A NONSTANDARD DELTA FUNCTION

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Abstract. We prove that the Dirac delta distribution has a kernel in the class of the pointwise nonstandard functions.

The purpose of this note is to prove the existence of a nonstandard function \( \Delta: \mathbb{R}^n \rightarrow \mathbb{C} \) such that

\[
\int_{\mathbb{R}^n} \Delta(x) \phi(x) \, dx = \phi(0)
\]

for all \( \phi \in C^0 \). Here \( C^0 \equiv C^0_0(\mathbb{R}^n) \) is the class of the continuous complex-valued functions defined by \( \mathbb{R}^n \), \( \mathbb{R} \) and \( \mathbb{C} \) are the sets of the nonstandard real and nonstandard complex numbers, respectively, and \( \phi: \mathbb{R}^n \rightarrow \mathbb{C} \) is the nonstandard extension of \( \phi \). For examples of nonstandard functions \( \Delta \) for which (1) holds merely "up to infinitesimals," we refer the reader to one of the many texts on nonstandard analysis, e.g. [2, p. 300]. Recall that there does not exist a standard function \( \Delta \) with the property mentioned above.

In what follows, we shall work in a nonstandard model with a set of individuals \( S \) that contains the complex numbers \( \mathbb{C} \) and degree of saturation \( k \) larger than \( 2^\kappa \) for \( \kappa = \text{card} \, C^0 \). In particular, any polysaturated model of \( \mathbb{C} \) will do [2].

Notation. For any \( \phi \in C^0 \), we define the functional \( F_\phi: \mathcal{D} \rightarrow \mathbb{C} \) by

\[
F_\phi(f) = \int_{\mathbb{R}^n} f(x) \phi(x) \, dx, \quad f \in \mathcal{D},
\]

where \( \mathcal{D} \equiv C_0^\infty(\mathbb{R}^n) \) is the class of all \( C^\infty \)-functions on \( \mathbb{R}^n \) with compact support. We write \( \ker F_\phi \) for the kernel of \( F_\phi \). For the nonstandard extension \( \mathcal{D} : \mathcal{D} \rightarrow \mathbb{C} \) of \( F_\phi \) for \( \phi \in C^0 \), we have the \( \ast \)-integral representation

\[
\ast F_\phi(f) = \int_{\mathbb{R}^n} f(x) \ast \phi(x) \, dx, \quad f \in \ast \mathcal{D}.
\]
Lemma. For any $k \in \mathbb{N}$ and any $\varphi_i \in C^0$, $i = 1, 2, \ldots, k$, the system of equations

$$(4) \quad F_{\varphi_i}(f) = \varphi_i(0), \quad i = 1, 2, \ldots, k,$$

has a solution $f$ in $\mathcal{D}$.

Proof. Consider first the case $k = 1$ of one equation:

$$(5) \quad F_{\varphi}(f) = \varphi(0).$$

If $\varphi = 0$, then any $f$ in $\mathcal{D}$ is a solution of (5). If $\varphi \neq 0$, then the set $\Phi \equiv \mathcal{D} - \ker F_{\varphi}$ is nonempty and the function

$$f = \frac{\varphi(0)}{F_{\varphi}(g)}g$$

satisfies (5) for any choice of $g \in \Phi$. Assume, now, that the statement is true for $k - 1$. If $\varphi_1, \ldots, \varphi_k$ are linearly dependent in $C^0$, then (4) is equivalent to a system of $k - 1$ equations and, by assumption, has a solution. If $\varphi_1, \ldots, \varphi_k$ are linearly independent, then the sets

$$\Phi_i \equiv \left( \bigcap_{j=1}^{k} \ker F_{\varphi_j} \right) - \ker F_{\varphi_i}, \quad i = 1, 2, \ldots, k,$$

are nonempty [1, vol. 3, Lemma 10, p. 421], and we can pick $g_i \in \Phi_i$. Now, the function

$$f = \sum_{i=1}^{k} \frac{\varphi_i(0)}{F_{\varphi_i}(g_i)}g_i$$

is obviously a solution of (4). The proof is complete. □

Proposition. There exists a nonstandard function $\Delta \in^* \mathcal{D}$ for which (1) holds for all $\varphi \in C^0$.

Proof. Define the family $\mathcal{A}_\varphi$, $\varphi \in C^0$, of subsets of $\mathcal{D}$ by

$$\mathcal{A}_\varphi = \{ f \in \mathcal{D} : F_{\varphi}(f) = \varphi(0) \},$$

and observe that, by the above lemma, it has the finite intersection property. Hence, by the saturation principle [2, 7.4.2(b), p. 181], the intersection

$$\mathcal{A} \equiv \bigcap_{\varphi \in C^0} \ ^*\mathcal{A}_\varphi$$

is nonempty, where $\ ^*\mathcal{A}_\varphi = \{ f \in^* \mathcal{D} : F_{\varphi}(f) = \varphi(0) \}$. Hence every $\Delta \in \mathcal{A}$ has the desired property. The proof is complete. □

References


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