Nonstandard Analysis in Topology: Nonstandard and Standard Compactifications

by

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Dedicated to Horst Herrlich on the occasion of his sixtieth anniversary.

**Abstract** Let $(X,T)$ be a topological space, and ${}^*X$ a non–standard extension of $X$. There is a natural "standard" topology ${}^ST$ on ${}^*X$ generated by ${}^*G$, where $G \in T$. The topological space $\left({}^*X, {}^S T\right)$ will be used to study, in a systematic way, compactifications of $(X,T)$.

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## 1. Introduction

Let $(X,T)$ be a topological space and $\left({}^*X, {}^S T\right)$ a non–standard enlargement, where ${}^ST$ is generated by ${}^*G$, where $G \in T$. The space $\left({}^*X, {}^S T\right)$ has many interesting topological features, but, as may be expected, is very poorly separated as far as points are concerned.

We show that a wide class of compactifications of $(X,T)$ may be obtained by rendering $\left({}^*X, {}^S T\right)$ "separated", thus illustrating the usefulness and effectiveness, and broad applicability of the non–standard compactification $\left({}^*X, {}^S T\right)$.

Conceptually, it is not very common to regard a non–standard model ${}^*X$ as a topological space, although this has been done: A. Robinson [16] and [17], W.A.J. Luxemburg [23], H. Gonshor [5], L. Haddad [6]. The $*$–open sets have always been part of non–standard techniques , but their role is more often at the level of the application of transfer principles than as basic open sets of a topological space $\left({}^*X, {}^S T\right)$.

In this paper we shall study the relationship between topological properties of $(X,T)$ and their counterparts in $\left({}^*X, {}^S T\right)$. This has led to a unification and, perhaps, simplification of the exposition concerning compactifications ([22], [14], [15], [23], [5], [6], [8]).

Arising from considerations related to the Theory of Frames, as well as from an interest in compactifications that are relevant to theoretical computer science, there has been an increasing interest in $T_0$–compactifications, described as "well compacted" ([20]) and "stably compact" ([21]). We shall show that these compactifications can also be obtained from non–standard compactifications in a canonical way. The relevant reference for Frames (and also stable compactness) is P. Johnstone's book Stone Spaces [10].

A methodological note is appropriate at this stage, concerning the role of the axiom of choice – the axiom is essential in topology, to yield the Stone-Čech compactification of Tychonoff spaces [12]; it is also essential in the non–standard approach by providing non–standard enlargements with adequate saturation ([8], observation before Łŏs Theorem 4.5, Chapter II).

For the topological notions and constructs, we refer to J.L. Kelley [12], and L. Gillman and M. Jerison [4]. For the relation between standard and non–standard methods in topology, we refer to L. Haddad [6]; for basic concepts, methods and further developments we refer to T. Lindstrøm [13]. For the notions concerning category theory, in particular reflections, we refer to [1], as well as [10].

## 2. Non–standard compactifications

For any topological space $(X,T)$ there is an enlargement ${}^*X$ which is saturated in the sense that if $\{F \subseteq X | \, i \in I\}$ is a family with the finite intersection property, then there is $\alpha$ in ${}^*X$ which is in every ${}^*F_i,\ i \in I$ (see, for example, [8], Chapter II, §8). Thus, $\left({}^*X,{}^S T\right)$ is a compact topological space.

The sets ${}^*G$, $G \in T$, constitute a base for ${}^S T$. For reference, we note that, for $A, B \subseteq X$, we have:

(i) ${}^*\phi = \phi,\ {}^*X = X$, (ii) ${}^*(A \cup B) ={}^* A \cup^* B$, (iii) ${}^*(A \cap B) = {}^*A \cap^* B$; (iv) ${}^*X -^* A =^* (X - A)$.

The monad of $x$ is $\mu(x) = \bigcap \{ {}^*G | \, G \in T, x \in G\}$. More generally, for $\alpha \in^* X$,

$\mu(\alpha) = \bigcap \{ {}^*G | \, G \in T, \alpha \in^* G\}$, similarly we may define $\mu(A)$, where $A \subseteq^* X$.

We recall A. Robinson's celebrated criterion for compactness: $(X,T)$ is compact if and only if ${}^*X = \bigcup \{ \mu(x) | \, x \in X\}$.

