here is another (trivial) reformulation of the above definition:

Lemma 4. *Let $\Phi \subset \mathcal{M}$. Then the following are equivalent*:

(i) Φ *is linearly v-independent over $\mathbb{C}$.*
(ii) $v(f) = 0$ *for all $f \in \mathrm{Span}(\Phi)$, $f = 0$.*
(iii) *There exists a vector subspace K of $\mathcal{M}$ such that $\Phi \subseteq K \subset C_v \cup \{0\}$.*

Example 4. The set of the complex numbers $\mathbb{C}$ is linearly v-independent (in a trivial way). If $f \in C_v \cup \{0\}$, then the set $\{f\}$ is also linearly v-independent over $\mathbb{C}$. In particular, the set $\{0\}$ is linearly v-independent over $\mathbb{C}$.

Lemma 5. *Each of the following sets is linearly v-independent*: $\Phi_1 = \{\sin x, \cos x\}$, $\Phi_2 = \{\ln x, \ln(\ln x), \ln(\ln(\ln x)), \ldots\}$, $\Phi_3 = \{\ln x, \ln^2 x, \ln^3 x, \ln^4 x, \ldots\}$, $\Phi_4 = \{e^{\pm i\pi x}, e^{\pm i\pi x}\ln x, e^{\pm i\pi x}\ln^2 x, e^{\pm i\pi x}\ln^3 x, \ldots\}$. *The set $\Phi_5 = \Phi_1 \cup \Phi_2 \cup \Phi_3 \cup \Phi_4$ is also linearly v-independent.*

Proof. We shall prove that Φ_2 is linearly v-independent. For the sake of convenience we denote $l_1(x) = \ln x$, $l_2(x) = \ln(\ln x)$, $l_3(x) = \ln(\ln(\ln x))$, $\ldots$. So, we have $\Phi_2 = \{l_n(x): n = 0, 1, 2, \ldots\}$. It suffices to prove that if $\sum_{k=1}^n \alpha_k l_k$ is a non-zero linear combination, then $v(\sum_{k=1}^n \alpha_k l_k) = 0$. First note that this is obviously true in the case $n = 1$, since each $l_n(x)$ has valuation 0. If the sum has more than one term, we just need to show that $\sum_{k=1}^n \alpha_k l_k$ has a "dominant term," as $x \to \infty$. (Technically, this is the greatest term in the polynomial with respect to the lexicographic ordering on the sequence of its exponents.) For this it suffices to show that $m > k$ implies $\lim_{x\to\infty}[(\lambda_m(x))^p/(\lambda_k(x))^q] = 0$ whenever $p > 0$ and $q > 0$, which is easily verified by l'Hopital's Rule. The rest of the sets are treated similarly and we leave the verification to the reader. $\square$

Remark 1 (*Linear independence vs. linear v-independence*). We observe that the sets in the above four examples are also linearly independent over $\mathbb{C}$ (in the usual sense). Notice that the set $\Phi = \{\ln x, 2\ln x\}$ is also linearly v-independent but it is clearly linearly dependent. Conversely, the functions $f(x) = 1$ and $g(x) = -1 + 1/x$ are linearly independent over $\mathbb{C}$, but f and g are not v-independent, since $v(f + g) = v(1/x) = 1$ (not $= 0$). We also observe that if a set Φ is linearly v-independent over $\mathbb{C}$, then the set $\Phi \cup \{0\}$ is also linearly v-independent over $\mathbb{C}$ (in sharp contrast to the case of linear independence).

4. Pseudostandard part

We prove the existence of a particular type of linear homomorphism $\widehat{\mathrm{st}}$ from $\mathcal{F}_v$ into $\mathcal{F}_v$, with range in $C_v \cup \{0\}$, which is an extension of $\lim_{x\to\infty}$ (considered also as a linear homomorphism). The applications of this construction appear in the next section.

Definition 4 (*Maximal vector spaces*). Let $\mathcal{C}$ be a vector subspace of $\mathcal{M}$. We say that $\mathcal{C}$ is *maximal in* $\mathcal{C}_v$ if $\mathbb{C} \subset \mathcal{C} \subset \mathcal{C}_v \cup \{0\}$ and there is no a vector subspace $\mathcal{K}$ of $\mathcal{M}$ such that $\mathcal{C} \subsetneqq \mathcal{K} \subset \mathcal{C}_v \cup \{0\}$.

Lemma 6 (Existence). *Let Φ be a linearly v-independent subset of $\mathcal{M}$ (Definition 3). Then there exists a vector subspace $\mathcal{C}$ of $\mathcal{M}$ which is maximal in $\mathcal{C}_v$ and which contains Φ.*

Proof. Let $\mathcal{U}(\Phi)$ denote the set of all vector subspaces $\mathcal{K}$ of $\mathcal{M}$ such that $\mathbb{C} \cup \Phi \subseteq \mathcal{K} \subset \mathcal{C}_v \cup \{0\}$ and let $\mathcal{U}(\Phi)$ be ordered by inclusion. Notice that $\mathcal{U}(\Phi)$ is non-empty since $\mathrm{Span}(\Phi) \in \mathcal{U}(\Phi)$. We observe that every monotonic subset (chain) $\mathcal{B}$ of $\mathcal{U}(\Phi)$ is bounded from above by $\bigcup_{B \in \mathcal{B}} B$. It follows that $\mathcal{U}(\Phi)$ has a maximal element $\mathcal{C}$ (as desired), by Zorn's lemma. $\square$

We shall sometimes refer to $\mathrm{Span}(\Phi)$ as the "explicit" part of the space $\mathcal{C}$ and to $\mathcal{C} \setminus \mathrm{Span}(\Phi)$ as the "implicit" part of $\mathcal{C}$.

Theorem 5 (v-Completeness). *Let $\mathcal{C}$ be a vector subspace of $\mathcal{M}$ which is maximal in $\mathcal{C}_v$. Then*:

(i) *$\mathcal{C}$ is v-complete in $\mathcal{F}_v$ in the sense that: $\mu_v(\mathcal{C}) = \mathcal{F}_v$, where $\mu_v(\mathcal{C})$ is the v-monad of $\mathcal{C}$ ((6), Section 2).*
(ii) *We have $\mathcal{F}_v = \mathcal{C} \oplus \mathcal{I}_v$ in the sense that every $f \in \mathcal{F}_v$ has a unique asymptotic expansion $f = \varphi + d\varphi$, where $\varphi \in \mathcal{C}$ and $d\varphi \in \mathcal{I}_v$ ((3), Section 2).*

Proof. (i) We have $\mu_v(\mathcal{C}) \subseteq \mathcal{F}_v$, since $v(\varphi + d\varphi) \quad \min\{v(\varphi), v(d\varphi)\} \quad 0$, by part (iii) of Theorem 1. To show $\mu_v(\mathcal{C}) \supseteq \mathcal{F}_v$, suppose (on the contrary) that there exists $g \in \mathcal{F}_v \setminus \mu_v(\mathcal{C})$, i.e., $v(g) \quad 0$ and $v(g - \varphi) \quad 0$ for all $\varphi \in \mathcal{C}$. By letting $\varphi = 0$, we conclude that $v(g) = 0$, i.e., $g \in \mathcal{C}_v$. It follows $v(g - \varphi) = 0$ for all $\varphi \in \mathcal{C}$ (by part (iii) of Theorem 1, again), i.e., $g - \varphi \in \mathcal{C}_v$ for all $\varphi \in \mathcal{C}$. Thus we have $\mathcal{C} \subsetneqq \mathrm{Span}(\mathcal{C} \cup \{g\}) \subset \mathcal{C}_v \cup \{0\}$, contradicting the maximality of $\mathcal{C}$.

