Internat. J. Math. & Math. Sci.
VOL. 21 NO. 3 (1998) 417-428

AN EMBEDDING OF SCHWARTZ DISTRIBUTIONS IN THE ALGEBRA OF ASYMPTOTIC FUNCTIONS

MICHAEL OBERGUGGENBERGER

Institut für Mathematik und Geometrie
Universität Innsbruck
A-6020 Inssbruck, AUSTRIA
michael@mat1.uibk.ac.at

and

TODOR TODOROV

Mathematics Department
California Polytechnic State University
San Luis Obispo, California 93407, USA
ttodorov@oboe.calpoly.edu

(Received October 28, 1996 and in revised form June 21, 1997)

ABSTRACT. We present a solution of the problem of multiplication of Schwartz distributions by embedding the space of distributions into a differential algebra of generalized functions, called in the paper "asymptotic function," similar to but different from J. F Colombeau's algebras of new generalized functions.

KEY WORDS AND PHRASES: Schwartz distributions, nonlinear theory of generalized functions, asymptotic expansion, nonstandard analysis, nonstandard asymptotic analysis
1991 AMS SUBJECT CLASSIFICATION CODES: 03H05, 12J25, 46F10, 46S20

1. INTRODUCTION

The main purpose of this paper is to prove the existence of an embedding $\Sigma_{D,\Omega}$ of the space of Schwartz distributions $\mathcal{D}'(\Omega)$ into the algebra of asymptotic functions ${}^\rho E(\Omega)$ which preserves all linear operations in $\mathcal{D}'(\Omega)$. Thus, we offer a solution of the problem of multiplication of Schwartz distributions since the multiplication within $\mathcal{D}'(\Omega)$ is impossible (L. Schwartz [1]).

The algebra ${}^\rho E(\Omega)$ is defined in the paper as a factor space of nonstandard smooth functions The field of the scalars ${}^\rho\mathbb{C}$ of the algebra ${}^\rho E(\Omega)$, coincides with the complex counterpart of A. Robinson's asymptotic numbers—known also as "Robinson's field with valuation" (see A Robinson [2]) and A. H. Lightstone and A. Robinson [3]). The embedding $\Sigma_{D,\Omega}$ is constructed in the form $\Sigma_{D,\Omega} = Q_\Omega \circ D * \Pi \cdot {}^*$ where (in backward order): * is the extension mapping (in the sense of nonstandard analysis), $\cdot$ is the Schwartz multiplication in $\mathcal{D}'(\Omega)$ (more precisely, its nonstandard extension in ${}^*\mathcal{D}'(\Omega)$), $*$ is the convolution operator (more precisely, its nonstandard extension), $\circ$ denotes "composition," Q_Ω is the quotient mapping (in the definition of the algebra of asymptotic functions) and D and Π_Ω are fixed nonstandard internal functions with special properties whose existence is proved in this paper.

Our interest in the algebra ${}^\rho E(\Omega)$ and the embedding $\mathcal{D}'(\Omega) \subset {}^\rho E(\Omega)$, is due to their role in the problem of multiplication of Schwartz distributions, the nonlinear theory of generalized functions and its applications to partial differential equations (M. Oberguggenberger [4]), (T. Todorov [5] and [6]). In particular, there is a strong similarity between the algebra of asymptotic functions ${}^\rho E(\Omega)$ and its generalized scalars ${}^\rho\mathbb{C}$, discussed in this paper, and the algebra of generalized functions $\mathcal{G}(\Omega)$ and their generalized scalars $\overline{\mathbb{C}}$, introduced by J. F. Colombeau in the framework of standard analysis (J. F. Colombeau [7], pp. 63, 138 and J. F. Colombeau [8], §8.3, pp. 161-166). We should mention that the involvement of nonstandard analysis has resulted in some improvements of the corresponding standard counterparts; one of them is that ${}^{\rho}\mathbb{C}$ is an algebraically closed field while its standard counterpart $\overline{\mathbb{C}}$ in J. F. Colombeau's theory is a ring with zero divisors.

This paper is a generalization of some results in [9] and [10] (by the authors of this paper, respectively) where only the embedding of the tempered distributions $\mathcal{S}'(\mathbb{R}^d)$ in ${}^{\rho}\mathcal{E}(\mathbb{R}^d)$ has been established. The embedding of all distributions $\mathcal{D}'(\Omega)$, discussed in this paper, presents an essentially different situation. We should mention that the algebra ${}^{\rho}\mathcal{E}(\mathbb{R}^d)$ was recently studied by R. F. Hoskins and J. Sousa Pinto [11].

Here Ω denotes an open set of $\mathbb{R}^d$ (d is a natural number), $\mathcal{E}(\Omega) = C^\infty(\Omega)$ and $\mathcal{D}(\Omega) = C_0^\infty(\Omega)$ denote the usual classes of C^∞-functions on Ω and C^∞-functions with compact support in Ω and $\mathcal{D}'(\Omega)$, and $\mathcal{E}'(\Omega)$ denote the classes of Schwartz distributions on Ω and Schwartz distributions with compact support in Ω, respectively. As usual, $\mathbb{N}$, $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{C}$ will be the systems of the natural, real, positive real and complex numbers, respectively, and we use also the notation $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$. For the partial derivatives we write ∂^α, $\alpha \in \mathbb{N}_0^d$. If $\alpha = (\alpha_1, ..., \alpha_d)$ for some $\alpha \in \mathbb{N}_0^d$, then we write $|\alpha| = \alpha_1 + ... + \alpha_d$ and if $x = (x_1, ..., x_d)$ for some $x \in \mathbb{R}^d$, then we write $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} ... x_d^{\alpha_2} ... x_d^{\alpha_d}$. For a general reference to distribution theory we refer to H. Bremermann [12] and V. Vladimirov [13].

Our framework is a nonstandard model of the complex numbers $\mathbb{C}$, with degree of saturation larger than card($\mathbb{N}$). We denote by ${}^*\mathbb{R}$, ${}^*\mathbb{R}_+$, ${}^*\mathbb{C}$, ${}^*\mathcal{E}(\Omega)$ and ${}^*\mathcal{D}(\Omega)$ the nonstandard extensions of $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{C}$, $\mathcal{E}(\Omega)$ and $\mathcal{D}(\Omega)$, respectively. If X is a set of complex numbers or a set of (standard) functions, then ${}^*X$ will be its nonstandard extension and if $f : X \to Y$ is a (standard) mapping, then ${}^*f : {}^*X \to {}^*Y$ will be its nonstandard extension. For integration in ${}^*\mathbb{R}^d$ we use the $*$-Lebesgue integral. We shall often use the same notation, $\|x\|$, for the Euclidean norm in $\mathbb{R}^d$ and its nonstandard extension in ${}^*\mathbb{R}^d$. For a short introduction to nonstandard analysis we refer to the Appendix in T. Todorov [6]. For a more detailed exposition we recommend T. Lindstrom [14], where the reader will find many references to the subject.

2. TEST FUNCTIONS AND THEIR MOMENTS

In this section we study some properties of the test functions in $\mathcal{D}(\mathbb{R}^d)$ (in a standard setting) which we shall use subsequently.

Following (J.F. Colombeau [7], p. 55), for any $k \in \mathbb{N}$ we define the set of test functions:

$$A_k = \Big\{\varphi \in \mathcal{D}(\mathbb{R}^d) : \varphi \text{ is real-valued}, \ \varphi(x) = 0 \text{ for } \|x\| \geq 1; \\ \int_{\mathbb{R}^d} \varphi(x)dx = 1 \text{ and } \int_{\mathbb{R}^d} x^\alpha \varphi(x)dx = 0 \text{ for } \alpha \in \mathbb{N}_0^d, \ 1 \leq |\alpha| \leq k\Big\}. \tag{2.1}$$

Obviously, $A_1 \supset A_2 \supset A_3 \supset ...$ Also, we have $A_k \neq \emptyset$ for all $k \in \mathbb{N}$ (J.F. Colombeau [7], Lemma (3.3.1), p. 55).