In general $\left({}^*X,{}^S T\right)$ is not a Hausdorff Topological space – the standard open sets are inadequate to separate the rich assortment of points in ${}^*X$. We shall provide and example, so as give an idea of the topological structure of $\left({}^*X,{}^S T\right)$.

**2.1 Example** Consider $\mathbb{N} = \{1, 2, 3, \ldots\}$ with the topology of upper semi–continuity, basic open sets being of the form $G_n = \{1, 2, \ldots, n\}$, $n \in \mathbb{N}$. Let $\alpha \in^* \mathbb{N}$, then $\alpha \in^* G_n$ if and only if $G_n \in \alpha$, hence $\alpha$ must be a principal ultrafilter since $G_n$ is finite. Thus, every $\alpha$ for which $\{H \subseteq \mathbb{N} | \alpha \in^* H\}$ is a free ultrafilter, necessarily has only one ${}^S T$–neighbourhood, ${}^*\mathbb{N}$ itself. Thus, no "non standard" $\alpha$'s can be separated by ${}^S T$–open sets.

The following observations are important because they give the functoriality of the ${}^*$–extension.

**2.2 Proposition** Let $(X, T)$ and $(X', T')$ be topological spaces and $\left({}^*X, {}^S T\right)$, $\left({}^*X', {}^S T'\right)$ non–standard compactifications. Then the function $f : (X, T) \to (X', T')$ is continuous if and only if ${}^*f : \left({}^*X, {}^S T\right) \to \left({}^*X', {}^S T'\right)$ is continuous.

**Proof** For any $G' \in T'$, we have that ${}^*G'$ is a basic ${}^*T'$ open set. Now $({}^*f)^{\leftarrow} [{}^*G'] = ({}^*f)^{\leftarrow} [G']$ and the result follows.

**2.3 Proposition** Let $f : (X, T) \to (X', J')$, $g : (X', T') \to (X'', T'')$, then ${}^*(g \circ f) = {}^*g \circ^* f$. *Also* ${}^*(\mathbb{I}_X) = \mathbb{I}_{{}^*X}$.

We have indicated why $\left({}^*X, {}^S T\right)$ is a compact topological space. In fact, more is true. Firstly some topological definitions and their non–standard description.

**2.4 Definition** A topological space $(X, T)$, not necessarily Hausdorff, is called **locally compact** if for every $x \in X$ and open $V \in T$ which is a neighbourhood of $x$, there is a compact neighbourhood of $x$, $W$, not necessarily open, such that $W \subseteq V$.

A simple non–standard description of local compactness follows from A. Robinson's compactness theorem mentioned above.

**Non–standard Local compactness:** For every neighbourhood $V$ of a given $x \in X$, there is a neighbourhood of $x$, $W$, such that

$$ {}^*W \subseteq \bigcup \{ \mu(x) | x \in W \} \subseteq^* V. $$

The following notion has been called **supersoberness** ([3], Chapter VII, 1.10 Definition). When applied to a compact space $(X, T)$, because it implies a precise form of compactness which specifies, not only that ultrafilters have adherences, but that these should be of a special form, we have taken the liberty of naming it **supercompactness.** In [3], examples illustrating the usefulness of supersoberness may be found in Chapter VII.

**2.5 Definition** Let $(X, T)$ be a topological space, not necessarily $T_0$. $(X, T)$ is **supercompact** if for every ultrafilter $\mathcal{U}$ on $X$ there is an essentially unique point $x$ such that the set of cluster points of $\mathcal{U}$, adh $\mathcal{U}$, is the closure of the point $x$:

$$\mathrm{adh}\mathcal{U} = \underset{T}{cl} X.$$

That $x$ is essentially unique means: if $x'$ is any other point with that same property, then $x$ and $x'$ have precisely the same $T$–neighbourhoods, i.e. $\mu(x) = \mu(x')$.

**2.6 Examples**

1. Every compact Hausdorff space is locally compact and supercompact.
2. Every supercompact $T_1$ space is compact Hausdorff.
3. Consider $\mathbb{N}_\infty = \mathbb{N} \cup \{\infty\}$, where basic $u$–open sets in $\mathbb{N}_\infty$ one of the form $G_n = \{0, 1, \ldots, n\}$ (thus, the only $u$–neighbourhood of the point at infinity is $\mathbb{N}_\infty$). Then $(\mathbb{N}_\infty, u)$ is a $T_0$ locally compact supercompact space. Indeed, it is the $T_0$–locally compact supercompact compactification $\beta_2(\mathbb{N}, u)$ of $(\mathbb{N}, u)$, see Proposition 4.4.

