

AN EMBEDDING OF SCHWARTZ DISTRIBUTIONS IN THE ALGEBRA OF ASYMPTOTIC FUNCTIONS

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ABSTRACT. We present a solution of the problem of multiplication of Schwartz distributions by embedding the space of distributions into a differential algebra of generalized functions, called in the paper "asymptotic function," similar to but different from J. F. Colombeau's algebras of new generalized functions.

KEY WORDS AND PHRASES: Schwartz distributions, nonlinear theory of generalized functions, asymptotic expansion, nonstandard analysis, nonstandard asymptotic analysis

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1. INTRODUCTION

The main purpose of this paper is to prove the existence of an embedding $\Sigma_{D,\Omega}$ of the space of Schwartz distributions $\mathcal{D}'(\Omega)$ into the algebra of asymptotic functions ${}^{\rho}E(\Omega)$ which preserves all linear operations in $\mathcal{D}'(\Omega)$. Thus, we offer a solution of the problem of multiplication of Schwartz distributions since the multiplication within $\mathcal{D}'(\Omega)$ is impossible (L. Schwartz [1]).

The algebra ${}^{\rho}E(\Omega)$ is defined in the paper as a factor space of nonstandard smooth functions. The field of the scalars ${}^{\rho}\mathbb{C}$ of the algebra ${}^{\rho}E(\Omega)$, coincides with the complex counterpart of A. Robinson's asymptotic numbers—known also as "Robinson's field with valuation" (see A. Robinson [2]) and A. H. Lightstone and A. Robinson [3]). The embedding $\Sigma_{D,\Omega}$ is constructed in the form $\Sigma_{D,\Omega} = Q_{\Omega} \circ D * \Pi \circ \bullet$ where (in backward order): \bullet is the extension mapping (in the sense of nonstandard analysis), \circ is the Schwartz multiplication in $\mathcal{D}'(\Omega)$ (more precisely, its nonstandard extension in ${}^{\rho}\mathcal{D}'(\Omega)$), $*$ is the convolution operator (more precisely, its nonstandard extension), \circ denotes "composition," Q_{Ω} is the quotient mapping (in the definition of the algebra of asymptotic functions) and D and Π_{Ω} are fixed nonstandard internal functions with special properties whose existence is proved in this paper.

Our interest in the algebra ${}^{\rho}E(\Omega)$ and the embedding $\mathcal{D}'(\Omega) \subset {}^{\rho}E(\Omega)$, is due to their role in the problem of multiplication of Schwartz distributions, the nonlinear theory of generalized functions and its applications to partial differential equations (M. Oberguggenberger [4]), (T. Todorov [5] and [6]). In particular, there is a strong similarity between the algebra of asymptotic functions ${}^{\rho}E(\Omega)$ and its generalized scalars ${}^{\rho}\mathbb{C}$, discussed in this paper, and the algebra of generalized functions $\mathcal{G}(\Omega)$ and their

generalized scalars $\overline{\mathbb{C}}$, introduced by J. F. Colombeau in the framework of standard analysis (J. F. Colombeau [7], pp. 63, 138 and J. F. Colombeau [8], §8.3, pp. 161-166). We should mention that the involvement of nonstandard analysis has resulted in some improvements of the corresponding standard counterparts; one of them is that ${}^{\rho}\mathbb{C}$ is an algebraically closed field while its standard counterpart $\overline{\mathbb{C}}$ in J. F. Colombeau's theory is a ring with zero divisors.

This paper is a generalization of some results in [9] and [10] (by the authors of this paper, respectively) where only the embedding of the tempered distributions $S'(\mathbb{R}^d)$ in ${}^{\rho}E(\mathbb{R}^d)$ has been established. The embedding of all distributions $\mathcal{D}'(\Omega)$, discussed in this paper, presents an essentially different situation. We should mention that the algebra ${}^{\rho}E(\mathbb{R}^d)$ was recently studied by R. F. Hoskins and J. Sousa Pinto [11].

Here Ω denotes an open set of \mathbb{R}^d (d is a natural number), $E(\Omega) = C^\infty(\Omega)$ and $\mathcal{D}(\Omega) = C_0^\infty(\Omega)$ denote the usual classes of C^∞ -functions on Ω and C^∞ -functions with compact support in Ω and $\mathcal{D}'(\Omega)$, and $E'(\Omega)$ denote the classes of Schwartz distributions on Ω and Schwartz distributions with compact support in Ω , respectively. As usual, \mathbb{N} , \mathbb{R} , \mathbb{R}_+ and \mathbb{C} will be the systems of the natural, real, positive real and complex numbers, respectively, and we use also the notation $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$. For the partial derivatives we write ∂^α , $\alpha \in \mathbb{N}_0^d$. If $\alpha = (\alpha_1, \dots, \alpha_d)$ for some $\alpha \in \mathbb{N}_0^d$, then we write $|\alpha| = \alpha_1 + \dots + \alpha_d$ and if $x = (x_1, \dots, x_d)$ for some $x \in \mathbb{R}^d$, then we write $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \dots x_d^{\alpha_d}$. For a general reference to distribution theory we refer to H. Bremermann [12] and V. Vladimirov [13].

Our framework is a nonstandard model of the complex numbers \mathbb{C} , with degree of saturation larger than $\text{card}(\mathbb{N})$. We denote by ${}^*\mathbb{R}$, ${}^*\mathbb{R}_+$, ${}^*\mathbb{C}$, ${}^*E(\Omega)$ and ${}^*\mathcal{D}(\Omega)$ the nonstandard extensions of \mathbb{R} , \mathbb{R}_+ , \mathbb{C} , $E(\Omega)$ and $\mathcal{D}(\Omega)$, respectively. If X is a set of complex numbers or a set of (standard) functions, then *X will be its nonstandard extension and if $f : X \rightarrow Y$ is a (standard) mapping, then ${}^*f : {}^*X \rightarrow {}^*Y$ will be its nonstandard extension. For integration in ${}^*\mathbb{R}^d$ we use the $*$ -Lebesgue integral. We shall often use the same notation, $\|x\|$, for the Euclidean norm in \mathbb{R}^d and its nonstandard extension in ${}^*\mathbb{R}^d$. For a short introduction to nonstandard analysis we refer to the Appendix in T. Todorov [6]. For a more detailed exposition we recommend T. Lindstrom [14], where the reader will find many references to the subject.

2. TEST FUNCTIONS AND THEIR MOMENTS

In this section we study some properties of the test functions in $\mathcal{D}(\mathbb{R}^d)$ (in a standard setting) which we shall use subsequently.

Following (J.F. Colombeau [7], p. 55), for any $k \in \mathbb{N}$ we define the set of test functions:

$$A_k = \left\{ \varphi \in \mathcal{D}(\mathbb{R}^d) : \varphi \text{ is real-valued, } \varphi(x) = 0 \text{ for } \|x\| \geq 1; \int_{\mathbb{R}^d} \varphi(x) dx = 1 \text{ and } \int_{\mathbb{R}^d} x^\alpha \varphi(x) dx = 0 \text{ for } \alpha \in \mathbb{N}_0^d, 1 \leq |\alpha| \leq k \right\}. \tag{2.1}$$

Obviously, $A_1 \supset A_2 \supset A_3 \supset \dots$. Also, we have $A_k \neq \emptyset$ for all $k \in \mathbb{N}$ (J.F. Colombeau [7], Lemma (3.3.1), p. 55).