(ii) The existence of the asymptotic expansion $f = \varphi + d\varphi$ follows from (i). To show the uniqueness, suppose that $\varphi + d\varphi = 0$. It follows $v(\varphi) = v(-d\varphi) = v(d\varphi) > 0$ implying $\varphi = d\varphi = 0$ since $\mathcal{C} \subset \mathcal{C}_v \cup \{0\}$. $\square$

The above result justifies the following definition.

Definition 5 (*Pseudostandard part mapping*). Let $\mathcal{C}$ be a vector subspace of $\mathcal{M}$ which is maximal in $\mathcal{C}_v$. We define $\widehat{\mathrm{st}}_\mathcal{C} : \mathcal{F}_v \to \mathcal{F}_v$, by $\widehat{\mathrm{st}}_\mathcal{C}(\varphi + d\varphi) = \varphi$, where $\varphi \in \mathcal{C}, d\varphi \in \mathcal{I}_v$. We say that $\widehat{\mathrm{st}}_\mathcal{C}$ is a *pseudostandard part mapping* determined by $\mathcal{C}$. We shall sometimes write simply $\widehat{\mathrm{st}}$ instead of the more precise $\widehat{\mathrm{st}}_\mathcal{C}$, suppressing the dependence on $\mathcal{C}$, when the choice of $\mathcal{C}$ is clear from the context.

Theorem 6 (Properties of $\widehat{\mathrm{st}}$). *Let $\mathcal{C}$ be a vector subspace of $\mathcal{M}$ which is maximal in $\mathcal{C}_v$. Then*:

(i) *The pseudostandard part mapping* $\widehat{\mathrm{st}}_\mathcal{C}$ *is a linear homomorphism from* $\mathcal{F}_v$ *into* $\mathcal{F}_v$ *with range* $\mathrm{ran}(\widehat{\mathrm{st}}_\mathcal{C}) = \mathcal{C}$, *i.e.,* $\widehat{\mathrm{st}}_\mathcal{C}(\alpha f + \beta g) = \alpha\widehat{\mathrm{st}}_\mathcal{C}(f) + \beta\widehat{\mathrm{st}}_\mathcal{C}(g)$, *for all* $f, g \in \mathcal{F}_v$ *and all* $\alpha, \beta \in \mathbb{C}$.

(ii) $\mathcal{C}$ *consists of the fixed points of* $\widehat{\mathrm{st}}_\mathcal{C}$ *in* $\mathcal{F}_v$, *i.e.,* $\mathcal{C} = \{f \in \mathcal{F}_v \mid \widehat{\mathrm{st}}_\mathcal{C}(\varphi) = \varphi\}$. *Consequently, we have* $\widehat{\mathrm{st}}_\mathcal{C} \circ \widehat{\mathrm{st}}_\mathcal{C} = \widehat{\mathrm{st}}_\mathcal{C}$.

(iii) *Suppose* (*in addition to the above*) *that* $\Phi \subset \mathcal{C}$ *for some linearly v-independent subset* Φ *of* $\mathcal{M}$ (*Definition* 3). *Then we have* $\widehat{\mathrm{st}}_\mathcal{C}(\varphi + d\varphi) = \varphi$ *for all* $\varphi \in \mathrm{Span}(\Phi)$ *and all* $d\varphi \in \mathcal{I}_v$. *Consequently, the restriction* $\widehat{\mathrm{st}}_\mathcal{C}|\mathbb{C} \oplus \mathcal{I}_v$ *is a* (*proper*) *standard part mapping* (*in the sense of non-standard analysis—A.H. Lightstone and A. Robinson* [9]), *i.e.,* $\widehat{\mathrm{st}}_\mathcal{C}(c + d\varphi) = c$ *for all* $c \in \mathbb{C}$ *and all* $d\varphi \in \mathcal{I}_v$ (*see the discussion after Lemma* 3). *Or, equivalently,* $\widehat{\mathrm{st}}_\mathcal{C}|\mathbb{C} \oplus \mathcal{I}_v$ *coincides with* $\lim_{x\to\infty}$ *in the sense that for every* $f \in \mathbb{C} \oplus \mathcal{I}_v$ *we have*

$$\widehat{\mathrm{st}}_\mathcal{C}(f) = \lim_{x\to\infty} f(x).$$

Proof. (i) We have $f = \varphi + d\varphi$ and $g = \psi + d\psi$ for some $\varphi, \psi \in \mathcal{C}, d\varphi, d\psi \in \mathcal{I}_v$, by Theorem 5. Also $\alpha f + \beta g = \alpha\varphi + \beta\psi + \alpha\, d\varphi + \beta\, d\psi$ and it is clear that $\alpha\varphi + \beta\psi \in \mathcal{C}$ (since $\mathcal{C}$ is a vector space) and $\alpha\, d\varphi + \beta\, d\psi \in \mathcal{I}_v$ (since $\mathcal{I}_v$ is an ideal in $\mathcal{F}_v$, by Theorem 2). Thus $\widehat{\mathrm{st}}_\mathcal{C}(\alpha f + \beta g) = \alpha\varphi + \beta\psi = \alpha\widehat{\mathrm{st}}(\varphi) + \beta\widehat{\mathrm{st}}(\psi) = \alpha\widehat{\mathrm{st}}_\mathcal{C}(f) + \beta\widehat{\mathrm{st}}_\mathcal{C}(g)$, as required, since $\widehat{\mathrm{st}}_\mathcal{C}(\varphi) = \varphi$ and $\widehat{\mathrm{st}}_\mathcal{C}(\psi) = \psi$, by the definition of $\widehat{\mathrm{st}}_\mathcal{C}$.

(ii) As we mentioned already, $\widehat{\mathrm{st}}_\mathcal{C}(\varphi) = \varphi$ for all $\varphi \in \mathcal{C}$, by the definition of $\widehat{\mathrm{st}}_\mathcal{C}$. Conversely, suppose that $\widehat{\mathrm{st}}_\mathcal{C}(f) = f$ for some $f \in \mathcal{F}_v$. It follows $f \in \mathcal{C}$ by the uniqueness of the asymptotic expansion $f = \varphi + d\varphi$ (part (ii) of Theorem 5).