**2.7 Theorem** $\left({}^*X, {}^S T\right)$ is a locally compact, supercompact enlargement of $(X, T)$.

**Proof** We first establish local compactness, by showing that, for $G \in T$, we have that ${}^*G$ is ${}^S T$–compact. Consider a filter $\mathcal{F}$ of closed sets ${}^*F_i,\ i \in I$ such that ${}^*F_i \cap {}^*G \neq \phi$, all $i$ in $I$. Then $\mathcal{F} = \{F_i \cap G|\, i \in I\}$ is a family of subsets of $X$ which is closed under finite intersections. Let $\mathcal{U}$ be an ultrafilter on $X$ which contains $\mathcal{F}$. By saturation, there exists $p \in^* X$ such that $p \in \quad \{{}^*(F_i \cap G)|\, i \in I\}$, i.e. $p \in (\quad \{{}^*F_i|\, i \in I\}) \cap^* G$. Thus, $p$ is a cluster point of $\{{}^*F_i \cap^* G|\, i \in I\}$ and belongs to ${}^*G$, as required. To prove supercompactness, let $\mathcal{V}$ be an ultrafilter on ${}^*X$. Note that $adh\mathcal{V} = \quad \left\{\bar{H}\middle|\, H \subseteq^* X,\ H \in \mathcal{V}\right\}$, so that $adh\mathcal{V} = \quad \{{}^*F|\, F \subseteq X,\ F$ is closed, and ${}^*F \in \mathcal{V}\}$. It is readily verified that $p = \{A \subseteq X|\ {}^*A \in \mathcal{V}\}$ is an ultrafilter on $X$. Hence $p \in^* X$. We show that $adh\mathcal{V}$ is the ${}^S T$–closure of $p$, thus exhibiting quite explicitly the special minimal point in the adherence. Firstly, $p$ is in the adherence of $\mathcal{V}$, since, given $G \in T$ with $p \in^* G$, we have ${}^*G \in \mathcal{V}$, by definition of $p$. Hence ${}^*G$ intersects every closed set in $\mathcal{V}$ showing that $p \in adh\mathcal{V}$. Let $\alpha \in adh\mathcal{V}$. If $\alpha$ is not in the ${}^S T$–closure of $p$, then there is $G \in T$ such that $a \in^* G$ and $p \notin^* G$. But then $p \in^* (X - G)$, hence ${}^*(X - G) \in \mathcal{V}$, so that ${}^*G \cap^* (X - G) = \phi$, contradicting the fact that $\alpha \in adh\mathcal{V}$.

**In Summary** – Every topological space $(X,T)$ may be embedded into a supercompact locally compact enlargement $\left({}^*X,{}^S T\right)$ with embedding map $\eta_X : (X,T) \to \left({}^*X,{}^S T\right)$. The assignment is functorial and $\eta$ provides, then, a natural transformation from the identity to $*$.

It is natural, and important, to determine the behaviour of $*$ on spaces that already compact. We shall examine two cases: the classical case, even in the non–standard sense, when $(X,T)$ is compact Hausdorff and the case where $(X,T)$ is a locally compact, supercompact $T_0$ space.

Before we do so, we shall examine further some separation properties of $\left({}^*X,{}^S T\right)$.

## 3. Non–standard compactifications and separation properties.

We shall describe conditions under which $\left({}^*X,{}^S T\right)$ is normal, or regular, or a $T_0$–space, in order to illustrate the nature of ${}^S T$ on ${}^*X$.