In addition to the above we have the following result:

LEMMA 2.2. For any $k \in \mathbb{N}$

$$\inf_{\varphi \in A_k} \left(\int_{\mathbb{R}^d} |\varphi(x)| dx \right) = 1. \tag{2.2}$$

More precisely, for any positive real δ there exists φ in A_k such that

$$1 \leq \int_{\mathbb{R}^d} |\varphi(x)| dx \leq 1 + \delta.$$

In addition, φ can be chosen symmetric.

PROOF. We consider the one dimensional case $d = 1$ first. Start with some fixed positive (real valued) ψ in $\mathcal{D}(\mathbb{R})$ such that $\psi(x) = 0$ for $|x| \geq 1$ and $\int_{\mathbb{R}} \psi(x)dx = 1$ (ψ can be also chosen symmetric if needed). We shall look for φ in the form:

$$\varphi(x) = \sum_{j=0}^{k} c_j \psi\Big(\frac{x}{\epsilon^j}\Big)$$

$x \in \mathbb{R}$, $\epsilon \in \mathbb{R}_+$. We have to find c_j for which $\varphi \in A_k$. Observing that

$$\int_{\mathbb{R}} x^i \psi\Big(\frac{x}{\epsilon^j}\Big)dx = \epsilon^{(i+1)j} \int_{\mathbb{R}} y^i \psi(y)dy$$

for $i = 0, 1, ..., k$, we derive the system for linear equations for c_j:

$$\begin{cases} \sum\limits_{j=0}^{k} c_j \epsilon^j = 1, \\ \Big(\int_{\mathbb{R}} y^i \psi(y)dy\Big) \sum\limits_{j=0}^{k} c_j \epsilon^{(i+1)j} = 0, \quad i = 1, ..., k. \end{cases}$$

The system is certainly satisfied, if

$$\begin{cases} \sum\limits_{j=0}^{k} c_j \epsilon^j = 1, \\ \sum\limits_{j=0}^{k} c_j \epsilon^{(1+1)j} = 0, \quad i = 1, ..., k, \end{cases}$$

which can be written in the matrix form $V_{k+1}(\epsilon)C = B$, where $V_{k+1}(\epsilon)$ is Vandermonde $(k+1) \times (k+1)$ matrix, C is the column of the unknowns c_j and B is the column whose top entry is 1 and all others are 0. For the determinant we have $\det V_{k+1}(\epsilon) \neq 0$ for $\epsilon \neq 1$, therefore, the system has a unique solution $(c_1, c_1, c_2, ..., c_k)$. Our next goal is to show that this solution is of the form:

$$c_j = \pm \frac{\epsilon^{\alpha_j}(1+\epsilon P_j(\epsilon))}{\epsilon^{\beta}(1+\epsilon P(\epsilon))} \tag{2.3}$$

where P_j and P are polynomials and

$$\alpha_j = \sum_{q=1}^{k-1} q(k+1-q) + \sum_{m=j}^{k-1} (k+1-m) \tag{2.4}$$

for $0 \leq j \leq k$, and

$$\beta = k + \sum_{q=1}^{k-1} q(k+1-q).$$

The coefficients $c_0, c_1, ..., c_k$ will be found by Cramer's rule. The formula for Vandermonde determinants gives

$$\begin{aligned} \prod_{m=1}^{k} \prod_{q=m+1}^{k+1} (\epsilon^q - \epsilon^m) &= \prod_{m=1}^{k} \Bigg(\prod_{q=m+1}^{k+1} \epsilon^m (\epsilon^{q-m} - 1) \Bigg) \\ &= \epsilon^{\beta} \prod_{m=1}^{k} \prod_{q=m+1}^{k+1} (\epsilon^{q-m} - 1) = \pm \epsilon^{\beta}(1+\epsilon P(\epsilon)) \end{aligned}$$

for some polynomial P, where

$$\beta = \sum_{m=1}^{k} m(k+1-m) = k + \sum_{m=1}^{k-1} m(k+1-m).$$

To calculate the numerator in (2.3), we have to replace the jth column of the matrix by the column B (whose top entry is 1 and all others are 0) and calculate the resulting determinant D_j. Consider first the case $1 \le j \le k-1$. By developing with respect to the jth column, we get

$$D_j = \pm \det \begin{pmatrix} 1, & \epsilon^2, & \dots & \epsilon^{2(j-1)}, & \epsilon^{2(j+1)}, & \dots & \epsilon^{2k} \\ 1, & \epsilon^3, & \dots & \epsilon^{3(j-1)}, & \epsilon^{3(j+1)}, & \dots & \epsilon^{3k} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 1, & \epsilon^{k+1}, & \dots & \epsilon^{(k+1)(j-1)}, & \epsilon^{(k+1)(j+1)}, & \dots & \epsilon^{(k+1)k} \end{pmatrix}.$$

We factor out $\epsilon^2, \epsilon^4, \dots, \epsilon^{2(j-1)}, \epsilon^{2(j+1)}, \dots, \epsilon^{2k}$ and obtain:

$$D_j = \pm \epsilon^{1\cdot 2}\epsilon^{2\cdot 2}\dots \epsilon^{(j-1)\cdot 2}\epsilon^{(j+1)\cdot 2}\dots \epsilon^{2k}$$
$$\times \det \begin{pmatrix} 1, & 1, & 1, & \dots & 1, & 1, & \dots & 1 \\ 1, & \epsilon, & \epsilon^2, & \dots & \epsilon^{(j-1)}, & \epsilon^{(j+1)}, & \dots & \epsilon^k \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 1, & \epsilon^{k-1}, & \epsilon^{2(k-1)}, & \dots & \epsilon^{(j-1)(k-1)}, & \epsilon^{(j+1)(k-1)}, & \dots & \epsilon^{k(k-1)} \end{pmatrix}.$$

The latter is a Vandermonde determinant again, and we have

$$\begin{aligned} D_j = \; & \pm \epsilon^{1\cdot 2+2\cdot 2+\dots+(j-1)2+(j+1)2+\dots k\cdot 2} \\ & \times (\epsilon-1)(\epsilon^2-1)(\epsilon^3-1)\dots(\epsilon^{j-1}-1)(\epsilon^{j+1}-1)\dots(\epsilon^k-1) \\ & \times (\epsilon^2-\epsilon)(\epsilon^3-\epsilon)\dots(\epsilon^{j-1}-\epsilon)(\epsilon^{j+1}-\epsilon)\dots(\epsilon^k-\epsilon) \\ & \times \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ & (\epsilon^{j-1}-\epsilon^{j-2})(\epsilon^{j+1}-\epsilon^{j-2})\dots(\epsilon^k-\epsilon^{j-2}) \\ & \times (\epsilon^{j+1}-\epsilon^{j-1})\dots(\epsilon^k-\epsilon^{j-1}) \\ & \dots\dots\dots\dots\dots\dots \\ & (\epsilon^k-\epsilon^{k-1}). \end{aligned}$$

Hence, factoring out $\epsilon^{(i-1)(k-i)}$ in the ith row above, we get $D_j = \pm \epsilon^{\alpha_j}(1+\epsilon P_j(\epsilon))$ for some polynomials $P_j(\epsilon)$ and

$$\begin{aligned} \alpha_j &= 1\cdot 2 + 2\cdot 2 + \dots + (j-1)\cdot 2 + (j+1)\cdot 2 + \dots + k\cdot 2 \\ &\quad + 1\cdot(k-2) + 2(k-3) + \dots + (j-1)(k-j) \\ &\quad + (j+1)(k-j-1) + \dots + (k-1)\cdot 1 \\ &= 1\cdot k + 2(k-1) + \dots + (j-1)(k-j+2) + (j+1)(k-j+1) + \dots + (k-1)\cdot 3 + k\cdot 2 \\ &= \sum_{q=1}^{j-1} q(k+1-q) + \sum_{m=j}^{k-1}(m+1)(k+1-m) = \sum_{q=1}^{k-1} q(k+1-q) + \sum_{m=j}^{k-1}(k+1-m) \end{aligned}$$

which coincides with the desired result (2.4) for α_j, in the case $1 \le j \le k-1$. For the extreme cases $j=0$ and $j=k$, we obtain

$$\alpha_0 = \sum_{m=0}^{k-1}(m+1)(k+1-m) = \sum_{q=1}^{k-1} q(k+1-q) + \sum_{m=0}^{k-1}(k+1-m)$$
$$\alpha_k = \sum_{q=1}^{k-1} q(k+1-q)$$

which both can be incorporated in the formula (2.4) for α_j. Finally, Cramer's rule gives the expression (2.3) for c_j.