In addition to the above we have the following result:

LEMMA 2.2. For any $k \in \mathbb{N}$

$$\inf_{\varphi \in A_k} \left(\int_{\mathbb{R}^d} |\varphi(x)| dx \right) = 1. \tag{2.2}$$

More precisely, for any positive real δ there exists φ in A_k such that

$$1 \leq \int_{\mathbb{R}^d} |\varphi(x)| dx \leq 1 + \delta.$$

In addition, φ can be chosen symmetric.

PROOF. We consider the one dimensional case $d = 1$ first. Start with some fixed positive (real valued) ψ in $\mathcal{D}(\mathbb{R})$ such that $\psi(x) = 0$ for $|x| \geq 1$ and $\int_{\mathbb{R}} \psi(x) dx = 1$ (ψ can be also chosen symmetric if needed). We shall look for φ in the form:

$$\varphi(x) = \sum_{j=0}^k c_j \psi\left(\frac{x}{\epsilon^j}\right)$$

$x \in \mathbb{R}, \epsilon \in \mathbb{R}_+$. We have to find c_j for which $\varphi \in A_k$. Observing that

$$\int_{\mathbb{R}} x^i \psi\left(\frac{x}{\epsilon^j}\right) dx = \epsilon^{(i+1)j} \int_{\mathbb{R}} y^i \psi(y) dy$$

for $i = 0, 1, \dots, k$, we derive the system for linear equations for c_j :

$$\begin{cases} \sum_{j=0}^k c_j \epsilon^j = 1, \\ \left(\int_{\mathbb{R}} y^i \psi(y) dy\right) \sum_{j=0}^k c_j \epsilon^{(i+1)j} = 0, \quad i = 1, \dots, k. \end{cases}$$

The system is certainly satisfied, if

$$\begin{cases} \sum_{j=0}^k c_j \epsilon^j = 1, \\ \sum_{j=0}^k c_j \epsilon^{(i+1)j} = 0, \quad i = 1, \dots, k, \end{cases}$$

which can be written in the matrix form $V_{k+1}(\epsilon)C = B$, where $V_{k+1}(\epsilon)$ is Vandermonde $(k+1) \times (k+1)$ matrix, C is the column of the unknowns c_j and B is the column whose top entry is 1 and all others are 0. For the determinant we have $\det V_{k+1}(\epsilon) \neq 0$ for $\epsilon \neq 1$, therefore, the system has a unique solution $(c_1, c_1, c_2, \dots, c_k)$. Our next goal is to show that this solution is of the form:

$$c_j = \pm \frac{\epsilon^{\alpha_j} (1 + \epsilon P_j(\epsilon))}{\epsilon^{\beta} (1 + \epsilon P(\epsilon))} \tag{2.3}$$

where P_j and P are polynomials and

$$\alpha_j = \sum_{q=1}^{k-1} q(k+1-q) + \sum_{m=j}^{k-1} (k+1-m) \tag{2.4}$$

for $0 \leq j \leq k$, and

$$\beta = k + \sum_{q=1}^{k-1} q(k+1-q).$$

The coefficients c_0, c_1, \dots, c_k will be found by Cramer's rule. The formula for Vandermonde determinants gives

$$\begin{aligned} \prod_{m=1}^k \prod_{q=m+1}^{k+1} (\epsilon^q - \epsilon^m) &= \prod_{m=1}^k \left(\prod_{q=m+1}^{k+1} \epsilon^m (\epsilon^{q-m} - 1) \right) \\ &= \epsilon^{\beta} \prod_{m=1}^k \prod_{q=m+1}^{k+1} (\epsilon^{q-m} - 1) = \pm \epsilon^{\beta} (1 + \epsilon P(\epsilon)) \end{aligned}$$

for some polynomial P , where

$$\beta = \sum_{m=1}^k m(k+1-m) = k + \sum_{m=1}^{k-1} m(k+1-m).$$

To calculate the numerator in (2.3), we have to replace the j th column of the matrix by the column B (whose top entry is 1 and all others are 0) and calculate the resulting determinant D_j . Consider first the case $1 \leq j \leq k - 1$. By developing with respect to the j th column, we get

$$D_j = \pm \det \begin{pmatrix} 1, & \epsilon^2, & \dots & \epsilon^{2(j-1)}, & \epsilon^{2(j+1)}, & \dots & \epsilon^{2k} \\ 1, & \epsilon^3, & \dots & \epsilon^{3(j-1)}, & \epsilon^{3(j+1)}, & \dots & \epsilon^{3k} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 1, & \epsilon^{k+1}, & \dots & \epsilon^{(k+1)(j-1)}, & \epsilon^{(k+1)(j+1)}, & \dots & \epsilon^{(k+1)k} \end{pmatrix}.$$

We factor out $\epsilon^2, \epsilon^4, \dots, \epsilon^{2(j-1)}, \epsilon^{2(j+1)}, \dots, \epsilon^{2k}$ and obtain:

$$D_j = \pm \epsilon^{1 \cdot 2} \epsilon^{2 \cdot 2} \dots \epsilon^{(j-1) \cdot 2} \epsilon^{(j+1) \cdot 2} \dots \epsilon^{2k} \times \det \begin{pmatrix} 1, & 1, & 1, & \dots & 1, & 1, & \dots & 1 \\ 1, & \epsilon, & \epsilon^2, & \dots & \epsilon^{(j-1)}, & \epsilon^{(j+1)}, & \dots & \epsilon^k \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 1, & \epsilon^{k-1}, & \epsilon^{2(k-1)}, & \dots & \epsilon^{(j-1)(k-1)}, & \epsilon^{(j+1)(k-1)}, & \dots & \epsilon^{k(k-1)} \end{pmatrix}.$$

The latter is a Vandermonde determinant again, and we have

$$D_j = \pm \epsilon^{1 \cdot 2 + 2 \cdot 2 + \dots + (j-1)2 + (j+1)2 + \dots + k \cdot 2} \times (\epsilon - 1)(\epsilon^2 - 1)(\epsilon^3 - 1) \dots (\epsilon^{j-1} - 1)(\epsilon^{j+1} - 1) \dots (\epsilon^k - 1) \times (\epsilon^2 - \epsilon)(\epsilon^3 - \epsilon) \dots (\epsilon^{j-1} - \epsilon)(\epsilon^{j+1} - \epsilon) \dots (\epsilon^k - \epsilon) \times \dots \times (\epsilon^{j-1} - \epsilon^{j-2})(\epsilon^{j+1} - \epsilon^{j-2}) \dots (\epsilon^k - \epsilon^{j-2}) \times (\epsilon^{j+1} - \epsilon^{j-1}) \dots (\epsilon^k - \epsilon^{j-1}) \times \dots \times (\epsilon^k - \epsilon^{k-1}).$$