**3.1 Proposition** $\left({}^*X,{}^S T\right)$ is normal if and only if $(X,T)$ is normal.

**Proof** Assume that $\left({}^*X,{}^S T\right)$ is normal and let $F_1, F_2$ be disjoint closed sets of $(X,T)$. Then ${}^*F_1$ and ${}^*F_2$ are disjoint closed sets of $\left({}^*X,{}^S T\right)$ so, by assumption, they can be included in disjoint ${}^S T$–open sets with disjoint closures. Restricting the open sets to $X$ provides two $T$–open sets $G_1, G_2$ with disjoint $T$–closures containing $F_1$ and $F_2$, respectively. Conversely, assume $(X,T)$ is normal. Let $A, B$ be disjoint closed sets in $\left({}^*X,{}^S T\right)$. By assumption, $A = \quad \{{}^*F | \, F \text{ closed } {}^*F \supseteq A\}$, $B = \quad \{{}^*H | \, H \text{ closed, } {}^*H \supseteq B\}$. Now ${}^*F \cap {}^*H = \phi$ for some ${}^*F \supseteq A$, ${}^*H \supseteq B$, where $F, H$ are closed in $X$, otherwise there is $\alpha$ in ${}^*X$ such that $\alpha \in^* (F \cap H)$, for all ${}^*F \supseteq A$, ${}^*H \supseteq B$. This would mean that $\alpha \in A \cap B$, which is impossible. Thus we have $F \cap H = \phi$ so there are two disjoint open sets $G_1, G_2$ such that $F \subseteq G_1$, $H \subseteq G_2$. Then $A \subseteq^* F \subseteq^* G_1$, $B \subseteq^* H \subseteq^* G_2$ and ${}^*G_1 \cap^* G_2 = \phi$, as required.

**3.2 Proposition** $\left({}^*X,{}^S T\right)$ is regular if and only if every open set in $(X,T)$ is closed.

**Proof** Suppose $(X,T)$ has the stated property and that $\alpha \in^* G$ for some $G \in T$. Since $G$ is open and closed, so is ${}^*G$, so $\left({}^*X,{}^S T\right)$ is regular. Conversely, assume $\left({}^*X,{}^S T\right)$ is regular and let $G \in T$. If $G$ were not closed, then there is $x \in \underset{T}{cl} G - G$. For each open neighbourhood of $x$, $H$, we have $H \cap G = \phi$, so the family $\{\, H \cap G | \, H \in \mathcal{N}_x\}$ has the finite intersection property. By saturation, there is $p \in^* X$ such that $p \in^* (H \cap G)$, for all $H \in \mathcal{N}_x$. Thus $p \in \quad ({}^*H \cap^* G)$, hence $p \in \quad \{{}^*H |\, H \in \mathcal{N}_x\} \quad {}^*G$. By regularity, there is $U \in T$ such that $p \in^* U \subseteq \underset{{}^S T}{cl}\, {}^*U \subseteq^* G$. But then $X - \underset{T}{cl} U = \left({}^*X - \underset{{}^S T}{cl}\, {}^*U\right) \cap X \in \mathcal{N}_x$, since $x \in \underset{T}{cl} G - G$. Letting $H = X - \underset{T}{cl} U$, we have $U \cap H = \phi$, hence ${}^*H \cap^* H = \phi$, which contradicts the fact that $p \in^* U$, and $p \in^* H$ (since $p \in^* W$ for all $W \in \mathcal{N}_x$).

**3.3 Corollary 1** $\left({}^*X, {}^S T\right)$ is a completely regular space (no $T_0$ separation assumed) if and only if every open set in $T$ is closed.

**Proof** If $\left({}^*X, {}^S T\right)$ is completely regular, then it is regular, hence $(X, T)$ has the desired property. Conversely, assume every open set is closed. Let $G \in T$ and $\alpha \in^* X$ be such that $\alpha \in^* G$. Because ${}^*G$ is open and closed, there is a continuous real valued function $f : \left({}^*X, {}^S T\right) \rightarrow (\mathbb{R}, m)$, where $m$ denotes the usual topology, such that $G = f^{\leftarrow}[0]$, $X - G = f^{\leftarrow}[1]$, the proof is complete.

**3.4 Corollary 2** Let $D$ denote the discrete topology on $\mathbb{N}$. $\left({}^*\mathbb{N}, {}^S D\right)$ is not a $T_0$–space.