Now, taking into account that $\psi \ge 0$, by assumption, and the fact that $|1+\epsilon P(\epsilon)| > |1-|\epsilon P(\epsilon)|| = 1-\epsilon|P(\epsilon)| > 0$ for all sufficiently small epsilon, we obtain

$$\int_{\mathbb{R}^d} |\varphi(x)|dx \leq \sum_{j=0}^{k} |c_j|\epsilon^j \leq \sum_{j=0}^{k} \frac{\epsilon^{j+\alpha_j}(1+\epsilon|P_j(\epsilon)|)}{\epsilon^{\beta}(1-\epsilon|P(\epsilon)|)}$$

and this latter expression can be made smaller than $1+\delta$ for sufficiently small ϵ if a) $j+\alpha_j-\beta>0$ for $0 \leq j \leq k-1$, and b) $k+\alpha_k-\beta=0$. Now, b) is obvious, as for a), we have:

$$j+\alpha_j-\beta = j+\sum_{m=j}^{k-1}(k+1-m)-k = \frac{1}{2}(k-j)(k-j+1) > 0,$$

for $0 \leq j \leq k-1$. To generalize the result for arbitrary dimension d, it suffices to consider a product of functions of one real variable. The proof is complete. □

3. NONSTANDARD DELTA FUNCTIONS

We prove the existence of a nonstandard function D in ${}^*\mathcal{D}(\mathbb{R}^d)$ with special properties. The proof is based on the result of Lemma 2.2 and the Saturation Principle (T. Todorov [6], p. 687). We also consider a type of nonstandard cut-off-functions which have close counterparts in standard analysis. The applications of these functions are left for the next sections.

LEMMA 3.1 (Nonstandard Mollifiers). For any positive infinitesimal ρ in ${}^*\mathbb{R}$ there exists a nonstandard function θ in ${}^*\mathcal{D}(\mathbb{R}^d)$ with values in ${}^*\mathbb{R}$, which is symmetric and which satisfies the following properties:

(i) $\theta(x) = 0$ for $x \in {}^*\mathbb{R}^d$, $\|x\| \geq 1$;

(ii) $\int_{{}^*\mathbb{R}^d} \theta(x)dx = 1$;

(iii) $\int_{{}^*\mathbb{R}^d} \theta(x)x^\alpha dx = 0$ for all $\alpha \in \mathbb{N}_0^d$, $\alpha \neq 0$;

(iv) $\int_{{}^*\mathbb{R}^d} |\theta(x)|dx \approx 1$;

(v) $|\ln\rho|^{-1}\left(\sup_{x\in{}^*\mathbb{R}^d} |\partial^\alpha\theta(x)|\right) \approx 0$ for all $\alpha \in \mathbb{N}_0^d$;

where $\approx$ is the infinitesimal relation in ${}^*\mathbb{C}$. We shall call this type of function *nonstandard ρ-mollifiers*.

PROOF. For any $k \in \mathbb{N}$, we define the set of test functions:

$$\begin{aligned}\overline{A}_k = \Big\{&\varphi \in \mathcal{D}(\mathbb{R}^d) : \varphi \text{ is real-valued and symmetric,}\\ &\varphi(x) = 0 \text{ for } \|x\| \geq 1,\ \int_{\mathbb{R}^d}\varphi(x)dx = 1,\\ &\int_{\mathbb{R}^d} x^\alpha\varphi(x)dx = 0 \text{ for } 1 \leq |\alpha| \leq k,\ \int_{\mathbb{R}^d}|\varphi(x)|dx < 1+\frac{1}{k}\Big\}\end{aligned}$$

and the internal subsets of ${}^*\mathcal{D}(\mathbb{R}^d)$:

$$\mathcal{A}_k = \left\{\varphi \in {}^*\left(\overline{A}_k\right) : |\ln\rho|^{-1}\left(\sup_{x\in{}^*\mathbb{R}^d}|\partial^\alpha({}^*\varphi(x))|\right) < \frac{1}{k} \text{ for } |\alpha| \leq k\right\}.$$

Obviously, we have $\overline{A}_1 \supset \overline{A}_2 \supset \overline{A}_3 \supset \ldots$ and $\mathcal{A}_1 \supset \mathcal{A}_2 \supset \mathcal{A}_3 \supset \ldots$. Also we have $\overline{A}_k \neq \emptyset$ for all $k \in \mathbb{N}$, by Lemma 2.2. On the other hand, we have $\overline{A}_k \subset \mathcal{A}_k$ in the sense that $\varphi \in \overline{A}_k$ implies ${}^*\varphi \in \mathcal{A}_k$, since

$$\sup_{x\in{}^*\mathbb{R}^d}|\partial^\alpha({}^*\varphi(x))| = \sup_{x\in\mathbb{R}^d}|\partial^\alpha\varphi(x)| = \sup_{x\leq 1}|\partial^\alpha\varphi(x)|$$

is a real (standard) number and, hence, $|\ln\rho|^{-1}\left(\sup_{x\in{}^*\mathbb{R}^d}|\partial^\alpha({}^*\varphi(x))|\right)$ is infinitesimal. Thus, we have $\mathcal{A}_k \neq \emptyset$ for all k in $\mathbb{N}$. By the Saturation Principle (T. Todorov [6], p. 687), the intersection $\mathcal{A} = \bigcap_{k\in\mathbb{N}}\mathcal{A}_k$ is non-empty and thus, any θ in $\mathcal{A}$ has the desired properties. □

DEFINITION 3.2 (ρ-Delta Function). Let ρ be a positive infinitesimal. A nonstandard function D in ${}^*\mathcal{D}(\mathbb{R}^d)$ is called a *ρ-delta function* if it takes values in ${}^*\mathbb{R}$, it is symmetric and it satisfies the following properties:

(i) $D(x) = 0 \quad \text{for} \quad x \in {}^*\mathbb{R}^d, \ \|x\| \geq \rho$,

(ii) $\int_{{}^*\mathbb{R}^d} D(x)dx = 1$,

(iii) $\int_{{}^*\mathbb{R}^d} D(x)x^\alpha dx = 0 \quad \text{for all} \quad \alpha \in \mathbb{N}_0^d, \ \alpha \neq 0$,

(iv) $\int_{{}^*\mathbb{R}^d} |D(x)|dx \approx 1$,

(v) $|\ln \rho|^{-1}\left(\rho^{d+|\alpha|} \sup_{x \in {}^*\mathbb{R}^d} |\partial^\alpha D(x)|\right) \approx 0 \quad \text{for all} \quad \alpha \in \mathbb{N}_0^d$.

THEOREM 3.3 (Existence). For any positive infinitesimal ρ in ${}^*\mathbb{R}$ there exists a ρ-delta function.