(iii) follows from the definition of $\widehat{\mathrm{st}}_\mathcal{C}$ taking into account that $\mathbb{C} \cup \Phi \subset \mathcal{C}$. □

Example 5 (*The case* $\Phi = \Phi_5$). Let $\mathcal{C}$ be a vector subspace of $\mathcal{M}$ which is maximal in $\mathcal{C}_v$ and which contains Φ_5, where Φ_5 is the set defined in Lemma 5 (Section 3). *For the sake of simplicity we shall write simply* $\widehat{\mathrm{st}}$ *instead of the more precise* $\widehat{\mathrm{st}}_\mathcal{C}$. By the above theorem, we have $\widehat{\mathrm{st}}(\varphi + d\varphi) = \varphi$ for all $\varphi \in \mathcal{C}$ (hence, for all $\varphi \in \mathrm{Span}(\Phi_5)$) and all $d\varphi \in \mathcal{I}_v$. In particular, for every $d\varphi \in \mathcal{I}_v$ and every $c \in \mathbb{C}$ we have: $\widehat{\mathrm{st}}(c + d\varphi) = c$; $\widehat{\mathrm{st}}(c + 1/\ln x) = c + 1/\ln x$, $\widehat{\mathrm{st}}(\sin x + d\varphi) = \sin x$, $\widehat{\mathrm{st}}(\cos x + d\varphi) = \cos x$, $\widehat{\mathrm{st}}(\sin x \cos x + d\varphi) = \sin x \cos x$, $\widehat{\mathrm{st}}(\ln x + d\varphi) = \ln x$, $\widehat{\mathrm{st}}(\ln(\ln x) + d\varphi) = \ln(\ln x)$, $\widehat{\mathrm{st}}(\ln^n x + d\varphi) = \ln^n x$, $\widehat{\mathrm{st}}(e^{\pm i\pi x}\ln^n x + d\varphi) = e^{\pm i\pi x}\ln^n x$, and $\widehat{\mathrm{st}}((\sin x)e^{\pm i\pi x}\ln^n x + d\varphi) = (\sin x)e^{\pm i\pi x}\ln^n x$, $n = 0, 1, 2, \ldots$.

5. v-Asymptotic series

We consider a type of infinite series in $\mathcal{M}$ which we call a v-asymptotic series and illustrate this concept by examples.

Recall that a series $\sum_{n=0}^\infty f_n$ is called *asymptotic* as $x \to \infty$, if $f_{n+1} = o(f_n)$ for all n. Notice that if an asymptotic series $\sum_{n=0}^\infty f_n$ is in $\mathcal{M}$ (i.e., $f_n \in \mathcal{M}$ for all n), then we have $v(f_n) \quad v(f_{n+1})$ for all n. In contrast, we have the following similar concept:

Definition 6 (v-*Asymptotic series*). Let $F = \sum_{n=0}^\infty f_n$ be a series in $\mathcal{M}$. Then:

(i) F is called a v-*asymptotic series*, as $x \to \infty$, if it has the following two properties:

(a) The set $\{x^{r_n} f_n \mid n \ \geq 0\}$ is linearly v-independent over $\mathbb{C}$ (Section 3), where $r_n = v(f_n)$ and we let (by convention) $x^\infty \cdot 0 = 0$.

(b) The restriction of the sequence (r_n) on the set $\{n \mid f_n = 0\}$ is a strictly increasing (finite or infinite) sequence in $\mathbb{R}$.

Every v-asymptotic series F can be written uniquely in its *canonical form*:

$$F = \sum_{n=0}^{\infty} \frac{\varphi_n(x)}{x^{r_n}}, \tag{7}$$

where $\varphi_n(x) = x^{r_n} f_n(x)$ and we let (by convention) $\frac{0}{x^\infty} = 0$. We call the sets $\mathrm{Supp}(F) = \{n \mid f_n = 0\} = \{n \mid \varphi_n = 0\}$, $\mathrm{Exp}(F) = \{r_n \mid n \ \geq 0\}$ and $\mathrm{Coef}(F) = \{x^{r_n} f_n \mid n \ \geq 0\} = \{\varphi_n \mid n \ \geq 0\}$ the *support*, the *set of exponents* and the *set of coefficients* of F, respectively.

In what follows we shall often *assume that the v-asymptotic series are already presented in their canonical form* (7). Notice that for the v-asymptotic series we have $\mathrm{Coef}(F) \subset \mathcal{C}_v \cup \{0\}$ (Section 3).

Notation 1. Let K be a subset of $\mathcal{C}_v \cup \{0\}$.

(i) We denote by $K(1/x^\omega)$ the set of all v-asymptotic series in the form (7) with coefficients φ_n in K.

(ii) We denote by $K1/x$ the set of all v-asymptotic series in the form (7) with coefficients φ_n in K and such that $\lim_{n \to \infty} r_n = \infty$.

We should notice that $K(1/x^\omega)$ consists of all Hahn's series [6] with coefficients in K and with a set of exponents which is an increasing sequence is $\mathbb{R}$ (hence, the origin of ω in the notation). We observe as well that in the particular case $K = \mathcal{C}_v \cup \{0\}$, the set $K(1/x^\omega)$ coincides with the set of all v-asymptotic series in $\mathcal{M}$, while $K1/x$ coincides with the set of all v-asymptotic series (7) in $\mathcal{M}$ with $\lim_{n \to \infty} r_n = \infty$.

Here are several examples:

Example 6. Let c_n be a (arbitrary) sequence in $\mathbb{C}$ and $m \in \mathbb{Z}$. Then the series $\sum_{n=0}^{\infty} \frac{c_n}{x^{m+n}}$ is both asymptotic and v-asymptotic.

Example 7. The series: $\sum_{n=0}^{\infty} \frac{\ln^n x}{x^n}$ $\sum_{n=0}^{\infty} \frac{e^{inx}}{x^n}$, $\sum_{n=1}^{\infty} \sqrt[n]{x}$, $\sum_{n=1}^{\infty} \ln^n x$ and $\ln x + \ln(\ln x) + \ln(\ln(\ln x)) + \cdots$ are all asymptotic. All but the last two are also v-asymptotic. The last two series are not v-asymptotic, since $v(\ln^n x) = 0$ and $v(\ln x) = v(\ln(\ln x)) = v(\ln(\ln(\ln x))) = \cdots = 0$.

Example 8. In contrast to the above the Sine-series:

$$\frac{\pi}{2} + \cos x \left(-\frac{1}{x} + \frac{2!}{x^3} - \frac{4!}{x^5} + \cdots + \frac{(2n)!(-1)^{n+1}}{x^{2n+1}} + \cdots \right)$$

$$+ \sin x \left(-\frac{1}{x^2} + \frac{3!}{x^4} - \frac{5!}{x^6} + \cdots + \frac{(2n+1)!(-1)^{n+1}}{x^{2n+2}} + \cdots \right),$$

is v-asymptotic (see the set Φ_1 in Lemma 5, Section 3), but not asymptotic since

$$\frac{(\sin x)(2n+1)!(-1)^{n+1}}{x^{2n+2}} = o\left(\frac{(\cos x)(2n)!(-1)^{n+1}}{x^{2n+1}}\right).$$

Example 9. The series $\sum_{n=0}^{\infty} \frac{1+n+1/\sqrt[n]{x}}{x^n}$ is asymptotic, but not v-asymptotic since the set of the coefficients $\{1+n+\frac{1}{\sqrt[n]{x}} \mid n \geq 0\}$ is not linearly v-independent over $\mathbb{C}$.