**Proof** If $\left({}^*\mathbb{N}, {}^S D\right)$ were $T_0$, then it would be $T_2$ since $\left({}^*\mathbb{N}, {}^S D\right)$ is regular, by above. In which case, since every bounded continuous real valued function on $(\mathbb{N}, D)$ admits an extension to $\left({}^*\mathbb{N}, {}^S D\right)$ and $(\mathbb{N}, D)$ is dense in $\left({}^*\mathbb{N}, {}^S D\right)$ (trivially, $(X, T)$ is always dense in $\left({}^*X, {}^S T\right)$, since ${}^*G \cap X = G$), it would follow that $\left({}^*\mathbb{N}, {}^S D\right)$ is the Stone–Čech compactification $\beta(\mathbb{N}, D)$ of $(\mathbb{N}, D)$. It is well known that this is impossible (see A. Robinson [16] page 582; or K.D. Stroyan and W.A.J. Luxemburg [23], 8.1.6, 8.1.7, 9.1); for a topologist, perhaps the easiest way is see this is to note that $\beta(\mathbb{N} \times \mathbb{N}) = \beta\mathbb{N} \times \beta\mathbb{N}$ (see, for example, [4]), whereas ${}^*(\mathbb{N} \times \mathbb{N}) =^* \mathbb{N} \times {}^*\mathbb{N}$.

**3.5 Corollary 3** There is no topology $T$ on $\mathbb{N}$ for which $\left({}^*\mathbb{N}, {}^S T\right)$ is a $T_0$–space.

**Proof** Suppose the contrary, that $\left({}^*\mathbb{N}, {}^S T\right)$ is a $T_0$–space for some topology $T$ on $\mathbb{N}$. The identity map i: $(\mathbb{N}, D) \rightarrow (\mathbb{N}, T)$ is continuous, hence so is its non–standard extension ${}^*i : \left({}^*\mathbb{N}, {}^S D\right) \rightarrow \left({}^*\mathbb{N}, {}^S T\right)$. Since ${}^*i$ is injective and $\left({}^*\mathbb{N}, {}^S T\right)$ is $T_0$, it follows that $\left({}^*\mathbb{N}, {}^S D\right)$ is $T_0$, which we know is impossible.

**3.6 Proposition** $\left({}^*X, {}^S T\right)$ is a $T_0$ space if and only if $X$ is finite.

**Proof** If $X$ were infinite and $\left({}^*X, {}^S T\right)$ a $T_0$–space , then there would be a countable subset $\mathbb{N}$ of $X$ with its relative topology, also denoted by $T$, giving $\left({}^*\mathbb{N}, {}^S T\right) \subseteq \left({}^*X, {}^S T\right)$. Thus $\left({}^*\mathbb{N}, {}^S T\right)$ is a $T_0$ space, which is impossible.

**3.7 Corollary** $\left({}^{*}X,{}^{S}T\right)$ is a Hausdorff space if and only if $(X,T)$ is a finite discrete space.

## 4. Non–standard compactifications of compact spaces and standard compactifications

We shall first discuss briefly the compact Hausdorff case.

**4.1 Proposition** Let $(X,T)$ be a compact Hausdorff space and $\left({}^{*}X,{}^{S}T\right)$ a non–standard extension of $(X,T)$. There is a continuous retraction $r_X : \left({}^{*}X,{}^{S}T\right) \to (X,T)$, with $r_X$ being the identity when restricted to $X$.

