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A NONSTANDARD DELTA FUNCTION

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ABSTRACT. We prove that the Dirac delta distribution has a kernel in the class of the pointwise nonstandard functions.

The purpose of this note is to prove the existence of a nonstandard function $\Delta: {}^*\mathbb{R}^n \to {}^*\mathbb{C}$ such that

$$\int_{{}^*\mathbb{R}^n} \Delta(x)\,{}^*\varphi(x)\,dx = \varphi(0)$$

for all $\varphi \in C^0$. Here $C^0 \equiv C^0(\mathbb{R}^n)$ is the class of the continuous complex-valued functions defined by $\mathbb{R}^n$, ${}^*\mathbb{R}$ and ${}^*\mathbb{C}$ are the sets of the nonstandard real and nonstandard complex numbers, respectively, and ${}^*\varphi: {}^*\mathbb{R}^n \to {}^*\mathbb{C}$ is the nonstandard extension of φ. For examples of nonstandard functions Δ for which (1) holds merely "up to infinitesimals," we refer the reader to one of the many texts on nonstandard analysis, e.g. [2, p. 300]. Recall that there does not exist a standard function Δ with the property mentioned above.

In what follows, we shall work in a nonstandard model with a set of individuals S that contains the complex numbers $\mathbb{C}$ and degree of saturation k larger than 2^κ for $\kappa = \operatorname{card} C^0$. In particular, any polysaturated model of $\mathbb{C}$ will do [2].

Notation. For any $\varphi \in C^0$, we define the functional $F_\varphi: \mathscr{D} \to \mathbb{C}$ by

$$F_\varphi(f) = \int_{\mathbb{R}^n} f(x)\varphi(x)\,dx, \quad f \in \mathscr{D}, \tag{2}$$

where $\mathscr{D} \equiv C_0^\infty(\mathbb{R}^n)$ is the class of all C^∞-functions on $\mathbb{R}^n$ with compact support. We write $\ker F_\varphi$ for the kernel of F_φ. For the nonstandard extension ${}^*F_{*\varphi}: {}^*\mathscr{D} \to {}^*\mathbb{C}$ of F_φ for $\varphi \in C^0$, we have the $*$-integral representation

$${}^*F_{*\varphi}(f) = \int_{{}^*\mathbb{R}^n} f(x)\,{}^*\varphi(x)\,dx, \quad f \in {}^*\mathscr{D}. \tag{3}$$

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Lemma. *For any* $k \in \mathbb{N}$ *and any* $\varphi_i \in C^0$, $i = 1, 2, \ldots, k$, *the system of equations*

$$F_{\varphi_i}(f) = \varphi_i(0), \qquad i = 1, 2, \ldots, k, \tag{4}$$

has a solution f *in* $\mathscr{D}$.

Proof. Consider first the case $k = 1$ of one equation:

$$F_\varphi(f) = \varphi(0). \tag{5}$$

If $\varphi = 0$, then any f in $\mathscr{D}$ is a solution of (5). If $\varphi \neq 0$, then the set $\Phi \equiv \mathscr{D} - \ker F_\varphi$ is nonempty and the function

$$f = \frac{\varphi(0)}{F_\varphi(g)} g$$

satisfies (5) for any choice of $g \in \Phi$. Assume, now, that the statement is true for $k-1$. If $\varphi_1, \ldots, \varphi_k$ are linearly dependent in C^0, then (4) is equivalent to a system of $k-1$ equations and, by assumption, has a solution. If $\varphi_1, \ldots, \varphi_k$ are linearly independent, then the sets

$$\Phi_i \equiv \left(\bigcap_{j=1}^{k} \ker F_{\varphi_j} \right) - \ker F_{\varphi_i}, \qquad i = 1, 2, \ldots, k,$$

are nonempty [1, vol. 3, Lemma 10, p. 421], and we can pick $g_i \in \Phi_i$. Now, the function

$$f = \sum_{i=1}^{k} \frac{\varphi_i(0)}{F_{\varphi_i}(g_i)} g_i$$

is obviously a solution of (4). The proof is complete. □

Proposition. *There exists a nonstandard function* $\Delta \in {}^*\mathscr{D}$ *for which* (1) *holds for all* $\varphi \in C^0$.

Proof. Define the family $\mathscr{A}_\varphi$, $\varphi \in C^0$, of subsets of $\mathscr{D}$ by

$$\mathscr{A}_\varphi = \{ f \in \mathscr{D} : F_\varphi(f) = \varphi(0) \},$$

and observe that, by the above lemma, it has the finite intersection property. Hence, by the saturation principle [2, 7.4.2(b), p. 181], the intersection

$$\mathscr{A} \equiv \bigcap_{\varphi \in C^0} {}^*\mathscr{A}_\varphi$$

is nonempty, where ${}^*\mathscr{A}_\varphi = \{ f \in {}^*\mathscr{D} : {}^*F_{{}^*\varphi}(f) = \varphi(0) \}$. Hence every $\Delta \in \mathscr{A}$ has the desired property. The proof is complete. □

References

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