Hence, factoring out $\epsilon^{(i-1)(k-i)}$ in the i th row above, we get $D_j = \pm \epsilon^{\alpha_j} (1 + \epsilon P_j(\epsilon))$ for some polynomials $P_j(\epsilon)$ and

$$\begin{aligned} \alpha_j &= 1 \cdot 2 + 2 \cdot 2 + \dots + (j - 1) \cdot 2 + (j + 1) \cdot 2 + \dots + k \cdot 2 \\ &\quad + 1 \cdot (k - 2) + 2(k - 3) + \dots + (j - 1)(k - j) \\ &\quad + (j + 1)(k - j - 1) + \dots + (k - 1) \cdot 1 \\ &= 1 \cdot k + 2(k - 1) + \dots + (j - 1)(k - j + 2) + (j + 1)(k - j + 1) + \dots + (k - 1) \cdot 3 + k \cdot 2 \\ &= \sum_{q=1}^{j-1} q(k + 1 - q) + \sum_{m=j}^{k-1} (m + 1)(k + 1 - m) = \sum_{q=1}^{k-1} q(k + 1 - q) + \sum_{m=j}^{k-1} (k + 1 - m) \end{aligned}$$

which coincides with the desired result (2.4) for α_j , in the case $1 \leq j \leq k - 1$. For the extreme cases $j = 0$ and $j = k$, we obtain

$$\begin{aligned} \alpha_0 &= \sum_{m=0}^{k-1} (m + 1)(k + 1 - m) = \sum_{q=1}^{k-1} q(k + 1 - q) + \sum_{m=0}^{k-1} (k + 1 - m) \\ \alpha_k &= \sum_{q=1}^{k-1} q(k + 1 - q) \end{aligned}$$

which both can be incorporated in the formula (2.4) for α_j . Finally, Cramer's rule gives the expression (2.3) for c_j .

Now, taking into account that $\psi \geq 0$, by assumption, and the fact that $|1 + \epsilon P(\epsilon)| > |1 - \epsilon P(\epsilon)| = 1 - \epsilon |P(\epsilon)| > 0$ for all sufficiently small epsilon, we obtain

$$\int_{\mathbb{R}^d} |\varphi(x)| dx \leq \sum_{j=0}^k |c_j| \epsilon^j \leq \sum_{j=0}^k \frac{\epsilon^{j+\alpha_j} (1 + \epsilon |P_j(\epsilon)|)}{\epsilon^\beta (1 - \epsilon |P(\epsilon)|)}$$

and this latter expression can be made smaller than $1 + \delta$ for sufficiently small ϵ if a) $j + \alpha_j - \beta > 0$ for $0 \leq j \leq k - 1$, and b) $k + \alpha_k - \beta = 0$. Now, b) is obvious, as for a), we have:

$$j + \alpha_j - \beta = j + \sum_{m=j}^{k-1} (k + 1 - m) - k = \frac{1}{2} (k - j)(k - j + 1) > 0,$$

for $0 \leq j \leq k - 1$. To generalize the result for arbitrary dimension d , it suffices to consider a product of functions of one real variable. The proof is complete. \square

3. NONSTANDARD DELTA FUNCTIONS

We prove the existence of a nonstandard function D in ${}^*\mathcal{D}(\mathbb{R}^d)$ with special properties. The proof is based on the result of Lemma 2.2 and the Saturation Principle (T. Todorov [6], p. 687). We also consider a type of nonstandard cut-off-functions which have close counterparts in standard analysis. The applications of these functions are left for the next sections.

LEMMA 3.1 (Nonstandard Mollifiers). For any positive infinitesimal ρ in ${}^*\mathbb{R}$ there exists a nonstandard function θ in ${}^*\mathcal{D}(\mathbb{R}^d)$ with values in ${}^*\mathbb{R}$, which is symmetric and which satisfies the following properties:

- (i) $\theta(x) = 0$ for $x \in {}^*\mathbb{R}^d, \|x\| \geq 1$;
- (ii) $\int_{\mathbb{R}^d} \theta(x) dx = 1$;
- (iii) $\int_{\mathbb{R}^d} \theta(x) x^\alpha dx = 0$ for all $\alpha \in \mathbb{N}_0^d, \alpha \neq 0$;
- (iv) $\int_{\mathbb{R}^d} |\theta(x)| dx \approx 1$;
- (v) $|\ln \rho|^{-1} \left(\sup_{x \in {}^*\mathbb{R}^d} |\partial^\alpha \theta(x)| \right) \approx 0$ for all $\alpha \in \mathbb{N}_0^d$;

where \approx is the infinitesimal relation in ${}^*\mathbb{C}$. We shall call this type of function *nonstandard ρ -mollifiers*.

PROOF. For any $k \in \mathbb{N}$, we define the set of test functions:

$$\begin{aligned} \bar{\mathcal{A}}_k = \{ & \varphi \in \mathcal{D}(\mathbb{R}^d) : \varphi \text{ is real-valued and symmetric,} \\ & \varphi(x) = 0 \text{ for } \|x\| \geq 1, \int_{\mathbb{R}^d} \varphi(x) dx = 1, \\ & \int_{\mathbb{R}^d} x^\alpha \varphi(x) dx = 0 \text{ for } 1 \leq |\alpha| \leq k, \int_{\mathbb{R}^d} |\varphi(x)| dx < 1 + \frac{1}{k} \} \end{aligned}$$

and the internal subsets of ${}^*\mathcal{D}(\mathbb{R}^d)$:

$$\mathcal{A}_k = \left\{ \varphi \in {}^*(\bar{\mathcal{A}}_k) : |\ln \rho|^{-1} \left(\sup_{x \in {}^*\mathbb{R}^d} |\partial^\alpha ({}^*\varphi(x))| \right) < \frac{1}{k} \text{ for } |\alpha| \leq k \right\}.$$

Obviously, we have $\bar{\mathcal{A}}_1 \supset \bar{\mathcal{A}}_2 \supset \bar{\mathcal{A}}_3 \supset \dots$ and $\mathcal{A}_1 \supset \mathcal{A}_2 \supset \mathcal{A}_3 \supset \dots$. Also we have $\bar{\mathcal{A}}_k \neq \emptyset$ for all $k \in \mathbb{N}$, by Lemma 2.2. On the other hand, we have $\bar{\mathcal{A}}_k \subset \mathcal{A}_k$ in the sense that $\varphi \in \bar{\mathcal{A}}_k$ implies ${}^*\varphi \in \mathcal{A}_k$, since

$$\sup_{x \in {}^*\mathbb{R}^d} |\partial^\alpha ({}^*\varphi(x))| = \sup_{x \in \mathbb{R}^d} |\partial^\alpha \varphi(x)| = \sup_{x \leq 1} |\partial^\alpha \varphi(x)|$$

is a real (standard) number and, hence, $|\ln \rho|^{-1} \left(\sup_{x \in {}^*\mathbb{R}^d} |\partial^\alpha ({}^*\varphi(x))| \right)$ is infinitesimal. Thus, we have