Example 10. Finally, the series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ and $\sum_{n=0}^{\infty} \frac{1}{\sqrt[n]{x}}$ are neither asymptotic, nor v-asymptotic as $x \to \infty$ (the first is convergent in $\mathbb{R}$, while the second is divergent in $\mathbb{R}$, but the convergence or divergence in $\mathbb{R}$ is irrelevant to their asymptotic properties).

6. v-Asymptotic expansion of a function

In this section we discuss the concept of the asymptotic expansion of a function of moderate growth at infinity in a v-asymptotic series.

Definition 7. Let $f \in \mathcal{M}$ (Definition 1) and let $\sum_{n=0}^{\infty} \frac{\varphi_n(x)}{x^{r_n}}$ be a v-asymptotic series in $\mathcal{M}$ (Section 5). We say that the series $\sum_{n=0}^{\infty} \frac{\varphi_n(x)}{x^{r_n}}$ is a v-*asymptotic expansion* of $f(x)$ (or, that $f(x)$ is an *asymptotic sum* of $\sum_{n=0}^{\infty} \frac{\varphi_n(x)}{x^{r_n}}$), as $x \to \infty$, in symbol, $f(x) \rightsquigarrow \sum_{n=0}^{\infty} \frac{\varphi_n(x)}{x^{r_n}}$, if for every integer $n \geq 0$ for which $\varphi_n \neq 0$, we have

$$f(x) - \sum_{k=0}^{n} \frac{\varphi_k(x)}{x^{r_k}} = o(1/x^{r_n}) \quad \text{as } x \to \infty. \tag{8}$$

Remark 2. Notice that $r_n = \infty$ iff $\varphi_n = \varphi_{n+1} = \varphi_{n+2} = \cdots = 0$ iff $f(x) - \sum_{k=0}^{n} \frac{\varphi_k(x)}{x^{r_k}} = f(x) - \sum_{k=0}^{n-1} \frac{\varphi_k(x)}{x^{r_k}} \in \mathcal{N}$ since, by convention, $\frac{0}{x^\infty} = 0$.

Lemma 7. *Let $f \in \mathcal{M}$ and $\sum_{n=0}^{\infty} \frac{\varphi_n(x)}{x^{r_n}}$ be v-asymptotic series in $\mathcal{M}$. Then the following are equivalent*:

(i) $f(x) \rightsquigarrow \sum_{n=0}^{\infty} \frac{\varphi_n(x)}{x^{r_n}}$.

(ii) $x^{r_n}[f(x) - \sum_{k=0}^{n} \frac{\varphi_k(x)}{x^{r_k}}] \in \mathcal{I}_v$ *for all* $n \geq 0$, $\varphi_n \neq 0$.

(iii) $x^{r_n}[f(x) - \sum_{k=0}^{n-1} \frac{\varphi_k(x)}{x^{r_k}}] \in \mathcal{C}_v$ *for all* $n \geq 0$, $\varphi_n \neq 0$.

Proof. (i) $\Rightarrow$ (ii): Suppose, first, that $\varphi_{n+1} \neq 0$. We have $f(x) - \sum_{k=0}^{n} \frac{\varphi_k(x)}{x^{r_k}} = \frac{\varphi_{n+1}(x)}{x^{r_{n+1}}} + o(1/x^{r_{n+1}})$. It follows

$$v\left(x^{r_n}\left[f(x) - \sum_{k=0}^{n} \frac{\varphi_k(x)}{x^{r_k}}\right]\right) = v\left(\frac{\varphi_{n+1}(x) + o(1)}{x^{r_{n+1}-r_n}}\right)$$

$$= v\big(\varphi_{n+1}(x) + o(1)\big) + r_{n+1} - r_n > 0,$$

as required, by Theorem 1, since $v(\varphi_{n+1}(x)+o(1)) \quad 0$ and $r_{n+1}-r_n > 0$ (by assumption). If $\varphi_{n+1} = 0$, then $f(x) - \sum_{k=0}^{n} \frac{\varphi_k(x)}{x^{r_k}} \in \mathcal{N}$ implying $x^{r_n}[f(x) - \sum_{k=0}^{n} \frac{\varphi_k(x)}{x^{r_k}}] \in \mathcal{N}$ and (ii) follows since $\mathcal{N} \subset \mathcal{I}_v$.

(i) $\Leftarrow$ (ii) follows directly by part (i) of Lemma 3.

(ii) $\Rightarrow$ (iii): We denote $x^{r_n}[f(x) - \sum_{k=0}^{n} \frac{\varphi_k(x)}{x^{r_k}}] = d\varphi_n(x)$ and observe that $v(x^{r_n}[f(x) - \sum_{k=0}^{n-1} \frac{\varphi_k(x)}{x^{r_k}}]) = v(\varphi_n(x) + d\varphi_n(x)) = 0$, as required, since $d\varphi_n \in \mathcal{I}_v$, by assumption.

(ii) $\Leftarrow$ (iii): Suppose, first, that $\varphi_{n+1} = 0$. We denote $x^{r_{n+1}}[f(x) - \sum_{k=0}^{n} \frac{\varphi_k(x)}{x^{r_k}}] = \psi_n(x)$ and observe that $\psi_n \in \mathcal{C}_v$, by assumption. It follows

$$x^{r_n} \quad f(x) - \sum_{k=0}^{n} \frac{\varphi_k(x)}{x^{r_k}} \quad = \frac{\psi_n(x)}{x^{r_{n+1}-r_n}} \in \mathcal{I}_v,$$

since $r_{n+1} - r_n > 0$. If $\varphi_{n+1} = 0$, then $f(x) - \sum_{k=0}^{n} \frac{\varphi_k(x)}{x^{r_k}} \in \mathcal{N}$ implying $x^{r_n}[f(x) - \sum_{k=0}^{n} \frac{\varphi_k(x)}{x^{r_k}}] \in \mathcal{N}$ and (ii) follows. $\square$

Theorem 7 (Convergent series in $\mathcal{M}$). *Let $f \in \mathcal{M}$ and $\sum_{n=0}^{\infty} \frac{\varphi_n(x)}{x^{r_n}}$ be a v-asymptotic series in $\mathcal{M}$ such that $\lim_{n\to\infty} r_n = \infty$. Then the following are equivalent:*

(i) $f(x) \quad \sum_{n=0}^{\infty} \frac{\varphi_n(x)}{x^{r_n}}$.

(ii) $f(x) = \sum_{n=0}^{\infty} \frac{\varphi_n(x)}{x^{r_n}}$ *in the sense that* $f(x) = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{\varphi_k(x)}{x^{r_k}}$ *in* $(\mathcal{M}, d_v)$ *(Definition 2, Section 2).*

Proof. (i) $\Rightarrow$ (ii): We have $f(x) - \sum_{k=0}^{n} \frac{\varphi_k(x)}{x^{r_k}} = o(1/x^{r_n})$ for all n which implies (ii), by Corollary 1 (Section 2), since $\lim_{n\to\infty} r_n = \infty$, by assumption.