PROOF. Let θ be a nonstandard ρ-mollifier of the type described in Lemma 3.1. Then the nonstandard function D in ${}^*\mathcal{D}(\mathbb{R}^d)$, defined by

$$D(x) = \rho^{-d}\theta(x/\rho), \quad x \in {}^*\mathbb{R}^d, \tag{3.1}$$

satisfies (i)-(v). □

REMARK. The existence of nonstandard functions D in ${}^*\mathcal{D}(\mathbb{R}^d)$ with the above properties is in sharp contrast with the situation in standard analysis where there is no D in $\mathcal{D}(\mathbb{R}^d)$ which satisfies both (ii) and (iii). Indeed, if we assume that D is in $\mathcal{D}(\mathbb{R}^d)$, then (iii) implies $\widehat{D}^{(n)}(0) = 0$, for all $n = 1, 2, \ldots$, where $\widehat{D}$ denotes the Fourier transform of D. It follows $\widehat{D} = \widehat{D}(0) = c$ for some constant c since $\widehat{D}$ is an entire function on $\mathbb{C}^d$, by the Paley-Wiener Theorem (H. Bremermann [12], Theorem 8.28, p. 97). On the other hand, $D \in \mathcal{D}(\mathbb{R}^d) \subset \mathcal{S}(\mathbb{R}^d)$ implies $\widehat{D}\,|\,\mathbb{R}^d \in \mathcal{S}(\mathbb{R}^d)$ since $\mathcal{S}(\mathbb{R}^d)$ is closed under Fourier transform. Thus, it follows $c = 0$, i.e. $\widehat{D} = 0$ which implies $D = 0$ contradicting (ii).

For other classes of nonstandard delta functions we refer to (A. Robinson [15], p. 133) and to (T. Todorov [16]).

Our next task is to show the existence of an internal *cut-off function*.

NOTATIONS. Let Ω be an open set of $\mathbb{R}^d$.

1) For any $\epsilon \in \mathbb{R}_+$ we define

$$B_\epsilon = \{x \in \mathbb{R}^d : \|x\| \leq \epsilon\} \quad \text{and} \quad \Omega_\epsilon = \{x \in \Omega : d(x, \partial\Omega) \geq \epsilon\},$$

where $\|x\|$ is the Euclidean norm in $\mathbb{R}^d$, $\partial\Omega$ is the boundary of Ω and $d(x, \partial\Omega)$ is the Euclidean distance between x and $\partial\Omega$. We also denote:

$$\mathcal{D}_\epsilon(\Omega) = \{\varphi \in \mathcal{D}(\Omega) : \operatorname{supp}\varphi \subseteq B_\epsilon\}, \ E'_\epsilon(\Omega) = \{T \in E'(\Omega) : \operatorname{supp} T \subseteq \Omega_\epsilon\}.$$

2) We shall use the same notation, $*$, for the convolution operator $* : \mathcal{D}'(\mathbb{R}^d) \times \mathcal{D}(\mathbb{R}^d) \to E(\mathbb{R}^d)$ (V. Vladimirov [13]) and its nonstandard extension $* : {}^*\mathcal{D}'(\mathbb{R}^d) \times {}^*\mathcal{D}(\mathbb{R}^d) \to {}^*E(\mathbb{R}^d)$ as well as for the convolution operator $* : E'_\epsilon(\Omega) \times \mathcal{D}_\epsilon(\Omega) \to \mathcal{D}(\Omega)$, defined for all sufficiently small $\epsilon \in \mathbb{R}_+$, and for its nonstandard extension: $* : {}^*E'_\epsilon(\Omega) \times {}^*\mathcal{D}_\epsilon(\Omega) \to {}^*\mathcal{D}(\Omega)$, $\epsilon \in {}^*\mathbb{R}_+$, $\epsilon \approx 0$.

3) Let τ be the usual Euclidean topology on $\mathbb{R}^d$. We denote by $\widetilde{\Omega}$ the set of the nearstandard points in ${}^*\Omega$, i.e.

$$\widetilde{\Omega} = \bigcup_{x \in \Omega} \mu(x), \tag{3.2}$$

where $\mu(x)$, $x \in \mathbb{R}^d$, is the system of monads of the topological space $(\mathbb{R}^d, \tau)$ (T. Todorov [6], p. 687). Recall that if $\xi \in {}^*\Omega$, then $\xi \in \widetilde{\Omega}$ if and only if ξ is a finite point whose standard part belongs to Ω.

LEMMA 3.4. For any positive infinitesimal ρ in ${}^*\mathbb{R}$ there exists a function Π in ${}^*\mathcal{D}(\Omega)$ (a *ρ-cut-off function*) such that:

a) $\Pi(x) = 1$ for all $x \in \widetilde{\Omega}$;

b) $\operatorname{supp}\Pi \subseteq {}^*\Omega_\rho$, where ${}^*\Omega_\rho = \{\xi \in {}^*\Omega : {}^*d(\xi, \partial\Omega) \geq \rho\}$.

PROOF. Let ρ be a positive infinitesimal in $^*\mathbb{R}$ and D be a ρ-delta function Define the internal set $X = \{\xi \in {}^*\Omega : {}^*\|\xi\| \leq 1/\rho,\ {}^*d(\xi, \partial\Omega) \geq 2\rho\}$ and let χ be its characteristic function. Then the function $\Pi = \chi * D$ has the desired property. $\square$

4. THE ALGEBRA OF ASYMPTOTIC FUNCTIONS

We define and study the algebra ${}^\rho E(\Omega)$ of asymptotic functions on an open set Ω of $\mathbb{R}^d$. The construction of the algebra ${}^\rho E(\Omega)$, presented here, is a generalization and a refinement of the constructions in [9] and [10] (by the authors of this paper, respectively), where the algebra ${}^\rho E(\mathbb{R}^d)$ was introduced by somewhat different but equivalent definitions. On the other hand, the algebra of asymptotic functions ${}^\rho E(\Omega)$ is somewhat similar to but different from the J. F. Colombeau [7], [8] algebras of new generalized functions. This essential difference between ${}^\rho E(\Omega)$ and J. F. Colombeau's algebras of generalized functions is the properties of the generalized scalars: the scalars of the algebra ${}^\rho E(\Omega)$ constitutes an algebraically closed field (as any scalars should do) while the scalars of J. F. Colombeau's algebras are rings with zero divisors (J. F. Colombeau [8], §2.1). This improvement compared with J. F. Colombeau's theory is due to the involvement of the nonstandard analysis.

Let Ω be an open set of $\mathbb{R}^d$ and $\rho \in {}^*\mathbb{R}$ be a positive infinitesimal. We shall keep Ω and ρ *fixed* in what follows.

Following A. Robinson [2], we define:

DEFINITION 4.1 (Robinson's Asymptotic Numbers). The field of the complex Robinson ρ-asymptotic numbers is defined as the factor space ${}^\rho\mathbb{C} = \mathbb{C}_M/\mathbb{C}_0$, where

$$\mathbb{C}_M = \{\xi \in {}^*\mathbb{C} : |\xi| < \rho^{-n} \text{ for some } n \in \mathbb{N}\},$$
$$\mathbb{C}_0 = \{\xi \in {}^*\mathbb{C} : |\xi| < \rho^{n} \text{ for all } n \in \mathbb{N}\},$$

("M" stands for "moderate"). We define the embedding $\mathbb{C} \subset {}^\rho\mathbb{C}$ by $c \to q(c)$, where $q : \mathbb{C}_M \to {}^\rho\mathbb{C}$ is the quotient mapping. The field of the real asymptotic numbers is defined by ${}^\rho\mathbb{R} = q({}^*\mathbb{R} \cap \mathbb{C}_M)$.

It is easy to check that $\mathbb{C}_0$ is a maximal ideal in $\mathbb{C}_M$ and hence ${}^\rho\mathbb{C}$ is a field. Also ${}^\rho\mathbb{R}$ is a real closed totally ordered nonarchimedean field (since $^*\mathbb{R}$ is a real closed totally ordered field) containing $\mathbb{R}$ as a totally ordered subfield. Thus, it follows that ${}^\rho\mathbb{C} = {}^\rho\mathbb{R}(i)$ is an algebraically closed field, where $i = \sqrt{-1}$.