$\mathcal{A}_k \neq \emptyset$ for all k in \mathbb{N} . By the Saturation Principle (T. Todorov [6], p. 687), the intersection $\mathcal{A} = \bigcap_{k \in \mathbb{N}} \mathcal{A}_k$ is non-empty and thus, any θ in \mathcal{A} has the desired properties. \square

DEFINITION 3.2 (ρ -Delta Function). Let ρ be a positive infinitesimal. A nonstandard function D in ${}^*\mathcal{D}(\mathbb{R}^d)$ is called a ρ -delta function if it takes values in ${}^*\mathbb{R}$, it is symmetric and it satisfies the following properties:

- (i) $D(x) = 0$ for $x \in {}^*\mathbb{R}^d, \|x\| \geq \rho,$
- (ii) $\int_{\mathbb{R}^d} D(x)dx = 1,$
- (iii) $\int_{\mathbb{R}^d} D(x)x^\alpha dx = 0$ for all $\alpha \in \mathbb{N}_0^d, \alpha \neq 0,$
- (iv) $\int_{\mathbb{R}^d} |D(x)|dx \approx 1,$
- (v) $|\ln \rho|^{-1} \left(\rho^{d+|\alpha|} \sup_{x \in {}^*\mathbb{R}^d} |\partial^\alpha D(x)| \right) \approx 0$ for all $\alpha \in \mathbb{N}_0^d.$

THEOREM 3.3 (Existence). For any positive infinitesimal ρ in ${}^*\mathbb{R}$ there exists a ρ -delta function.

PROOF. Let θ be a nonstandard ρ -mollifier of the type described in Lemma 3.1. Then the nonstandard function D in ${}^*\mathcal{D}(\mathbb{R}^d),$ defined by

$$D(x) = \rho^{-d}\theta(x/\rho), \quad x \in {}^*\mathbb{R}^d, \tag{3.1}$$

satisfies (i)-(v). \square

REMARK. The existence of nonstandard functions D in ${}^*\mathcal{D}(\mathbb{R}^d)$ with the above properties is in sharp contrast with the situation in standard analysis where there is no D in $\mathcal{D}(\mathbb{R}^d)$ which satisfies both (ii) and (iii). Indeed, if we assume that D is in $\mathcal{D}(\mathbb{R}^d),$ then (iii) implies $\widehat{D}^{(n)}(0) = 0,$ for all $n = 1, 2, \dots,$ where \widehat{D} denotes the Fourier transform of $D.$ It follows $\widehat{D} = \widehat{D}(0) = c$ for some constant c since \widehat{D} is an entire function on $\mathbb{C}^d,$ by the Paley-Wiener Theorem (H. Bremermann [12], Theorem 8.28, p. 97). On the other hand, $D \in \mathcal{D}(\mathbb{R}^d) \subset \mathcal{S}(\mathbb{R}^d)$ implies $\widehat{D}|_{\mathbb{R}^d} \in \mathcal{S}(\mathbb{R}^d)$ since $\mathcal{S}(\mathbb{R}^d)$ is closed under Fourier transform. Thus, it follows $c = 0,$ i.e. $\widehat{D} = 0$ which implies $D = 0$ contradicting (ii).

For other classes of nonstandard delta functions we refer to (A. Robinson [15], p. 133) and to (T. Todorov [16]).

Our next task is to show the existence of an internal *cut-off function*.

NOTATIONS. Let Ω be an open set of $\mathbb{R}^d.$

- 1) For any $\epsilon \in \mathbb{R}_+$ we define

$$B_\epsilon = \{x \in \mathbb{R}^d : \|x\| \leq \epsilon\} \quad \text{and} \quad \Omega_\epsilon = \{x \in \Omega : d(x, \partial\Omega) \geq \epsilon\},$$

where $\|x\|$ is the Euclidean norm in $\mathbb{R}^d, \partial\Omega$ is the boundary of Ω and $d(x, \partial\Omega)$ is the Euclidean distance between x and $\partial\Omega.$ We also denote:

$$\mathcal{D}_\epsilon(\Omega) = \{\varphi \in \mathcal{D}(\Omega) : \text{supp } \varphi \subseteq B_\epsilon\}, \quad E'_\epsilon(\Omega) = \{T \in E'(\Omega) : \text{supp } T \subseteq \Omega_\epsilon\}.$$

- 2) We shall use the same notation, $*$, for the convolution operator $*$: $\mathcal{D}'(\mathbb{R}^d) \times \mathcal{D}(\mathbb{R}^d) \rightarrow E(\mathbb{R}^d)$ (V. Vladimirov [13]) and its nonstandard extension $*$: ${}^*\mathcal{D}'(\mathbb{R}^d) \times {}^*\mathcal{D}(\mathbb{R}^d) \rightarrow {}^*E(\mathbb{R}^d)$ as well as for the convolution operator $*$: $E'_\epsilon(\Omega) \times \mathcal{D}_\epsilon(\Omega) \rightarrow \mathcal{D}(\Omega),$ defined for all sufficiently small $\epsilon \in \mathbb{R}_+,$ and for its nonstandard extension: $*$: ${}^*E'_\epsilon(\Omega) \times {}^*\mathcal{D}_\epsilon(\Omega) \rightarrow {}^*\mathcal{D}(\Omega), \epsilon \in {}^*\mathbb{R}_+, \epsilon \approx 0.$

- 3) Let τ be the usual Euclidean topology on $\mathbb{R}^d.$ We denote by $\widetilde{\Omega}$ the set of the nearstandard points in ${}^*\Omega,$ i.e.

$$\widetilde{\Omega} = \bigcup_{x \in \Omega} \mu(x), \tag{3.2}$$

where $\mu(x), x \in \mathbb{R}^d,$ is the system of monads of the topological space (\mathbb{R}^d, τ) (T. Todorov [6], p. 687). Recall that if $\xi \in {}^*\Omega,$ then $\xi \in \widetilde{\Omega}$ if and only if ξ is a finite point whose standard part belongs to $\Omega.$

LEMMA 3.4. For any positive infinitesimal ρ in ${}^*\mathbb{R}$ there exists a function Π in ${}^*\mathcal{D}(\Omega)$ (a ρ -cut-off function) such that:

- a) $\Pi(x) = 1$ for all $x \in \widetilde{\Omega};$
- b) $\text{supp } \Pi \subseteq {}^*\Omega_\rho,$ where ${}^*\Omega_\rho = \{\xi \in {}^*\Omega : {}^*d(\xi, \partial\Omega) \geq \rho\}.$

PROOF. Let ρ be a positive infinitesimal in ${}^*\mathbb{R}$ and D be a ρ -delta function. Define the internal set $X = \{\xi \in {}^*\Omega : \|\xi\| \leq 1/\rho, {}^*d(\xi, \partial\Omega) \geq 2\rho\}$ and let χ be its characteristic function. Then the function $\Pi = \chi * D$ has the desired property. \square

4. THE ALGEBRA OF ASYMPTOTIC FUNCTIONS

We define and study the algebra ${}^\rho E(\Omega)$ of asymptotic functions on an open set Ω of \mathbb{R}^d . The construction of the algebra ${}^\rho E(\Omega)$, presented here, is a generalization and a refinement of the constructions in [9] and [10] (by the authors of this paper, respectively), where the algebra ${}^\rho E(\mathbb{R}^d)$ was introduced by somewhat different but equivalent definitions. On the other hand, the algebra of asymptotic functions ${}^\rho E(\Omega)$ is somewhat similar to but different from the J. F. Colombeau [7], [8] algebras of new generalized functions. This essential difference between ${}^\rho E(\Omega)$ and J. F. Colombeau's algebras of generalized functions is the properties of the generalized scalars: the scalars of the algebra ${}^\rho E(\Omega)$ constitutes an algebraically closed field (as any scalars should do) while the scalars of J. F. Colombeau's algebras are rings with zero divisors (J. F. Colombeau [8], §2.1). This improvement compared with J. F. Colombeau's theory is due to the involvement of the nonstandard analysis.