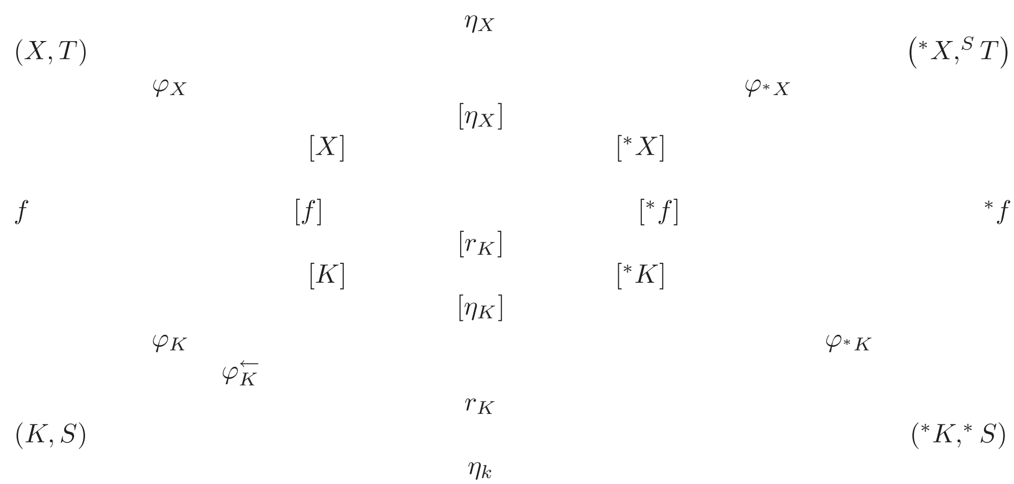
**Proof** By Robinson's characterization, ${}^{*}X = \bigcup \{\mu(x) | \, x \in X\}$. Thus, given $\alpha \in^{*} X$, there is $x$ such that $\alpha \in \mu(x)$. Since $(X,T)$ is a Hausdorff space, if $x = x'$, we have $\mu(x) \cap \mu(x') = \phi$, hence $\alpha \in \mu(x)$ **for a unique** $x$. Define $\underset{X}{r}(\alpha)$ to be that $x$. Clearly, $\underset{X}{r}(x) = x$ for all $x$ in $X$. If $G \in T$ and $x \in G$, then $\alpha \in \mu(x)$ gives $\alpha \in^{*} H$ for all $H \in T$ that contain $x$. In particular $\alpha \in^{*} H$, where $H \in T$ is such that $x \in H \subseteq \bar{H} \subseteq G$, ($H$ exists by regularity of $(X,T)$). If $\beta \in^{*} H$, and $\underset{X}{r}(\beta) = x'$, then $x' \in \bar{H}$, otherwise $X - \bar{H}$ is an open set containing $x'$, hence, by definition of $\underset{X}{r}$, ${}^{*}\left(X - \bar{H}\right) =^{*} X -^{*} \left(\bar{H}\right)$ contains $\beta$, which is impossible since $\beta \in^{*} H$. Thus $\underset{X}{r}$ is a continuous retraction, as stated.

We now show that the Stone–Čech compactification of a Tychonoff space $(X,T)$ is simply $\left({}^{*}\mathbf{X},{}^{\mathbf{S}}\mathbf{T}\right)$ **made Hausdorff.** More precisely, let $[(X,T)]$ denote the $T_2$–reflection of $(X,T)$ ([1]) then $\beta X = \left[\left({}^{*}X,{}^{S}T\right)\right]$.

**4.2 Theorem** Let $(X,T)$ be a Tychonoff space. Then $\beta(X,T) = \left[\left({}^{*}X,{}^{S}T\right)\right]$.

**Proof** $(X,T)$ is dense in $\left({}^{*}X,{}^{S}T\right)$ and the reflection map $\varphi_X : \left({}^{*}X,{}^{S}T\right) \to \left[\left({}^{*}X,{}^{S}T\right)\right]$ is continuous, so $\varphi_X(X)$ is dense in the compact Hausdorff space $\left[\left({}^{*}X,{}^{S}T\right)\right]$. Consider $f : (X,T) \to (K,S)$ where $(K,S)$ is a compact Hausdorff space. We show that there is a map $F$, necessarily unique, such that $F : \left[\left({}^{*}X,{}^{S}T\right)\right] \to (K,S)$ and $F \circ \varphi_X \circ \eta_X = f$, where $\eta_X : (X,T) \to \left({}^{*}X,{}^{S}T\right)$ is the embedding map of $(X,T)$ into the non–standard compactification $\left({}^{*}X,{}^{S}T\right)$. The result then follows.

Naturality, in the categorical sense, of the constructs is best expressed as a commutative diagram, given below, from which one can read off the required $F$. For convenience, we write $[X]$ in place of $[(X,T)]$, and $[f]$ for the reflected map.




Observe that $\varphi_K$ has an inverse $\varphi_K^{\leftarrow}$, since $(K,S)$ is already Hausdorff. The required map is: $F = \varphi_K^{\leftarrow} \circ [r_K] \circ [{}^*f]$, since

$$\begin{aligned} F \circ \varphi_X \circ \eta_X &= \varphi_K^{\leftarrow} \circ [r_K] \circ ([{}^*f] \cdot \varphi_X) \cdot \eta_X \\ &= \varphi_K^{\leftarrow} \circ ([r_K] \cdot \varphi_{{}^*K}) \circ ({}^*f \circ \eta_X) \\ &= \varphi_K^{\leftarrow} \circ \varphi_K \circ r_K \circ \eta_K \circ f \\ &= \mathbb{I}_K \circ \mathbb{I}_K \circ f = f. \end{aligned}$$

We now consider $T_0$ locally compact supercompact spaces.