The algebra of "asymptotic functions" is, in a sense, a C^∞-counterpart of A. Robinson's asymptotic numbers ${}^\rho\mathbb{C}$:

DEFINITION 4.2 (Asymptotic Functions on Ω). (i) We define the class ${}^\rho E(\Omega)$ of the *ρ-asymptotic functions on* Ω (or simply, *asymptotic functions on* Ω if no confusion could arise) as the factor space ${}^\rho E(\Omega)$=$E_M(\Omega)/E_0(\Omega)$, where

$$E_M(\Omega) = \{f \in {}^*E(\Omega) : \partial^\alpha f(\xi) \in \mathbb{C}_M, \text{ for all } \alpha \in \mathbb{N}_0^d \text{ and all } \xi \in \widetilde{\Omega}\},$$
$$E_0(\Omega) = \{f \in {}^*E(\Omega) : \partial^\alpha f(\xi)\} \in \mathbb{C}_0, \text{ for all } \alpha \in \mathbb{N}_0^d \text{ and all } \xi \in \widetilde{\Omega}\},$$

and $\widetilde{\Omega}$ is the set of the nearstandard points of ${}^*\Omega$ (3.2). The functions in $E_M(\Omega)$ are called *ρ-moderate* (or, simply, *moderate*) and those in $E_0(\Omega)$ are called *ρ-null functions* (or, simply, *null functions*).

(ii) The pairing between ${}^\rho E(\Omega)$ and $\mathcal{D}(\Omega)$ with values in ${}^\rho\mathbb{C}$, is defined by

$$\langle Q_\Omega(f), \varphi\rangle = q\left(\int_{{}^*\Omega} f(x)\, {}^*\varphi(x) dx\right),$$

where $q : \mathbb{C}_M \to {}^\rho\mathbb{C}$ and $Q_\Omega : E_M(\Omega) \to {}^\rho E(\Omega)$ are the corresponding quotient mappings, φ is in $\mathcal{D}(\Omega)$ and ${}^*\varphi$ is its nonstandard extension.

(iii) We define the *canonical embedding* $E(\Omega) \subset {}^\rho E(\Omega)$ by the mapping $\sigma_\Omega : f \to Q_\Omega({}^*f)$, where ${}^*f$ is the nonstandard extension of f.

EXAMPLE 4.3. Let D be a nonstandard ρ-delta function in the sense of Definition 3.2. Then $D \in E_M(\mathbb{R}^d)$. In addition, $D \mid {}^*\Omega \in E_M(\Omega)$, where $D \mid {}^*\Omega$ denotes the pointwise restriction of D on ${}^*\Omega$. To show this, denote $|\ln\rho|^{-1}\left(\rho^{d+|\alpha|} \sup_{x\in {}^*\mathbb{R}^d} |\partial^\alpha D(x)|\right) = h_\alpha$ and observe that $h_\alpha \approx 0$ for all $\alpha \in \mathbb{N}_0^d$, by the definition of D. Thus, for any (finite) x in ${}^*\mathbb{R}^d$ and any $\alpha \in \mathbb{N}_0^d$ we have $|\partial^\alpha D(x)| \leq \sup_{x\in {}^*\mathbb{R}^d} |\partial^\alpha D(x)| = \frac{h_\alpha |\ln\rho|}{\rho^{d+|\alpha|}} < \rho^{-n}$, for $n = d + |\alpha| + 1$, thus, $D \in E_M(\mathbb{R}^d)$. On the other hand, $D \mid {}^*\Omega \in E_M(\Omega)$ follows immediately from the fact that $\widetilde{\Omega}$ consists of finite points in ${}^*\mathbb{R}^d$ only.

THEOREM 4.4 (Differential Algebra). (i) The class of asymptotic functions ${}^\rho E(\Omega)$ is a *differential algebra* over the field of the complex asymptotic numbers ${}^\rho\mathbb{C}$.

(ii) $E(\Omega)$ is a *differential subalgebra* of ${}^\rho E(\Omega)$ over the scalars $\mathbb{C}$ under the canonical embedding σ_Ω. In addition, σ_Ω preserves the pairing in the sense that $\langle f, \varphi\rangle = \langle \sigma_\Omega(f), \varphi\rangle$ for all f in $E(\Omega)$ and for all φ in $\mathcal{D}(\Omega)$, where $\langle f, \varphi\rangle = \int_\Omega f(x)\varphi(x)dx$ is the usual pairing between $E(\Omega)$ and $\mathcal{D}(\Omega)$.

PROOF. (i) It is clear that $E_M(\Omega)$ is a differential ring and $E_0(\Omega)$ is a differential ideal in $E_M(\Omega)$ since $\mathbb{C}_M$ is a ring and $\mathbb{C}_0$ is an ideal in $\mathbb{C}_M$ and, on the other hand, both $E_M(\Omega)$ and $E_0(\Omega)$ are closed under differential, by definition. Hence, the factor space ${}^\rho E(\Omega)$ is also a differential ring. It is clear that, $E_M(\Omega)$ is a module over the ring $\mathbb{C}_M$ and, in addition, the annihilator $\{c \in \mathbb{C}_M : cf \in E_0(\Omega), f \in E_M(\Omega)\}$ of $\mathbb{C}_M$ coincides with the ideal $\mathbb{C}_0$. Thus, ${}^\rho E(\Omega)$ becomes an algebra over the field of the complex asymptotic numbers ${}^\rho\mathbb{C}$.

(ii) Assume that $\sigma_\Omega({}^*f) = 0$ in ${}^\rho E(\Omega)$, i.e. ${}^*f \in E_0(\Omega)$. By the definition of $E_0(\Omega)$ (applied for $\alpha = 0$ and $n = 1$), it follows $f = 0$ since ${}^*f$ is an extension of f and ρ is an infinitesimal. Thus, the mapping $f \to \sigma_\Omega(f)$ is injective. It preserves the algebraic operations since the mapping $f \to {}^*f$ preserves them. The preserving of the pairing follows immediately from the fact that $\int_{{}^*\Omega} {}^*f(x)dx = \int_\Omega f(x)dx$, by the Transfer Principle (T. Todorov [6], p. 686). The proof is complete. $\square$

5. EMBEDDING OF SCHWARTZ DISTRIBUTIONS

Let Ω be (as before) an open set of $\mathbb{R}^d$. Recall that the *Schwartz embedding* $L_\Omega : \mathcal{L}_{\mathrm{loc}}(\Omega) \to \mathcal{D}'(\Omega)$ from $\mathcal{L}_{\mathrm{loc}}(\Omega)$ into $\mathcal{D}'(\Omega)$ is defined by the formula:

$$\langle L_\Omega(f), \varphi\rangle = \int_\Omega f(x)\varphi(x)dx, \quad \varphi \in \mathcal{D}(\Omega). \tag{5.1}$$

Here $\mathcal{L}_{\mathrm{loc}}(\Omega)$ denotes, as usual, the space of the locally (Lebesgue) integrable complex valued functions on Ω (V. Vladimirov [13]). The Schwartz embedding L_Ω preserves the addition and multiplication by a complex number, hence, the space $\mathcal{L}_{\mathrm{loc}}(\Omega)$ can be considered as a linear subspace of $\mathcal{D}'(\Omega)$. In addition, the restriction $L_\Omega \mid E(\Omega)$ of L_Ω on $E(\Omega)$ (often denoted also by L_Ω) preserves the partial differentiation of any order and in this sense $E(\Omega)$ is a differential linear subspace of $\mathcal{D}'(\Omega)$. In short, we have the chain of linear embeddings: $\mathcal{L}_{\mathrm{loc}}(\Omega) \subset E(\Omega) \subset \mathcal{D}'(\Omega)$.