Let Ω be an open set of \mathbb{R}^d and $\rho \in {}^*\mathbb{R}$ be a positive infinitesimal. We shall keep Ω and ρ fixed in what follows.

Following A. Robinson [2], we define:

DEFINITION 4.1 (Robinson's Asymptotic Numbers). The field of the complex Robinson ρ -asymptotic numbers is defined as the factor space ${}^\rho\mathbb{C} = \mathbb{C}_M/\mathbb{C}_0$, where

$$\begin{aligned} \mathbb{C}_M &= \{\xi \in {}^*\mathbb{C} : |\xi| < \rho^{-n} \text{ for some } n \in \mathbb{N}\}, \\ \mathbb{C}_0 &= \{\xi \in {}^*\mathbb{C} : |\xi| < \rho^n \text{ for all } n \in \mathbb{N}\}, \end{aligned}$$

("M" stands for "moderate"). We define the embedding $\mathbb{C} \subset {}^\rho\mathbb{C}$ by $c \rightarrow q(c)$, where $q : \mathbb{C}_M \rightarrow {}^\rho\mathbb{C}$ is the quotient mapping. The field of the real asymptotic numbers is defined by ${}^\rho\mathbb{R} = q({}^*\mathbb{R} \cap \mathbb{C}_M)$.

It is easy to check that \mathbb{C}_0 is a maximal ideal in \mathbb{C}_M and hence ${}^\rho\mathbb{C}$ is a field. Also ${}^\rho\mathbb{R}$ is a real closed totally ordered nonarchimedean field (since ${}^*\mathbb{R}$ is a real closed totally ordered field) containing \mathbb{R} as a totally ordered subfield. Thus, it follows that ${}^\rho\mathbb{C} = {}^\rho\mathbb{R}(i)$ is an algebraically closed field, where $i = \sqrt{-1}$.

The algebra of "asymptotic functions" is, in a sense, a C^∞ -counterpart of A. Robinson's asymptotic numbers ${}^\rho\mathbb{C}$:

DEFINITION 4.2 (Asymptotic Functions on Ω). (i) We define the class ${}^\rho E(\Omega)$ of the ρ -asymptotic functions on Ω (or simply, *asymptotic functions on Ω* if no confusion could arise) as the factor space ${}^\rho E(\Omega) = E_M(\Omega)/E_0(\Omega)$, where

$$\begin{aligned} E_M(\Omega) &= \{f \in {}^*E(\Omega) : \partial^\alpha f(\xi) \in \mathbb{C}_M, \text{ for all } \alpha \in \mathbb{N}_0^d \text{ and all } \xi \in \tilde{\Omega}\}, \\ E_0(\Omega) &= \{f \in {}^*E(\Omega) : \partial^\alpha f(\xi) \in \mathbb{C}_0, \text{ for all } \alpha \in \mathbb{N}_0^d \text{ and all } \xi \in \tilde{\Omega}\}, \end{aligned}$$

and $\tilde{\Omega}$ is the set of the nearstandard points of ${}^*\Omega$ (3.2). The functions in $E_M(\Omega)$ are called ρ -moderate (or, simply, *moderate*) and those in $E_0(\Omega)$ are called ρ -null functions (or, simply, *null functions*).

(ii) The pairing between ${}^\rho E(\Omega)$ and $\mathcal{D}(\Omega)$ with values in ${}^\rho\mathbb{C}$, is defined by

$$\langle Q_\Omega(f), \varphi \rangle = q\left(\int_{-\Omega} f(x) * \varphi(x) dx\right),$$

where $q : \mathbb{C}_M \rightarrow {}^\rho\mathbb{C}$ and $Q_\Omega : E_M(\Omega) \rightarrow {}^\rho E(\Omega)$ are the corresponding quotient mappings, φ is in $\mathcal{D}(\Omega)$ and $*\varphi$ is its nonstandard extension.

(iii) We define the *canonical embedding* $E(\Omega) \subset {}^\rho E(\Omega)$ by the mapping $\sigma_\Omega : f \rightarrow Q_\Omega(*f)$, where $*f$ is the nonstandard extension of f .

EXAMPLE 4.3. Let D be a nonstandard ρ -delta function in the sense of Definition 3.2. Then $D \in E_M(\mathbb{R}^d)$. In addition, $D|{}^*\Omega \in E_M(\Omega)$, where $D|{}^*\Omega$ denotes the pointwise restriction of D on ${}^*\Omega$. To show this, denote $|\ln \rho|^{-1} \left(\rho^{d+|\alpha|} \sup_{x \in {}^*\mathbb{R}^d} |\partial^\alpha D(x)| \right) = h_\alpha$ and observe that $h_\alpha \approx 0$ for all $\alpha \in \mathbb{N}_0^d$, by the definition of D . Thus, for any (finite) x in ${}^*\mathbb{R}^d$ and any $\alpha \in \mathbb{N}_0^d$ we have $|\partial^\alpha D(x)| \leq \sup_{x \in {}^*\mathbb{R}^d} |\partial^\alpha D(x)| = \frac{h_\alpha |\ln \rho|}{\rho^{d+|\alpha|}} < \rho^{-n}$, for $n = d + |\alpha| + 1$, thus, $D \in E_M(\mathbb{R}^d)$. On the other hand, $D|{}^*\Omega \in E_M(\Omega)$ follows immediately from the fact that $\tilde{\Omega}$ consists of finite points in ${}^*\mathbb{R}^d$ only.

THEOREM 4.4 (Differential Algebra). (i) The class of asymptotic functions ${}^\rho E(\Omega)$ is a *differential algebra* over the field of the complex asymptotic numbers ${}^\rho \mathbb{C}$.

(ii) $E(\Omega)$ is a *differential subalgebra* of ${}^\rho E(\Omega)$ over the scalars \mathbb{C} under the canonical embedding σ_Ω . In addition, σ_Ω preserves the pairing in the sense that $\langle f, \varphi \rangle = \langle \sigma_\Omega(f), \varphi \rangle$ for all f in $E(\Omega)$ and for all φ in $\mathcal{D}(\Omega)$, where $\langle f, \varphi \rangle = \int_\Omega f(x)\varphi(x)dx$ is the usual pairing between $E(\Omega)$ and $\mathcal{D}(\Omega)$.