(i) $\Leftarrow$ (ii): Suppose, first, that $f \in \mathcal{N}$ (in particular, $f = 0$). It follows $\lim_{n\to\infty} v \times (\sum_{k=0}^{n} \frac{\varphi_k(x)}{x^{r_k}}) = \infty$, by part (i) of Theorem 4. It follows $r_0 = r_1 = r_2 = \cdots = \infty$ and $\varphi_0(x) = \varphi_1(x) = \varphi_2(x) = \cdots = 0$, thus, (i) hods (in the form $f(x) \quad 0 + 0 + \cdots$). Suppose that $f \notin \mathcal{N}$. It follows $v(f) = v(\sum_{k=0}^{n} \frac{\varphi_k(x)}{x^{r_k}})$ for all sufficiently large n, by part (ii) of Theorem 4 (Section 2). Thus $v(f) = r_0$ and more generally, $v(f(x) - \sum_{k=0}^{n} \frac{\varphi_k(x)}{x^{r_k}}) = r_{n+1}$ for all n. Similarly, we observe that $\widehat{\mathrm{st}}_{\mathcal{C}}(x^{r_0} f(x)) = \varphi_0(x)$ and

$$\widehat{\mathrm{st}}_{\mathcal{C}} \quad x^{r_{n+1}} \quad f(x) - \sum_{k=0}^{n} \frac{\varphi_k(x)}{x^{r_k}} \quad = \varphi_{n+1}(x)$$

for all n, where $\mathcal{C}$ is a maximal vector space containing the set $\{\varphi_n \mid n \in \mathbb{N}\}$. It follows that for each n we have $f(x) - \sum_{k=0}^{n} \frac{\varphi_k(x)}{x^{r_k}} \quad \sum_{k=n+1}^{\infty} \frac{\varphi_k(x)}{x^{r_k}}$ which implies (i), as required. $\square$

7. Existence and uniqueness result

The purpose of this section is to show that every function with moderate growth at infinity has a v-asymptotic expansion in $\mathcal{M}$.

Theorem 8 (Existence and uniqueness). *Every function with moderate growth at infinity has a v-asymptotic expansion in $\mathcal{M}$. More precisely, let $\mathcal{C}$ be a vector subspace of $\mathcal{M}$*

which is maximal in C_v (Definition 4, Section 4). Then every function f in $\mathcal{M}$ has a unique v-asymptotic expansion (as $x \to \infty$):

$$f(x) \sim \sum_{n=0}^{\infty} \frac{\varphi_n(x)}{x^{r_n}}, \tag{9}$$

with coefficients φ_n in $\mathcal{C}$ (that is, within the set $\mathcal{C}(1/x^{\omega})$, Notation 1, Section 5).

Proof. (A) *Existence*: Our first goal is to show the existence of a v-asymptotic series in (9). Let $\widehat{\mathrm{st}}_{\mathcal{C}} : \mathcal{F}_v \to \mathcal{F}_v$ be the pseudostandard part mapping determined by $\mathcal{C}$ (Definition 5, Section 4). The function $f(x)$ determines the sequences (φ_n) and (r_n) by the following *recursive formulas*:

$$r_0 = v(f), \qquad\qquad \varphi_0(x) = \widehat{\mathrm{st}}_{\mathcal{C}}\left(x^{r_0} f(x) \right),$$

$$r_1 = v\left(f(x) - \frac{\varphi_0(x)}{x^{r_0}} \right), \qquad \varphi_1(x) = \widehat{\mathrm{st}}_{\mathcal{C}}\left(x^{r_1}\left[f(x) - \frac{\varphi_0(x)}{x^{r_0}} \right] \right),$$

$$\ldots, \quad \text{etc.,} \tag{10}$$

$$r_n = v\left(f(x) - \sum_{k=0}^{n-1} \frac{\varphi_k(x)}{x^{r_k}} \right), \qquad \varphi_n(x) = \widehat{\mathrm{st}}_{\mathcal{C}}\left(x^{r_n}\left[f(x) - \sum_{k=0}^{n-1} \frac{\varphi_k(x)}{x^{r_k}} \right] \right),$$

$$r_{n+1} = v\left(f(x) - \sum_{k=0}^{n} \frac{\varphi_k(x)}{x^{r_k}} \right), \quad \varphi_{n+1}(x) = \widehat{\mathrm{st}}_{\mathcal{C}}\left(x^{r_{n+1}}\left[f(x) - \sum_{k=0}^{n} \frac{\varphi_k(x)}{x^{r_k}} \right] \right), \tag{11}$$

$n = 0, 1, 2, \ldots$, where, by convention, we let $x^{\infty} \cdot \varphi = 0$ whenever $\varphi \in \mathcal{N}$. Next, we observe that $\varphi_n \in \mathcal{C}$ (since $\mathrm{ran}(\widehat{\mathrm{st}}_{\mathcal{C}}) = \mathcal{C}$) and that (r_n) is a non-decreasing sequence in $\mathbb{R} \cup \{\infty\}$. Our next goal is to show that the series on the RHS of (9) is a v-asymptotic series. Suppose, first, that $r_n = \infty$ for some n. It follows $f(x) - \sum_{k=0}^{n-1} \frac{\varphi_k(x)}{x^{r_k}} \in \mathcal{N}$ thus $x^{\infty} \cdot [f(x) - \sum_{k=0}^{n-1} \frac{\varphi_k(x)}{x^{r_k}}] = 0$ (by convention), implying that $\varphi_n = 0$ (since $\mathcal{N} \subset \mathcal{I}_v$). Consequently, it follows that $r_k = \infty$ and $\varphi_k = 0$ for all $k \geq n$, as required (which means that in this case the series is a finite sum). Suppose now that $r_{n+1} = \infty$ for some n. It follows that $r_n = \infty$ as well, by what was just proved above. We have to show that $r_n < r_{n+1}$. We observe that the second formula in (11) implies

$$f(x) = \sum_{k=0}^{n} \frac{\varphi_k(x)}{x^{r_k}} + \frac{d\varphi_n(x)}{x^{r_n}} \tag{12}$$

for some $d\varphi_n(x) \in \mathcal{I}_v$. It follows

$$r_{n+1} = v\left(f(x) - \sum_{k=0}^{n} \frac{\varphi_k(x)}{x^{r_k}} \right) = v\left(\frac{d\varphi_n(x)}{x^{r_n}} \right) = v(d\varphi_n) + v(1/x^{r_n}) > r_n,$$

by Theorem 1, since $v(d\varphi_n) > 0$ and $v(1/x^{r_n}) = r_n$. Next, we observe that the set $\Phi = \{\varphi_n \mid n \geq 0\}$ is linearly v-independent over $\mathbb{C}$, by Lemma 4 (Section 3). Thus the series on the RHS of (9) is a v-asymptotic series. To show that the asymptotic expansion (9) holds, suppose that $\varphi_n = 0$ (hence, $r_n = \infty$). The formula (12) implies $x^{r_n}[f(x) - \sum_{k=0}^{n} \frac{\varphi_k(x)}{x^{r_k}}] =$

$d\varphi_n(x)$ which implies (9), by Lemma 7, since $d\varphi_n(x) \in \mathcal{I}_v$. This proves the existence of a v-asymptotic expansion of f within the set of series $\mathcal{C}(1/x^\omega)$.