The purpose of this section is to show that the algebra of asymptotic functions ${}^\rho E(\Omega)$ contains an isomorphic copy of the space of Schwartz distributions $\mathcal{D}'(\Omega)$ and, hence, to offer *a solution of the Problem of Multiplication of Schwartz Distributions*. This result is a generalization of some results in [9] and [10] (by the authors of this paper, respectively) where only the embedding of the tempered distributions $\mathcal{S}'(\mathbb{R}^d)$ in ${}^\rho E(\mathbb{R}^d)$ has been established. The embedding of all distributions $\mathcal{D}'(\Omega)$, discussed here, presents an essentially different situation.

The spaces $\widetilde{E}(\Omega)$ and $\widetilde{\mathcal{D}}(\Omega)$, defined below, are immediate generalizations of the spaces $\widetilde{E}(\mathbb{R}^d)$ and $\widetilde{\mathcal{D}}(\mathbb{R}^d)$, introduced in (K. D. Stroyan and W. A. Luxemburg [17], (10.4), p. 299):

$$\widetilde{E}(\Omega) = \{\varphi \in {}^*E(\Omega) : \partial^\alpha\varphi(x) \text{ is a finite number in } {}^*\mathbb{C} \text{ for all } x \in \widetilde{\Omega} \text{ and all } \alpha \in \mathbb{N}_0^d\}, \tag{5.2}$$

$$\widetilde{\mathcal{D}}(\Omega) = \{\varphi \in {}^*\mathcal{E}(\Omega) : \ \partial^\alpha\varphi(x) \text{ is a finite number in } {}^*\mathbb{C} \text{ for all } x \in \widetilde{\Omega},\ \alpha \in \mathbb{N}_0^d \text{ and } \varphi(x) = 0 \text{ for all } x \in {}^*\Omega\backslash\widetilde{\Omega}\}, \tag{5.3}$$

Obviously, we have $\widetilde{\mathcal{D}}(\Omega) \subset \widetilde{\mathcal{E}}(\Omega) \subset \mathcal{E}_M(\Omega)$. Notice as well that $\varphi \in \widetilde{\mathcal{D}}(\Omega)$ implies $\varphi \in {}^*\mathcal{D}(G)$ for some open relatively compact set G of Ω. We have also the following simple result:

LEMMA 5.1. If $T \in \mathcal{D}'(\Omega)$ and $\varphi \in \mathcal{E}_0(\Omega) \bigcap \widetilde{\mathcal{D}}(\Omega)$, then $\langle {}^*T, \varphi\rangle \in \mathbb{C}_0$.

PROOF. Observe that $\mathcal{E}_0(\Omega) \bigcap \widetilde{\mathcal{D}}(\Omega)$ implies $\varphi \in \mathcal{E}_0(\Omega) \bigcap {}^*\mathcal{D}(G)$ for some open relatively compact set G of Ω. By the continuity of T (and Transfer Principle) there exist constants $M \in \mathbb{R}_+$ and $m \in \mathbb{N}_0$ such that

$$|\langle {}^*T, \varphi\rangle| \leq M \sum_{|\mu|\leq m} \sup_{x\in{}^*G} |\partial^\mu\varphi(x)|.$$

On the other hand, $M \sum_{|\mu|\leq m} \sup_{x\in{}^*G} |\partial^\mu\varphi(x)| < \rho^n$ for all $n \in \mathbb{N}$, since $\varphi \in \mathcal{E}_0(\Omega)$, by assumption. Thus, $|\langle {}^*T, \varphi\rangle| < \rho^n$ for all $n \in \mathbb{N}$. □

Let D be a ρ-delta function in the sense of Definition 3.2. We shall keep D (along with Ω and ρ) *fixed* in what follows.

DEFINITION 5.2 (*Embedding of Schwartz Distributions*). We define the embedding $\mathcal{D}'(\Omega) \subset {}^\rho\mathcal{E}(\Omega)$ by $\Sigma_{D,\Omega} : T \to Q_\Omega(({}^*T\,\Pi_\Omega) * D)$, where ${}^*T$ is the nonstandard extension of T, Π_Ω is a (an arbitrarily chosen) ρ-cut-off function for Ω (Lemma 3.4), ${}^*T\,\Pi_\Omega$ is the Schwartz product between ${}^*T$ and Π_Ω in ${}^*\mathcal{D}'(\Omega)$ (defined by Transfer Principle), $*$ is the convolution operator and $Q_\Omega : \mathcal{E}_M(\Omega) \to {}^\rho\mathcal{E}(\Omega)$ is the quotient mapping in the definition of ${}^\rho\mathcal{E}(\Omega)$ (Definition 4.2).

The cut-off function Π_Ω can be dropped in the above definition, i.e. $\Sigma_{D,\Omega} : T \to Q_\Omega({}^*T * D)$, in some particular cases; e.g. when:

a) T has a compact support in Ω;

b) $\Omega = \mathbb{R}^d$.

PROPOSITION 5.3 (Correctness). $T \in \mathcal{D}'(\Omega)$ implies $({}^*T\,\Pi_\Omega) * D \in \mathcal{E}_M(\Omega)$.

PROOF. Choose $\alpha \in \mathbb{N}_0^d$ and all $x \in \widetilde{\Omega}$. Since we have $\partial^\alpha((\Pi_\Omega\,{}^*T) * D)(x) = (\partial^\alpha({}^*T) * D)(x)$ (by the definition of Π_Ω), we need to show that $\partial^\alpha({}^*T * D)(x) \in \mathbb{C}_M$ only, i.e. that $|\partial^\alpha({}^*T * D)(x)| < \rho^{-m}$ for some $m \in \mathbb{N}$ (m might depend on α). We start with the case $\alpha = 0$ Denote $D_x(\xi) = D(\xi - x)$, $\xi \in {}^*\mathbb{R}$ and observe that $\mathrm{supp}(D_x) \subseteq {}^*G$ for some open relatively compact set G of Ω, since D_x vanishes on ${}^*\Omega\backslash\widetilde{\Omega}$. Next, by the continuity of T (and the Transfer Principle), there exist constants $m \in \mathbb{N}_0$ and $M \in \mathbb{R}_+$ such that

$$|({}^*T * D)(x)| = |\langle {}^*T, D_x \,|\, {}^*\Omega\rangle| \leq M \sum_{|\mu|\leq m} \sup_{\xi\in{}^*G} \left|\partial_\xi^\mu D(x - \xi)\right|.$$

Finally, there exists $n \in \mathbb{N}$ such that $\sum_{|\mu|\leq m} \sup_{\xi\in{}^*G} \left|\partial_\xi^\mu D(x - \xi)\right| < \rho^{-n}$, since $D \,|\, {}^*G$ is a ρ-moderate function (Example 4.3). Combining these arguments, we have: $|({}^*T * D)(x)| \leq M\rho^{-n} < \rho^{-(n+1)}$, as required. The generalization for arbitrary multiindex α follows immediately since $\partial^\alpha({}^*T * D) = (\partial^\alpha({}^*T)) * D = {}^*(\partial^\alpha T) * D$, by Transfer Principle, a $\partial^\alpha T$ is (also) in $\mathcal{D}'(\Omega)$. □

PROPOSITION 5.4. $f \in \widetilde{\mathcal{E}}(\Omega)$ implies $(f\,\Pi_\Omega) * D - f \in \mathcal{E}_0(\Omega)$.