PROOF. (i) It is clear that $E_M(\Omega)$ is a differential ring and $E_0(\Omega)$ is a differential ideal in $E_M(\Omega)$ since \mathbb{C}_M is a ring and \mathbb{C}_0 is an ideal in \mathbb{C}_M and, on the other hand, both $E_M(\Omega)$ and $E_0(\Omega)$ are closed under differential, by definition. Hence, the factor space ${}^\rho E(\Omega)$ is also a differential ring. It is clear that, $E_M(\Omega)$ is a module over the ring \mathbb{C}_M and, in addition, the annihilator $\{c \in \mathbb{C}_M : c f \in E_0(\Omega), f \in E_M(\Omega)\}$ of \mathbb{C}_M coincides with the ideal \mathbb{C}_0 . Thus, ${}^\rho E(\Omega)$ becomes an algebra over the field of the complex asymptotic numbers ${}^\rho \mathbb{C}$.

(ii) Assume that $\sigma_\Omega({}^*f) = 0$ in ${}^\rho E(\Omega)$, i.e. ${}^*f \in E_0(\Omega)$. By the definition of $E_0(\Omega)$ (applied for $\alpha = 0$ and $n = 1$), it follows $f = 0$ since *f is an extension of f and ρ is an infinitesimal. Thus, the mapping $f \rightarrow \sigma_\Omega(f)$ is injective. It preserves the algebraic operations since the mapping $f \rightarrow {}^*f$ preserves them. The preserving of the pairing follows immediately from the fact that $\int_\Omega {}^*f(x)dx = \int_\Omega f(x)dx$, by the Transfer Principle (T. Todorov [6], p. 686). The proof is complete. \square

5. EMBEDDING OF SCHWARTZ DISTRIBUTIONS

Let Ω be (as before) an open set of \mathbb{R}^d . Recall that the *Schwartz embedding* $L_\Omega : \mathcal{L}_{loc}(\Omega) \rightarrow \mathcal{D}'(\Omega)$ from $\mathcal{L}_{loc}(\Omega)$ into $\mathcal{D}'(\Omega)$ is defined by the formula:

$$\langle L_\Omega(f), \varphi \rangle = \int_\Omega f(x)\varphi(x)dx, \quad \varphi \in \mathcal{D}(\Omega). \tag{5.1}$$

Here $\mathcal{L}_{loc}(\Omega)$ denotes, as usual, the space of the locally (Lebesgue) integrable complex valued functions on Ω (V. Vladimirov [13]). The Schwartz embedding L_Ω preserves the addition and multiplication by a complex number, hence, the space $\mathcal{L}_{loc}(\Omega)$ can be considered as a linear subspace of $\mathcal{D}'(\Omega)$. In addition, the restriction $L_\Omega|E(\Omega)$ of L_Ω on $E(\Omega)$ (often denoted also by L_Ω) preserves the partial differentiation of any order and in this sense $E(\Omega)$ is a differential linear subspace of $\mathcal{D}'(\Omega)$. In short, we have the chain of linear embeddings: $\mathcal{L}_{loc}(\Omega) \subset E(\Omega) \subset \mathcal{D}'(\Omega)$.

The purpose of this section is to show that the algebra of asymptotic functions ${}^\rho E(\Omega)$ contains an isomorphic copy of the space of Schwartz distributions $\mathcal{D}'(\Omega)$ and, hence, to offer a *solution of the Problem of Multiplication of Schwartz Distributions*. This result is a generalization of some results in [9] and [10] (by the authors of this paper, respectively) where only the embedding of the tempered distributions $\mathcal{S}'(\mathbb{R}^d)$ in ${}^\rho E(\mathbb{R}^d)$ has been established. The embedding of all distributions $\mathcal{D}'(\Omega)$, discussed here, presents an essentially different situation.

The spaces $\tilde{E}(\Omega)$ and $\tilde{\mathcal{D}}(\Omega)$, defined below, are immediate generalizations of the spaces $\tilde{E}(\mathbb{R}^d)$ and $\tilde{\mathcal{D}}(\mathbb{R}^d)$, introduced in (K. D. Stroyan and W. A. Luxemburg [17], (10.4), p. 299):

$$\tilde{E}(\Omega) = \{ \varphi \in {}^*E(\Omega) : \partial^\alpha \varphi(x) \text{ is a finite number in } {}^*\mathbb{C} \text{ for all } x \in \tilde{\Omega} \text{ and all } \alpha \in \mathbb{N}_0^d \}, \tag{5.2}$$

$$\tilde{\mathcal{D}}(\Omega) = \{\varphi \in {}^*E(\Omega) : \partial^\alpha \varphi(x) \text{ is a finite number in } {}^*\mathbb{C} \text{ for all } x \in \tilde{\Omega}, \alpha \in \mathbb{N}_0^d \text{ and } \varphi(x) = 0 \text{ for all } x \in {}^*\Omega \setminus \tilde{\Omega}\}, \quad (5.3)$$

Obviously, we have $\tilde{\mathcal{D}}(\Omega) \subset \tilde{\mathcal{E}}(\Omega) \subset E_M(\Omega)$. Notice as well that $\varphi \in \tilde{\mathcal{D}}(\Omega)$ implies $\varphi \in {}^*\mathcal{D}(G)$ for some open relatively compact set G of Ω . We have also the following simple result:

LEMMA 5.1. If $T \in \mathcal{D}'(\Omega)$ and $\varphi \in E_0(\Omega) \cap \tilde{\mathcal{D}}(\Omega)$, then $({}^*T, \varphi) \in \mathbb{C}_0$.

PROOF. Observe that $E_0(\Omega) \cap \tilde{\mathcal{D}}(\Omega)$ implies $\varphi \in E_0(\Omega) \cap {}^*\mathcal{D}(G)$ for some open relatively compact set G of Ω . By the continuity of T (and Transfer Principle) there exist constants $M \in \mathbb{R}_+$ and $m \in \mathbb{N}_0$ such that

$$|({}^*T, \varphi)| \leq M \sum_{|\mu| \leq m} \sup_{x \in {}^*G} |\partial^\mu \varphi(x)|.$$

On the other hand, $M \sum_{|\mu| \leq m} \sup_{x \in {}^*G} |\partial^\mu \varphi(x)| < \rho^n$ for all $n \in \mathbb{N}$, since $\varphi \in E_0(\Omega)$, by assumption. Thus, $|({}^*T, \varphi)| < \rho^n$ for all $n \in \mathbb{N}$. \square

Let D be a ρ -delta function in the sense of Definition 3.2. We shall keep D (along with Ω and ρ) fixed in what follows.

DEFINITION 5.2 (Embedding of Schwartz Distributions). We define the embedding $\mathcal{D}'(\Omega) \subset {}^\rho E(\Omega)$ by $\Sigma_{D,\Omega} : T \rightarrow Q_\Omega({}^*T \Pi_\Omega * D)$, where *T is the nonstandard extension of T , Π_Ω is a (an arbitrarily chosen) ρ -cut-off function for Ω (Lemma 3.4), ${}^*T \Pi_\Omega$ is the Schwartz product between *T and Π_Ω in ${}^*\mathcal{D}'(\Omega)$ (defined by Transfer Principle), $*$ is the convolution operator and $Q_\Omega : E_M(\Omega) \rightarrow {}^\rho E(\Omega)$ is the quotient mapping in the definition of ${}^\rho E(\Omega)$ (Definition 4.2).