(B) *Uniqueness*: Suppose that

$$f(x) \sim \sum_{n=0}^{\infty} \frac{\psi_n(x)}{x^{s_n}}, \quad \text{as } x \to \infty, \tag{13}$$

for another v-asymptotic series $\sum_{n=0}^{\infty} \frac{\psi_n(x)}{x^{s_n}}$, where $\psi_n \in \mathcal{C}$. Suppose as well that $\psi_n = 0$ (hence, $s_n = \infty$) for some n. It follows that $x^{s_n}[f(x) - \sum_{k=0}^{n-1} \frac{\psi_k(x)}{x^{s_k}}] \in \mathcal{C}_v$, by part (iii) of Lemma 7. Thus

$$s_n = v\left(f(x) - \sum_{k=0}^{n-1} \frac{\psi_k(x)}{x^{s_k}}\right), \tag{14}$$

by part (ii) of Theorem 1. Similarly, we have $x^{s_n}[f(x) - \sum_{k=0}^{n} \frac{\psi_k(x)}{x^{s_k}}] \in \mathcal{I}_v$, by part (ii) of Lemma 7. It follows

$$\psi_n(x) = \widehat{\text{st}}_\mathcal{C}\left(x^{s_n}\left[f(x) - \sum_{k=0}^{n-1} \frac{\psi_k(x)}{x^{s_k}}\right]\right), \tag{15}$$

since $\widehat{\text{st}}_\mathcal{C}(\psi_n) = \psi_n$. Since $\psi_k = 0$ for all $k \geq n$ we can start from $k = 0$. Formula (14) gives $s_0 = v(f(x)) = r_0$ and formula (15) gives $\psi_0(x) = \widehat{\text{st}}_\mathcal{C}(x^{s_0} f(x)) = \widehat{\text{st}}_\mathcal{C}(x^{r_0} f(x)) = \varphi_0(x)$. Similarly, $k = 1$ gives

$$s_1 = v\left(f(x) - \frac{\psi_0(x)}{x^{s_0}}\right) = v\left(f(x) - \frac{\varphi_0(x)}{x^{r_0}}\right) = r_1,$$

$$\psi_1(x) = \widehat{\text{st}}_\mathcal{C}\left(x^{s_1}\left[f(x) - \frac{\psi_0(x)}{x^{s_0}}\right]\right) = \widehat{\text{st}}_\mathcal{C}\left(x^{r_1}\left[f(x) - \frac{\varphi_0(x)}{x^{r_0}}\right]\right) = \varphi_1(x).$$

Continuing we obtain $s_k = r_k$ and $\psi_k(x) = \varphi_k(x)$ for all $k \leq n$. Suppose, now, that $\psi_n = 0$ for some n which implies $s_{n+1} = s_{n+2} = \cdots = \infty$ and $\psi_n = \psi_{n+1} = \psi_{n+2} = \cdots = 0$. Without loss of generality we can assume that $\psi_{n-1} = 0$, hence, $s_{n-1} = \infty$. So, we have $s_k = r_k$ and $\psi_k(x) = \varphi_k(x)$ for all $k \leq n - 1$. It follows

$$f(x) - \sum_{k=0}^{n-1} \frac{\varphi_k(x)}{x^{r_k}} = f(x) - \sum_{k=0}^{n-1} \frac{\psi_k(x)}{x^{s_k}} \in \mathcal{N},$$

which, on its turn, implies $r_n = r_{n+1} = r_{n+2} = \cdots = \infty$ and $\varphi_n = \varphi_{n+1} = \varphi_{n+2} = \cdots = 0$. Summarizing, we have $s_k = r_k$ and $\psi_k = \varphi_k$ for all $k \geq 0$ which proves the uniqueness of the v-asymptotic expansion within $\mathcal{C}(1/x^\omega)$ (Notation 1, Section 5). To complete the proof we observe that there exist maximal linear spaces $\mathcal{C}$ in $\mathcal{M}$, by Lemma 6, Section 4 (in particular, there exists $\mathcal{C}$ such that $\Phi_5 \subset \mathcal{C}$, where Φ_5 is defined in Lemma 5). $\square$

The next corollary is written for those readers who do not feel comfortable with the concept of "maximal linear space."

Corollary 2 (On uniqueness again). *Let* (9) *holds for some moderate function* $f \in \mathcal{M}$ *and some v-asymptotic series* $\sum_{n=0}^{\infty} \frac{\varphi_n(x)}{x^{r_n}}$. *Let* $\mathcal{K}$ *be a linear subspace of* $\mathcal{M}$ *such that* $\Phi \subseteq \mathcal{K} \subset \mathcal{C}_v \cup \{0\}$, *where* $\Phi = \{\varphi_n \mid n = 0, 1, 2, \ldots\}$. *Then* (9) *is the only v-asymptotic expansion of* f *within the set of v-asymptotic series* $\mathcal{K}(1/x^{\omega})$ (*Notation 1, Section 5*).

Proof. Let $\mathcal{C}$ be a vector subspace of $\mathcal{M}$ which is maximal in $\mathcal{C}_v$ and which contains $\mathcal{K}$. Notice that the existence of $\mathcal{C}$ is guaranteed by Lemma 6 (Section 4). The asymptotic expansion (9) is unique in $\mathcal{C}(1/x^{\omega})$, by the above theorem. The latter implies the uniqueness of (9) in $\mathcal{K}(1/x^{\omega})$. $\square$

In the next example we explain how to use Theorem 8 in asymptotic analysis.

Example 11 (*An integral with a large parameter*). Let $f \in C^{\infty}[-\pi, \pi]$ and define:

$$I(x) = \int_{-\pi}^{\pi} e^{ixy - y^2 \ln x} f(y) \, dy. \tag{16}$$

Integrals of the form (16) arise in quantum statistical mechanics and random matrix theory (M.L. Mehta [10]), where they are related to the probability that no eigenvalue of a large Hermitian matrix lies in the interval $(-x, x)$ (see also E. Basor [4] and C.P. Hughes, J.P. Keating, N. O'Connell [7]). Since $I(x)$ is a function with moderate growth at infinity (Section 2), it follows that $I(x)$ has a v-asymptotic expansion in $\mathcal{M}$, by Theorem 8. Notice that Theorem 8 is used to guarantee the existence of a v-asymptotic expansion in advance; before plunging into possibly hard and time consuming calculations. Next, we note that the recursive formulas (10) (used in the proof of Theorem 8) are rarely efficient for explicit asymptotic calculations (just as the recursive formulas (1) in the Introduction are rarely efficient for explicit calculations). The explicit calculations should be done by any of the methods known to asymptotic analysis. In the case of $I(x)$ the "integrating by parts" produces the following v-asymptotic expansion:

$$I(x) \quad \sum_{n=0}^{\infty} \frac{\varphi_n(x)}{x^{\pi^2 + 1 + n}}, \tag{17}$$

where the sequence (φ_n) is determined by the recursive formulas:

$$\varphi_0(x) = \frac{1}{i}\left[f(\pi)e^{i\pi x} - f(-\pi)e^{-i\pi x} \right],$$

$$\varphi_1(x) = f'(\pi)e^{i\pi x} - f'(-\pi)e^{-i\pi x} - (2\pi \ln x)\left[f(\pi)e^{i\pi x} + f(-\pi)e^{-i\pi x} \right],$$

and, more generally,

$$\varphi_n(x) = \frac{(-1)^n}{i^{n+1}} \sum_{k=0}^{[\frac{n}{2}]} \binom{n}{2k} (2\pi \ln x)^{2k} \left[f^{(n-2k)}(\pi)e^{i\pi x} - f^{(n-2k)}(-\pi)e^{-i\pi x} \right]$$

$$- \frac{(-1)^n}{i^{n+1}} \sum_{k=0}^{[\frac{n-1}{2}]} \binom{n}{2k+1} (2\pi \ln x)^{2k+1}$$

$$\times \quad f^{(n-2k-1)}(\pi)e^{i\pi x} + f^{(n-2k-1)}(-\pi)e^{-i\pi x} \quad .$$

Now is the time to establish our uniqueness result. First, we observe that $\mathrm{Span}(\{\varphi_n \mid n = 0, 1, 2, \ldots\}) = \mathrm{Span}(\Phi_4)$, where Φ_4 is the set of functions defined in Lemma 5 (Section 3). We conclude that the series in (17) is a v-asymptotic series since Φ_4 is a linearly v-independent set and the sequence of exponents $r_n = \pi^2 + 1 + n$ is strictly increasing. Let $\mathcal{K}$ be a vector subspace of $\mathcal{M}$ such that $\Phi_4 \subset \mathcal{K} \subset \mathcal{C}_v \cup \{0\}$ and let $\mathcal{K}(1/x^\omega)$ denote the set of all v-asymptotic series with coefficients in $\mathcal{K}$ (Notation 1, Section 5). We conclude that the v-asymptotic expansion (17) of $I(x)$ is unique within the set of series $\mathcal{K}(1/x^\omega)$, by Corollary 2. In particular, let $\mathcal{C}$ be a vector subspace of $\mathcal{M}$ which is maximal in $\mathcal{C}_v$ and which contains Φ_4. Then the v-asymptotic expansion (17) of $I(x)$ is unique within set of series $\mathcal{C}(1/x^\omega)$. Finally, it is curious to observe that $I(x) = \sum_{n=0}^{\infty} \frac{\varphi_n(x)}{x^{\pi^2+1+n}}$, by Theorem 7, since $\lim_{n\to\infty}(\pi^2 + 1 + n) = \infty$. That means that the series (17) is convergent and $I(x) = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{\varphi_k(x)}{x^{\pi^2+1+k}}$ in the metric space $(\mathcal{M}, d_v)$ (Definition 2, Section 2).

8. Colombeau's generalized numbers

As an application to the non-linear theory of generalized functions (J.F. Colombeau [2, 3]), we show that every Colombeau's generalized number has a v-asymptotic expansion. Since the Colombeau generalized functions can be characterized as pointwise functions in the ring of Colombeau's generalized numbers (M. Kunzinger and M. Oberguggenberger [8]), it follows that every Colombeau's generalized function (including every Schwartz distribution) has (pointwisely) a v-asymptotic expansion. This result has an importance for finding weak generalized solutions of some partial differential equations with singularities (M. Oberguggenberger [11]).

Definition 8 (*Colombeau's ring*).

(i) We call the factor-ring $\bar{\mathbb{C}} = \mathcal{M}/\mathcal{N}$ Colombeau's ring of *complex generalized numbers* (Section 2). We denote by $q : \mathcal{M} \to \bar{\mathbb{C}}$ the corresponding quotient mapping. The Colombeau's ring of *real generalized numbers* is

$$\bar{\mathbb{R}} = \quad q(f) \mid f \in \mathcal{M}, \, f(x) \in \mathbb{R} \text{ for all sufficiently large } x \quad .$$

The real generalized number $\lambda = q(\mathrm{id})$ is called the *scale* of $\bar{\mathbb{C}}$, where $\mathrm{id}(x) = x$ for all $x \in \mathbb{R}$.

(ii) We define a pseudovaluation $v : \bar{\mathbb{C}} \to \mathbb{R} \cup \{\infty\}$ and a metric $d_v : \bar{\mathbb{C}} \times \bar{\mathbb{C}} \to \mathbb{R}$ inherited from $\mathcal{M}$, i.e., by $v(q(f)) = v(v)$ and $d_v(q(f), v(g)) = d_v(f, g)$, respectively. We denote by $(\bar{\mathbb{C}}, d_v)$ the corresponding metric space.

(iii) We denote by $\mathcal{I}_v(\bar{\mathbb{C}}), \mathcal{C}_v(\bar{\mathbb{C}})$ and $\mathcal{F}_v(\bar{\mathbb{C}})$ the subsets of $\bar{\mathbb{C}}$ with positive, zero and non-negative pseudovaluation, respectively.

(iv) Let C be a vector subspace of $\bar{\mathbb{C}}$. We say that C is *maximal in* $\mathcal{C}_v(\bar{\mathbb{C}})$ if $\mathbb{C} \subset C \subset \mathcal{C}_v(\bar{\mathbb{C}}) \cup \{0\}$ and there is no a vector subspace $\mathbb{K}$ of $\bar{\mathbb{C}}$ such that $C \subsetneq \mathbb{K} \subset \mathcal{C}_v(\bar{\mathbb{C}}) \cup \{0\}$.

(v) We say that a set $\mathbb{F} \subset \bar{\mathbb{C}}$ is *linearly v-independent over* $\mathbb{C}$ if $v(a) = 0$ for all $a \in \mathrm{Span}(\mathbb{F}), a = 0$.

Remark 3. The ring $\bar{\mathbb{C}}$ defined above is, actually, isomorphic to the original Colombeau ring of generalized numbers (under the mapping $x \to 1/\varepsilon$), introduced in (J.F. Colombeau [3, Section 2.1]). We shall ignore the difference between these two isomorphic rings. Notice that both $\bar{\mathbb{R}}$ and $\bar{\mathbb{C}}$ are (like $\mathcal{M}$) partially ordered rings with zero-divisors and we have $\bar{\mathbb{R}} \subset \bar{\mathbb{C}}$.

Due to the factorization we have: $v(a) = \infty$ iff $a = 0$ in $\bar{\mathbb{C}}$. Consequently, $\bar{\mathbb{C}}$ is a metric space (in contrast to $\mathcal{M}$ is a pseudometric space). The next two results follow immediately from the corresponding properties of $\mathcal{M}$ (Section 2):

Theorem 9 (Metric). *Colombeau's ring* $(\bar{\mathbb{C}}, d_v)$ *is a ultrametric space, i.e., for every* $a, b, c \in \bar{\mathbb{C}}$ *we have*:

(i) $d_v(a, b) = 0$ *iff* $a = b$.
(ii) $d_v(a, b) = d_v(b, a)$.
(iii) $d_v(a, b) \quad \max\{d_v(a, c), d_v(c, g)\} \quad d_v(a, c) + d_v(c, b)$.