PROOF. Let $x \in \widetilde{\Omega}$ and $\alpha \in \mathbb{N}_0^d$. Since we have $\partial^\alpha[((f\,\Pi_\Omega) * D)(x) - f(x)] = \partial^\alpha[(f * D)(x) - f(x)]$ (by the definition of Π_Ω), we need to show that $\partial^\alpha[(f * D)(x) - f(x)] \in \mathbb{C}_0$ only. Choose $n \in \mathbb{N}$. We need to show that $|\partial^\alpha[(f * D)(x) - f(x)]| < \rho^n$. We start first with the case $\alpha = 0$. By Taylor's formula (applied by transfer), we have

$$f(x-\xi)-f(x)=\sum_{|\beta|=1}^{n}\frac{(-1)^{|\beta|}\partial^{\beta}f(x)}{\beta!}\,\xi^{\beta}+\frac{(-1)^{n+1}}{(n+1)!}\sum_{|\beta|=n+1}\partial^{\beta}f(\eta(\xi))\xi^{\beta}$$

for any $\xi \in \widetilde{\Omega}$, where $\eta(\xi)$ is a point in ${}^*\Omega$ "between x and ξ." Notice that the point $\eta(\xi)$ is also in $\widetilde{\Omega}$ It follows

$$(f*D)(x)-f(x)=\int_{\|\xi\|\le\rho}D(\xi)[f(x-\xi)-f(x)]d\xi=\frac{(-1)^{n+1}}{(n+1)!}\sum_{|\beta|=n+1}\int_{\|\xi\|\le\rho}D(\xi)\xi^{\beta}\partial^{\beta}f(\eta(\xi))d\xi,$$

since $\int_{\|\xi\|\le\rho}D(\xi)\xi^{\beta}d\xi=0$, by the definition of D. Thus, we have

$$|(f*D)(x)-f(x)|\le\frac{\rho^{n+1}}{(n+1)!}\left(\int_{{}^*\mathbb{R}^d}|D(x)|dx\right)\left(\sum_{|\beta|=n+1}\sup_{\|\xi\|\le\rho}\left|\partial^{\beta}f(\eta(\xi))\right|\right)<\rho^{n},$$

as desired, since, on one hand, $\int_{{}^*\mathbb{R}^d}|D(x)|dx\approx 1$, by the definition of D and on the other hand, the above sum is a finite number because ${}^*\partial^{\beta}f(\eta(\xi))$ are all finite due to $\eta(\xi)\in\widetilde{\Omega}$. The generalization for an arbitrary α is immediate since $\partial^{\alpha}[(f*D)(x)-f(x)]=(\partial^{\alpha}f*D)(x)-\partial^{\alpha}f(x)$, by the Transfer Principle. □

COROLLARY 5.5. (i) $f\in\mathcal{E}(\Omega)$ implies $({}^*f\,\Pi_\Omega)*D-{}^*f\in\mathcal{E}_0(\Omega)$.

(ii) $\varphi\in\mathcal{D}(\Omega)$ implies $({}^*\varphi\,\Pi_\Omega)*D-{}^*\varphi\in\mathcal{E}_0(\Omega)\bigcap\widetilde{\mathcal{D}}(\Omega)$.

PROOF. (i) follows immediately from the above proposition since $f\in\mathcal{E}(\Omega)$ implies ${}^*f\in\widetilde{\mathcal{E}}(\Omega)$.

(ii) Both $({}^*\varphi\,\Pi_\Omega)*D$ and ${}^*\varphi$ vanish on ${}^*\Omega\backslash\widetilde{\Omega}$ since their supports are within an open relatively compact neighborhood G of supp(φ) and the latter is a compact set of Ω, by assumption. Thus,

$$({}^*\varphi\,\Pi_\Omega)*D-{}^*\varphi\in{}^*\mathcal{D}(G)\subset\widetilde{\mathcal{D}}(\Omega),\quad\text{as required.}\quad\square$$

Denote $\check{D}(x)=D(-x)$ and recall that $\check{D}=D$ since D is symmetric (Definition 3.2).

PROPOSITION 5.6. If $T\in\mathcal{D}'(\Omega)$ and $\varphi\in\mathcal{D}(\Omega)$, then

$$\int_{{}^*\Omega}(({}^*T\,\Pi_\Omega)*D)(x)\,{}^*\varphi(x)dx-\langle T,\varphi\rangle\in\mathbb{C}_0.$$

PROOF. Using the properties of the convolution operator (applied by transfer), we have

$$\begin{aligned}&\int_{{}^*\Omega}(({}^*T\,\Pi_\Omega)*D)(x)\,{}^*\varphi(x)dx-\langle T,\varphi\rangle\\&\quad=\langle({}^*T\,\Pi_\Omega)*D,{}^*\varphi\rangle-\langle{}^*T,{}^*\varphi\rangle=\langle{}^*T\,\Pi_\Omega,{}^*\varphi*\check{D}\rangle-\langle{}^*T\,\Pi_\Omega,{}^*\varphi\rangle\\&\quad=\langle{}^*T\,\Pi_\Omega,{}^*\varphi*\check{D}-{}^*\varphi\rangle=\langle{}^*T,{}^*\varphi*D-{}^*\varphi\rangle\in\mathbb{C}_0,\end{aligned}$$

by Lemma 5.1 since ${}^*\varphi*D-{}^*\varphi\in\mathcal{E}_0(\Omega)\bigcap\widetilde{\mathcal{D}}(\Omega)$, by Corollary 5.5. □

We are ready to state our *main result*:

THEOREM 5.7 (Properties of $\Sigma_{D,\Omega}$). (i) $\Sigma_{D,\Omega}$ *preserves the pairing* in the sense that for all T in $\mathcal{D}'(\Omega)$ and all φ in $\mathcal{D}(\Omega)$ we have $\langle T,\varphi\rangle=\langle\Sigma_{D,\Omega}(T),\varphi\rangle$, where the left hand side is the (usual) pairing of T and φ in $\mathcal{D}'(\Omega)$, while the right hand side is the pairing of $\Sigma_{D,\Omega}(T)$ and φ in ${}^\rho\mathcal{E}(\Omega)$ (Definition 4.2).

(ii) $\Sigma_{D,\Omega}$ is *injective* and it *preservers all linear operations* in $\mathcal{D}'(\Omega)$: the addition, multiplication by (standard) complex numbers and the partial differentiation of any (standard) order.

(iii) $\Sigma_{D,\Omega}$ is *an extension of the canonical embedding* σ_Ω defined earlier in Definition 4.2 in the sense that $\sigma_\Omega=\Sigma_{D,\Omega}\circ L_\Omega$, where L_Ω is the *Schwartz embedding* (5.1) restricted on $\mathcal{E}(\Omega)$ and $\circ$ denotes *composition.* Or, equivalently, the following diagram is commutative:

$$\begin{array}{ccc} & & \mathcal{D}'(\Omega) \\ & L_\Omega \nearrow & \\ \mathcal{E}(\Omega) & & \downarrow \Sigma_{D,\Omega} \\ & \sigma_\Omega \searrow & \\ & & {}^\rho\mathcal{E}(\Omega). \end{array} \tag{5.4}$$

PROOF. (i) Denote (as before) $\check{D}(x) = D(-x)$ and recall that $\check{D}(x) = D$ (Definition 3.2). We have

$$\begin{aligned} \langle \Sigma_{D,\Omega}(T), \varphi \rangle &= \langle Q_\Omega(({}^*T\Pi_\Omega) * D), \varphi \rangle - \langle T, \varphi \rangle \\ &= q\left(\int_{{}^*\Omega} ((\Pi_\Omega {}^*T) * D)(x)\, {}^*\varphi(x) dx \right) - q(\langle T, \varphi \rangle) \\ &= q\left(\int_{{}^*\Omega} (({}^*T\Pi_\Omega) * D)(x)\, {}^*\varphi(x) dx - \langle T, \varphi \rangle \right) = 0, \end{aligned}$$

because $\int_{{}^*\Omega} (({}^*T\Pi_\Omega) * D)(x)\ {}^*\varphi(x) dx - \langle T, \varphi \rangle \in \mathbb{C}_0$, by Proposition 5.6. Here $\langle T, \varphi \rangle = q(\langle T, \varphi \rangle)$ holds because $\langle T, \varphi \rangle$ is a standard (complex) number.