The cut-off function Π_Ω can be dropped in the above definition, i.e. $\Sigma_{D,\Omega} : T \rightarrow Q_\Omega({}^*T * D)$, in some particular cases; e.g. when:

- a) T has a compact support in Ω ;
- b) $\Omega = \mathbb{R}^d$.

PROPOSITION 5.3 (Correctness). $T \in \mathcal{D}'(\Omega)$ implies $({}^*T \Pi_\Omega) * D \in E_M(\Omega)$.

PROOF. Choose $\alpha \in \mathbb{N}_0^d$ and all $x \in \tilde{\Omega}$. Since we have $\partial^\alpha((\Pi_\Omega * T) * D)(x) = (\partial^\alpha({}^*T) * D)(x)$ (by the definition of Π_Ω), we need to show that $\partial^\alpha({}^*T * D)(x) \in \mathbb{C}_M$ only, i.e. that $|\partial^\alpha({}^*T * D)(x)| < \rho^{-m}$ for some $m \in \mathbb{N}$ (m might depend on α). We start with the case $\alpha = 0$. Denote $D_x(\xi) = D(\xi - x)$, $\xi \in {}^*\mathbb{R}$ and observe that $\text{supp}(D_x) \subseteq {}^*G$ for some open relatively compact set G of Ω , since D_x vanishes on ${}^*\Omega \setminus \tilde{\Omega}$. Next, by the continuity of T (and the Transfer Principle), there exist constants $m \in \mathbb{N}_0$ and $M \in \mathbb{R}_+$ such that

$$|({}^*T * D)(x)| = |({}^*T, D_x | {}^*\Omega)| \leq M \sum_{|\mu| \leq m} \sup_{\xi \in {}^*G} |\partial_\xi^\mu D(x - \xi)|.$$

Finally, there exists $n \in \mathbb{N}$ such that $\sum_{|\mu| \leq m} \sup_{\xi \in {}^*G} |\partial_\xi^\mu D(x - \xi)| < \rho^{-n}$, since $D|{}^*G$ is a ρ -moderate function (Example 4.3). Combining these arguments, we have: $|({}^*T * D)(x)| \leq M \rho^{-n} < \rho^{-(n+1)}$, as required. The generalization for arbitrary multiindex α follows immediately since $\partial^\alpha({}^*T * D) = (\partial^\alpha({}^*T)) * D = ({}^*\partial^\alpha T) * D$, by Transfer Principle, a $\partial^\alpha T$ is (also) in $\mathcal{D}'(\Omega)$. \square

PROPOSITION 5.4. $f \in \tilde{\mathcal{E}}(\Omega)$ implies $(f \Pi_\Omega) * D - f \in E_0(\Omega)$.

PROOF. Let $x \in \tilde{\Omega}$ and $\alpha \in \mathbb{N}_0^d$. Since we have $\partial^\alpha[(f \Pi_\Omega) * D](x) - f(x) = \partial^\alpha[(f * D)(x) - f(x)]$ (by the definition of Π_Ω), we need to show that $\partial^\alpha[(f * D)(x) - f(x)] \in \mathbb{C}_0$ only. Choose $n \in \mathbb{N}$. We need to show that $|\partial^\alpha[(f * D)(x) - f(x)]| < \rho^n$. We start first with the case $\alpha = 0$. By Taylor's formula (applied by transfer), we have

$$f(x - \xi) - f(x) = \sum_{|\beta|=1}^n \frac{(-1)^{|\beta|} \partial^\beta f(x)}{\beta!} \xi^\beta + \frac{(-1)^{n+1}}{(n+1)!} \sum_{|\beta|=n+1} \partial^\beta f(\eta(\xi)) \xi^\beta$$

for any $\xi \in \tilde{\Omega}$, where $\eta(\xi)$ is a point in ${}^*\Omega$ "between x and ξ ." Notice that the point $\eta(\xi)$ is also in $\tilde{\Omega}$. It follows

$$(f * D)(x) - f(x) = \int_{\|\xi\| \leq \rho} D(\xi) [f(x - \xi) - f(x)] d\xi = \frac{(-1)^{n+1}}{(n+1)!} \sum_{|\beta|=n+1} \int_{\|\xi\| \leq \rho} D(\xi) \xi^\beta \partial^\beta f(\eta(\xi)) d\xi,$$

since $\int_{\|\xi\| \leq \rho} D(\xi) \xi^\beta d\xi = 0$, by the definition of D . Thus, we have

$$|(f * D)(x) - f(x)| \leq \frac{\rho^{n+1}}{(n+1)!} \left(\int_{\mathbb{R}^d} |D(x)| dx \right) \left(\sum_{|\beta|=n+1} \sup_{\|\xi\| \leq \rho} |\partial^\beta f(\eta(\xi))| \right) < \rho^n,$$

as desired, since, on one hand, $\int_{\mathbb{R}^d} |D(x)| dx \approx 1$, by the definition of D and on the other hand, the above sum is a finite number because ${}^*\partial^\beta f(\eta(\xi))$ are all finite due to $\eta(\xi) \in \tilde{\Omega}$. The generalization for an arbitrary α is immediate since $\partial^\alpha [(f * D)(x) - f(x)] = (\partial^\alpha f * D)(x) - \partial^\alpha f(x)$, by the Transfer Principle. \square

COROLLARY 5.5. (i) $f \in E(\Omega)$ implies $({}^*f \Pi_\Omega) * D - {}^*f \in E_0(\Omega)$.

(ii) $\varphi \in \mathcal{D}(\Omega)$ implies $({}^*\varphi \Pi_\Omega) * D - {}^*\varphi \in E_0(\Omega) \cap \tilde{\mathcal{D}}(\Omega)$.

PROOF. (i) follows immediately from the above proposition since $f \in E(\Omega)$ implies ${}^*f \in \tilde{E}(\Omega)$.

(ii) Both $({}^*\varphi \Pi_\Omega) * D$ and ${}^*\varphi$ vanish on ${}^*\Omega \setminus \tilde{\Omega}$ since their supports are within an open relatively compact neighborhood G of $\text{supp}(\varphi)$ and the latter is a compact set of Ω , by assumption. Thus,

$$({}^*\varphi \Pi_\Omega) * D - {}^*\varphi \in {}^*\mathcal{D}(G) \subset \tilde{\mathcal{D}}(\Omega), \quad \text{as required. } \square$$

Denote $\check{D}(x) = D(-x)$ and recall that $\check{D} = D$ since D is symmetric (Definition 3.2).