**4.3 Proposition** Let $(X,T)$ be a $T_0$ locally compact supercompact space and $\left({}^*X,{}^S T\right)$ a non–standard compactification. There is a continuous retraction $r_X : \left({}^*X,{}^S T\right) \to (X,T)$, which is the identity when restricted to $X$.

**Proof** Let $\alpha \in^* X$. $\alpha$ determines an ultrafilter $\mathcal{U}_\alpha$ on $X$. As usual, $\mathcal{U}_\alpha = \{A \subseteq X \,|\, \alpha \in^* A\}$. By supercompactness, there is $x$ such that $adh\mathcal{U}_\alpha = \underset{T}{cl}x$. Define $r_X(\alpha)$ to be that $x$, which is, in fact, unique, as $(X,T)$ is a $T_0$ topological space. To prove continuity of $r_X : \left({}^*X,{}^S T\right) \to (X,T)$, consider $\alpha \in^* X$ and $x = r_X(\alpha)$. Given

$V \in T$ with $x \in V$, local compactness ensures that there is $W \in T$ and $K$ compact such that $x \in W \subseteq K \subseteq V$. Now $\alpha \in^* W$, otherwise $\alpha \in^* (X - W)$, so that $X - W \in U_\alpha$, contradicting $x \in adh\mathcal{U}_\alpha$. Consider now $\beta \in^* W$. We have, then, that $\beta \in \mu(x'')$, for some $x''$ in $K$, since ${}^*W \subseteq \cup \{ \mu(x'') | \, x'' \in K\} \subseteq^* V$. Hence $x'' \in adh\mathcal{U}_\beta$, so that $x'' \in cl_T x'$, where $x' = r_X(\beta)$. Now $x'' \in V$, since $\mu(x'') \subseteq^* V$, hence $x' \in G$, since $x'' \in cl_T x'$, proving that $r_X : (X,T) \to \left({}^*X, {}^S T\right)$ is continuous.

As in the compact Hausdorff case, an entirely analogous proof will show that the $T_0$ locally compact supercompact reflection of $(X,T)$, $\beta_2(X,T)$, is "$\left({}^*\mathbf{X}, {}^S\mathbf{T}\right)$ **made** $\mathbf{T}_0$". More precisely, let $[\ ]_0$ denote the $T_0$–reflector, we have:

**4.4 Proposition** Let $(X,T)$ be a $T_0$–space. Then $\beta_2(X,T) = \left[\left({}^*X, {}^S T\right)\right]_0$.

There is a note of warning that should be mentioned here – The notion of "reflection" that is relevant is the notion of weak reflection of H. Herrlich, and does not require the uniqueness of the map that solves the extension problem: A subcategory $\mathbf{R}$ of a category $\mathbf{A}$ is reflective, if for every $A$ in $\mathbf{A}$, there is $R = R(A)$ in $\mathbf{R}$ and $\eta_A : A \to R(A)$ such that if $A' \in \mathbf{A}$ and $f : A \to A'$, then there is $F$, not necessarily unique, such that $F : R(A) \to A'$ and

$$F \circ \eta_A = f.$$

## 5. All Compactifications

Firstly, a brief reference to compact Hausdorff compactifications of a given Tychonoff space $(X,T)$. These may all be obtained by a uniform method, as described in ([8], page 158). The method given above does not refer to continuous real valued functions on $(X,T)$. However the $T_2$ reflection of $\left({}^*X, {}^S T\right)$ can be seen to be induced by the ${}^*f's$ where $f$ ranges through the bounded continuous real valued function, thus establishing a relationship between the two approaches.

Obtaining all $T_0$–locally compact supercompactifications can also be achieved by considering an analogue of the $Q$–compactifications referred to above ([8], page 158) – one considers families of continuous real valued functions into the Sierpiński dyad $\mathbb{D} = \{0,1\}$, with Topology $u = \{\phi, \{0\}, \{0,1\}\}$.

Lest the reader become too optimistic, it should be mentioned that it is not possible to obtain **all** $T_0$ compactifications of a $T_0$ space as a type of quotient of $\left({}^*X, {}^S T\right)$ – if it were possible, then the category of compact $T_0$ spaces would be (weakly) reflective in the category of $T_0$ spaces, which it is **not,** as shown by Miroslav Hušek ([9]; see also [7] for further developments) in response to a problem posed by Horst Herrlich.

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