(ii) The injectivity of $\Sigma_{D,\Omega}$ follows from (i): $\Sigma_{D,\Omega}(T) = 0$ in ${}^\rho\mathcal{E}(\Omega)$ implies $\langle \Sigma_{D,\Omega}(T), \varphi \rangle = 0$ for all $\varphi \in \mathcal{D}(\Omega)$, which is equivalent to $\langle T, \varphi \rangle = 0$ for all $\varphi \in \mathcal{D}(\Omega)$, by (i), thus, $T = 0$ in $\mathcal{D}'(\Omega)$, as required. The preserving of the linear operations follows from the fact that both the extension mapping * and the convolution $*$ (applied by Transfer Principle) are linear operators.

(iii) For any $f \in \mathcal{E}(\Omega)$ we have $\sigma(f) = Q_\Omega({}^*f) = Q_\Omega(({}^*f\Pi_\Omega) * D) = Q_\Omega(({}^*L(f)\Pi_\Omega) * D) = \Sigma_{D,\Omega}(L(f))$, as required, since ${}^*f - ({}^*f\Pi_\Omega) * D \in \mathcal{E}_0(\Omega)$, by Corollary 5.5. □

REMARK 5.8 (Multiplication of Distributions). As a consequence of the above result, the Schwartz distributions in $\mathcal{D}'(\Omega)$ can be multiplied within the associative and commutative differential algebra ${}^\rho\mathcal{E}(\Omega)$ (something impossible in $\mathcal{D}'(\Omega)$ itself). By the property (iii) above, the multiplication in ${}^\rho\mathcal{E}(\Omega)$ coincides on $\mathcal{E}(\Omega)$ with the usual (pointwise) multiplication in $\mathcal{E}(\Omega)$. Thus, the class ${}^\rho\mathcal{E}(\Omega)$, endowed with an embedding $\Sigma_{D,\Omega}$, presents *a solution of the problem of multiplication of Schwartz distributions* which, in a sense, is optimal, in view of the Schwartz impossibility results (L. Schwartz [1]) (for a discussion we refer also to J. F. Colombeau [7], §2.4 and M. Oberguggenberg [18], §2). We should mention that the existence of an embedding of $\mathcal{D}'(\mathbb{R}^d)$ into ${}^\rho\mathcal{E}(\mathbb{R}^d)$ can be proved also by sheaf-theoretical arguments as indicated in (M. Oberguggenberger [18], §23).

REMARK 5.9 (Nonstandard Asymptotic Analysis). We sometimes refer to the area connected directly or indirectly with the fields ${}^\rho\mathbb{R}$ as *Nonstandard Asymptotic Analysis*. The fields ${}^\rho\mathbb{R}$ were introduced by A. Robinson [2] and are sometimes known as "Robinson's nonarchimedean valuation fields." The terminology "Robinson's asymptotic numbers," chosen in this paper, is due to the role of ${}^\rho\mathbb{R}$ for the asymptotic expansions of classical functions (A. H. Lightstone and A. Robinson [3]) and also to stress the fact that in our approach ${}^\rho\mathbb{C}$ plays the role of the scalars of the algebra ${}^\rho\mathcal{E}(\Omega)$. Linear spaces over the field ${}^\rho\mathbb{R}$ has been studied by W. A. J. Luxemburg [19] in order to establish a connection between nonstandard and nonarchimedean analysis. More recently ${}^\rho\mathbb{R}$ has been used by V. Pestov [20] for studying Banach spaces. The field ${}^\rho\mathbb{R}$ has been exploited by Li Bang-He [21] for multiplication of Schwartz distributions.

REFERENCES

[1] SCHWARTZ, L., Sur l'impossibilité de la multiplication des distributions, *C.R. Acad. Sci.*, Paris, **239** (1954), 847-848.

[2] ROBINSON, A., Function theory on some nonarchimedean fields, *Amer. Math. Monthly* **80** (6), Part II: Papers in the Foundations of Mathematics (1973), 87-109.

[3] LIGHTSTONE, A.H. and ROBINSON, A., *Nonarchimedean Fields and Asymptotic Expansions*, North-Holland Mathematical Library, 13, North Holland, Amsterdam-Oxford/American Elsevier, New York, 1975.

[4] OBERGUGGENBERGER, M., Contributions of nonstandard analysis to partial differential equations, in *Development in Nonstandard Mathematics* (Eds. N.J. Cutland, V. Neves, F. Oliveira and J. Sousa-Pinto), Longman-Harlow, 1995, 130-150.

[5] TODOROV, T., An existence result for a class of partial differential equations with smooth coefficients, in *Advances in Analysis, Probability and Mathematical Physics - Contributions to Nonstandard Analysis* (Eds. S. Albeverio, W.A.J. Luxemburg and M.P.H. Wolff), Kluwer Academic Publishers (Mathematics and Its Applications), Dordrecht/Boston/London, Vol. 314, 1995, 107-121.

[6] TODOROV, T., An existence of solutions for linear partial differential equations with C^∞-coefficients in an algebra of generalized functions, in *Trans. Am. Math. Soc.*, **348**, 2 (1996).

[7] COLOMBEAU, J.F., *New Generalized Functions and Multiplication of Distributions*, North-Holland Math. Studies 84, North-Holland, Amsterdam, 1984.

[8] COLOMBEAU, J.F., *Elementary Introduction to New Generalized Functions*, North-Holland, Amsterdam, 1985.

[9] OBERGUGGENBERGER, M., Products of distributions: nonstandard methods, *Z. Anal. Anwendungen* 7 (1988), 347-365., Corrections: ibid. **10** (1991), 263-264.

[10] TODOROV, T., Colombeau's new generalized function and non-standard analysis, in *Generalised Functions, Convergence Structures and their Applications* (Eds. B. Stankovic, E. Pap, S. Pilipovic and V.S. Vladimirov), Plenum Press, New York (1988), 327-339.

[11] HOSKINS, R.F. and PINTO, J.S., Nonstandard treatments of new generalised functions, in *Generalized Functions and Their Applications* (Ed. R.S. Pathak), Plenum Press, New York, 1993, 95-108.

[12] BREMERMANN, H., *Distributions, Complex Variables, and Fourier Transforms*, Addison-Wesley Publ. Co., Inc., Pal Alto, 1965.

[13] VLADIMIROV, V., *Generalized Functions in Mathematical Physics*, Mir-Publisher, Moscow, 1979.

[14] LINDSTROM, T., An invitation to nonstandard analysis, in *Nonstandard Analysis and its Applications* (Ed. N. Cutland), London Mathematical Society Student Texts 10, Cambridge University Press, Cambridge, 1988, 1-105.

[15] ROBINSON, A., *Non-Standard Analysis*, North-Holland, Amsterdam, 1966.

[16] TODOROV, T., A nonstandard delta function, *Proc. Amer. Math. Soc.*, **110** (1990), 1143-1144.

[17] STROYAN, K.D. and LUXEMBURG, W.A.J., *Introduction to the Theory of Infinitesimals*, Academic Press, New York, 1976.

[18] OBERGUGGENBERGER, M., Multiplication of distributions and applications to partial differential equations, *Pitman Research Notes Math.*, **259**, Longman, Harlow, 1992.

[19] LUXEMBURG, W.A.J., On a class of valuation fields introduced by A. Robinson, *Israel Journal Math.*, **25** (1976), 189-201.

[20] PESTOV, V., On a valuation field invented by A. Robinson and certain structures connected with it, *Israel Journal Math.*, **74** (1991), 65-79.

[21] BANG-HE, L., Non-standard analysis and multiplication of distributions, *Sci. Sinica*, **21** (1978), 561-585.