PROPOSITION 5.6. If $T \in \mathcal{D}'(\Omega)$ and $\varphi \in \mathcal{D}(\Omega)$, then

$$\int_{\mathbb{R}^d} (({}^*T \Pi_\Omega) * D)(x) {}^*\varphi(x) dx - \langle T, \varphi \rangle \in C_0.$$

PROOF. Using the properties of the convolution operator (applied by transfer), we have

$$\begin{aligned} & \int_{\mathbb{R}^d} (({}^*T \Pi_\Omega) * D)(x) {}^*\varphi(x) dx - \langle T, \varphi \rangle \\ &= \langle ({}^*T \Pi_\Omega) * D, {}^*\varphi \rangle - \langle {}^*T, {}^*\varphi \rangle = \langle {}^*T \Pi_\Omega, {}^*\varphi * \check{D} \rangle - \langle {}^*T \Pi_\Omega, {}^*\varphi \rangle \\ &= \langle {}^*T \Pi_\Omega, {}^*\varphi * \check{D} - {}^*\varphi \rangle = \langle {}^*T, {}^*\varphi * D - {}^*\varphi \rangle \in C_0, \end{aligned}$$

by Lemma 5.1 since ${}^*\varphi * D - {}^*\varphi \in E_0(\Omega) \cap \tilde{\mathcal{D}}(\Omega)$, by Corollary 5.5. \square

We are ready to state our *main result*:

THEOREM 5.7 (Properties of $\Sigma_{D,\Omega}$). (i) $\Sigma_{D,\Omega}$ *preserves the pairing* in the sense that for all T in $\mathcal{D}'(\Omega)$ and all φ in $\mathcal{D}(\Omega)$ we have $\langle T, \varphi \rangle = \langle \Sigma_{D,\Omega}(T), \varphi \rangle$, where the left hand side is the (usual) pairing of T and φ in $\mathcal{D}'(\Omega)$, while the right hand side is the pairing of $\Sigma_{D,\Omega}(T)$ and φ in ${}^*E(\Omega)$ (Definition 4.2).

(ii) $\Sigma_{D,\Omega}$ is *injective* and it *preserves all linear operations* in $\mathcal{D}'(\Omega)$: the addition, multiplication by (standard) complex numbers and the partial differentiation of any (standard) order.

(iii) $\Sigma_{D,\Omega}$ is *an extension of the canonical embedding* σ_Ω defined earlier in Definition 4.2 in the sense that $\sigma_\Omega = \Sigma_{D,\Omega} \circ L_\Omega$, where L_Ω is the *Schwartz embedding* (5.1) restricted on $E(\Omega)$ and \circ denotes *composition*. Or, equivalently, the following diagram is commutative:

$$\begin{array}{ccc}
 & \mathcal{D}'(\Omega) & \\
 & L_\Omega \nearrow & \\
 E(\Omega) & \downarrow \Sigma_{D,\Omega} & \\
 & \sigma_\Omega \searrow & \\
 & {}^\rho E(\Omega) &
 \end{array} \tag{5.4}$$

PROOF. (i) Denote (as before) $\check{D}(x) = D(-x)$ and recall that $\check{D}(x) = D$ (Definition 3.2). We have

$$\begin{aligned}
 \langle \Sigma_{D,\Omega}(T), \varphi \rangle &= \langle Q_\Omega({}^*T\Pi_\Omega) * D, \varphi \rangle - \langle T, \varphi \rangle \\
 &= q \left(\int_{-\Omega} ((\Pi_\Omega * T) * D)(x) * \varphi(x) dx \right) - q(\langle T, \varphi \rangle) \\
 &= q \left(\int_{-\Omega} ({}^*T\Pi_\Omega) * D(x) * \varphi(x) dx - \langle T, \varphi \rangle \right) = 0,
 \end{aligned}$$

because $\int_{-\Omega} ({}^*T\Pi_\Omega) * D(x) * \varphi(x) dx - \langle T, \varphi \rangle \in \mathbb{C}_0$, by Proposition 5.6. Here $\langle T, \varphi \rangle = q(\langle T, \varphi \rangle)$ holds because $\langle T, \varphi \rangle$ is a standard (complex) number.

(ii) The injectivity of $\Sigma_{D,\Omega}$ follows from (i): $\Sigma_{D,\Omega}(T) = 0$ in ${}^\rho E(\Omega)$ implies $\langle \Sigma_{D,\Omega}(T), \varphi \rangle = 0$ for all $\varphi \in \mathcal{D}(\Omega)$, which is equivalent to $\langle T, \varphi \rangle = 0$ for all $\varphi \in \mathcal{D}(\Omega)$, by (i), thus, $T = 0$ in $\mathcal{D}'(\Omega)$, as required. The preserving of the linear operations follows from the fact that both the extension mapping $*$ and the convolution $*$ (applied by Transfer Principle) are linear operators.

(iii) For any $f \in E(\Omega)$ we have $\sigma(f) = Q_\Omega(*f) = Q_\Omega({}^*f\Pi_\Omega) * D = Q_\Omega({}^*L(f)\Pi_\Omega) * D = \Sigma_{D,\Omega}(L(f))$, as required, since $*f - ({}^*f\Pi_\Omega) * D \in E_0(\Omega)$, by Corollary 5.5. \square

REMARK 5.8 (Multiplication of Distributions). As a consequence of the above result, the Schwartz distributions in $\mathcal{D}'(\Omega)$ can be multiplied within the associative and commutative differential algebra ${}^\rho E(\Omega)$ (something impossible in $\mathcal{D}'(\Omega)$ itself). By the property (iii) above, the multiplication in ${}^\rho E(\Omega)$ coincides on $E(\Omega)$ with the usual (pointwise) multiplication in $E(\Omega)$. Thus, the class ${}^\rho E(\Omega)$, endowed with an embedding $\Sigma_{D,\Omega}$, presents a solution of the problem of multiplication of Schwartz distributions which, in a sense, is optimal, in view of the Schwartz impossibility results (L. Schwartz [1]) (for a discussion we refer also to J. F. Colombeau [7], §2.4 and M. Oberguggenberger [18], §2). We should mention that the existence of an embedding of $\mathcal{D}'(\mathbb{R}^d)$ into ${}^\rho E(\mathbb{R}^d)$ can be proved also by sheaf-theoretical arguments as indicated in (M. Oberguggenberger [18], §23).

REMARK 5.9 (Nonstandard Asymptotic Analysis). We sometimes refer to the area connected directly or indirectly with the fields ${}^\rho\mathbb{R}$ as *Nonstandard Asymptotic Analysis*. The fields ${}^\rho\mathbb{R}$ were introduced by A. Robinson [2] and are sometimes known as "Robinson's nonarchimedean valuation fields." The terminology "Robinson's asymptotic numbers," chosen in this paper, is due to the role of ${}^\rho\mathbb{R}$ for the asymptotic expansions of classical functions (A. H. Lightstone and A. Robinson [3]) and also to stress the fact that in our approach ${}^\rho\mathbb{C}$ plays the role of the scalars of the algebra ${}^\rho E(\Omega)$. Linear spaces over the field ${}^\rho\mathbb{R}$ has been studied by W. A. J. Luxemburg [19] in order to establish a connection between nonstandard and nonarchimedean analysis. More recently ${}^\rho\mathbb{R}$ has been used by V. Pestov [20] for studying Banach spaces. The field ${}^\rho\mathbb{R}$ has been exploited by Li Bang-He [21] for multiplication of Schwartz distributions.

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