Theorem 10 (Convergence in $\bar{\mathbb{C}}$). *Let* (c_n) *be a sequence in* $\bar{\mathbb{C}}$ *and* $c \in \bar{\mathbb{C}}$. *Then*:

(i) $\lim_{n \to \infty} c_n = 0$ *in* $(\bar{\mathbb{C}}, d_v)$ *iff* $\lim_{n \to \infty} v(c_n) = \infty$ *in* $\mathbb{R} \cup \{\infty\}$.
(ii) $\lim_{n \to \infty} c_n = c = 0$ *in* $(\bar{\mathbb{C}}, d_v)$ *implies that* $v(c_n) = v(c)$ *for all sufficiently large* n.

The concept of a v-asymptotic series in $\mathcal{M}$ can be adapted to $\bar{\mathbb{C}}$. In what follows λ is the scale of $\bar{\mathbb{C}}$ (Definition 8).

Definition 9 (v-*Asymptotic series in* $\bar{\mathbb{C}}$).

(i) A series $\sum_{n=0}^{\infty} \frac{a_n}{\lambda^{r_n}}$ in $\bar{\mathbb{C}}$ is called v-*asymptotic* if:
   (a) The set of the coefficients $\{a_n \mid n \in \mathbb{N}\}$ is linearly v-independent over $\mathbb{C}$.
   (b) The restriction of the sequence (r_n) on the set $\{n \mid a_n = 0\}$ is a strictly increasing (finite or infinite) sequence in $\mathbb{R}$.
   If $\mathbb{K}$ is a subset of $\mathcal{C}_v(\bar{\mathbb{C}}) \cup \{0\}$, then $\mathbb{K}(1/\lambda^{\omega})$ denotes the set of all v-asymptotic series $\sum_{n=0}^{\infty} \frac{a_n}{\lambda^{r_n}}$ in $\bar{\mathbb{C}}$ with coefficients a_n in $\mathbb{K}$.
(ii) We say that $\sum_{n=0}^{\infty} \frac{a_n}{\lambda^{r_n}}$ is a v-*asymptotic expansion* of a generalized number $a \in \bar{\mathbb{C}}$ (or, that a is an *asymptotic sum* of $\sum_{n=0}^{\infty} \frac{a_n}{\lambda^{r_n}}$), in symbol $a \quad \sum_{n=0}^{\infty} \frac{a_n}{\lambda^{r_n}}$, if $\sum_{n=0}^{\infty} \frac{a_n}{\lambda^{r_n}}$ is a v-asymptotic series in $\bar{\mathbb{C}}$ and for every $n \quad 0, a_n = 0$, we have $v(a - \sum_{k=0}^{n} \frac{a_k}{\lambda^{r_n}}) > r_n$.

Theorem 11 (Convergent series in $\bar{\mathbb{C}}$). *Let* $f \in \mathcal{M}$ *and let* $\sum_{n=0}^{\infty} \frac{a_n}{x^{r_n}}$ *be a* v-*asymptotic series in* $\bar{\mathbb{C}}$ *such that* $\lim_{n \to \infty} r_n = \infty$. *Then the following are equivalent*:

(i) $a \quad \sum_{n=0}^{\infty} \frac{a_n}{\lambda^{r_n}}$ *in* $\bar{\mathbb{C}}$.
(ii) $a = \sum_{n=0}^{\infty} \frac{a_n}{\lambda^{r_n}}$ *in the sense that* $a = \lim_{n \to \infty} \sum_{k=0}^{n} \frac{a_k}{x^{r_k}}$ *in* $(\bar{\mathbb{C}}, d_v)$.

Proof. The result follows immediately from part (i) of Theorem 10.   □

Theorem 12 (*v-Asymptotic expansion in* $\bar{\mathbb{C}}$). *Every Colombeau's generalized number has a v-asymptotic expansion in* $\bar{\mathbb{C}}$. *More precisely, let C be a vector subspace of* $\bar{\mathbb{C}}$ *(over* $\mathbb{C}$*) which is maximal in* $\mathcal{C}_v(\bar{\mathbb{C}})$. *Then every* $a \in \bar{\mathbb{C}}$ *has a unique v-asymptotic expansion*:

$$a \quad \sum_{n=0}^{\infty} \frac{a_n}{\lambda^{r_n}}, \tag{18}$$

with coefficients a_n *in C.*

Proof. We have $a = q(f)$ for some $f \in \mathcal{M}$. We observe that $\mathcal{C} = q^{-1}[C]$ is a vector subspace of $\mathcal{M}$ which is maximal in $\mathcal{C}_v$ (Section 4). It follows that f has a unique v-asymptotic expansion $f(x) \quad \sum_{n=0}^{\infty} \frac{\varphi_n(x)}{x^{r_n}}$ in $\mathcal{C}(1/x^\omega)$, by Theorem 8 (Section 7). Thus (18) follows for $a_n = q(\varphi_n)$. $\quad\square$

If Ω is an open subset of $\mathbb{R}^n$, then we denote

$$\mu_v(\Omega) = \quad \omega + d\omega \quad \omega \in \Omega, \ d\omega \in \bar{\mathbb{R}}^n, \ v \ \|d\omega\| \ > 0 \ .$$

In what follows $\mathcal{G}(\Omega)$ denotes the algebra of Colombeau's generalized functions on Ω. We recall that $\mathcal{G}(\Omega)$ contains a canonical copy of the space of Schwartz's distributions, in symbol, $\mathcal{D}'(\Omega) \subset \mathcal{G}(\Omega)$ (J.F. Colombeau [2]).

Corollary 3. *Let* $f \in \mathcal{G}(\Omega)$ *be a Colombeau's generalized function (in particular, a Schwartz distribution) and let* $f : \mu_v(\Omega) \to \bar{\mathbb{C}}$ *be its graph. Then there exist* $f_n : \mu_v(\Omega) \to \bar{\mathbb{C}}$, $f_n = 0$, *and* $r_n : \mu_v(\Omega) \to \mathbb{R}$, *such that for every* $\xi \in \mu_v(\Omega)$:

(a) $\sum_{n=0}^{\infty} \frac{f_n(\xi)}{\lambda^{r_n(\xi)}}$ *is a v-asymptotic series in* $\bar{\mathbb{C}}$ *(which, among other things, implies that* $v(f_n(\xi)) = 0$);
(b) $f(\xi) \quad \sum_{n=0}^{\infty} \frac{f_n(\xi)}{\lambda^{r_n(\xi)}}$ *in* $\bar{\mathbb{C}}$.

Proof. The result follows immediately from Theorem 12 and the identification between the Colombeau generalized functions with theirs graphs (M. Kunzinger and M. Oberguggenberger [8]). $\quad\square$

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