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POINTWISE KERNELS OF SCHWARTZ DISTRIBUTIONS

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ABSTRACT. We show that Schwartz distributions have kernels in the class of the pointwise nonstandard functions.

The main purpose of this note is to show that every Schwartz distribution $T \in \mathscr{D}'$ has a kernel $f\colon {}^*\mathbb{R}^d \to {}^*\mathbb{C}$ in the class of the pointwise nonstandard functions in the sense that

$$(1) \qquad \langle T, \varphi \rangle = \int_{{}^*\mathbb{R}^d} f(x)^*\varphi(x)\,dx$$

for all $\varphi \in \mathscr{D}$, where ${}^*\varphi$ is the nonstandard extension of φ. Recall that, in general, the Schwartz distributions do not have kernels in the class of the standard pointwise functions (Schwartz [2]).

We denote the usual classes of the C^∞-functions, C^∞-functions with compact supports, and continuous complex-valued functions defined on $\mathbb{R}^d$ (d is a natural number) by $\mathscr{E} \equiv \mathscr{E}(\mathbb{R}^d) \equiv C^\infty(\mathbb{R}^d)$, $\mathscr{D} \equiv \mathscr{D}(\mathbb{R}^d) \equiv C_0^\infty(\mathbb{R}^d)$, and $C^0 \equiv C^0(\mathbb{R}^d) \equiv C(\mathbb{R}^d)$, respectively, and the class of Schwartz distributions by $\mathscr{D}' \equiv \mathscr{D}'(\mathbb{R}^d)$. Let $\mathscr{P}$ be the ring of standard complex-valued polynomials defined on $\mathbb{R}^d$. As usual, $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{C}$ will be the systems of the natural, real, and complex numbers, respectively, and we use also the notations $\mathbb{N}_0 \equiv \{0\} \cup \mathbb{N}$ and $\check{f}(x) \equiv f(-x)$.

In what follows, we shall work in a nonstandard model with a set of individuals J that contains the complex numbers $\mathbb{C}$ and degree of saturation k larger than 2^κ for $\kappa \equiv \operatorname{card} C^0$. In particular, any polysaturated model of $\mathbb{C}$ will suffice (Stroyan and Luxemburg [3]). If X is a set of complex numbers or a set of (standard) functions, then ${}^*X$ will be its nonstandard extension, and if $f\colon X \to Y$ is a (standard) mapping, then ${}^*f\colon {}^*X \to {}^*Y$ will be its nonstandard extension. We shall use the same notation, $*$, for the convolution operator $*\colon \mathscr{D}' \times \mathscr{D} \to \mathscr{E}$ and for its nonstandard extension $*\colon {}^*\mathscr{D}' \times {}^*\mathscr{D} \to {}^*\mathscr{E}$.

Lemma. *There exists Δ in ${}^*\mathscr{D}$ such that for all φ in C^0 we have*

$$\int_{{}^*\mathbb{R}^d} \Delta(x)^*\varphi(x)\,dx = \int_{{}^*\mathbb{R}^d} \check{\Delta}(x)^*\varphi(x)\,dx = \varphi(0).$$

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For the proof we refer the reader to [4].

Proposition. *If T is a Schwartz distribution, then* (1) *holds for $f = {}^*T * \Delta$ and all φ in $\mathscr{D}$.*

Proof. Using the properties of the convolution operator, the transfer principle, and the above lemma (since $T * \check{\varphi}$ is in $\mathscr{E}$), we obtain

$$\begin{aligned}\int_{{}^*\mathbb{R}^d} ({}^*T * \Delta)(x)\,{}^*\varphi(x)\,dx &= \langle {}^*T * \Delta, {}^*\varphi\rangle = \langle {}^*T, {}^*\varphi * \check{\Delta}\rangle \\ &= \langle {}^*T * ({}^*\check{\varphi}), \check{\Delta}\rangle = \int_{{}^*\mathbb{R}^d} \check{\Delta}(x)\,{}^*(T * \check{\varphi})(x)\,dx \\ &= (T * \check{\varphi})(0) = \langle T, \varphi\rangle\end{aligned}$$

as required. □

We shall keep Δ fixed in what follows.

Corollary. (i) *The mapping from $\mathscr{D}'$ into ${}^*\mathscr{E}$ defined by $T \to {}^*T * \Delta$ is injective and preserves the addition, multiplication by a complex* (*standard*) *number, and partial differentiation in $\mathscr{D}'$.*

(ii) *There exists an infinitely large natural number $\nu \in {}^*\mathbb{N}$ such that $P * \Delta = P$ holds for all* (*nonstandard, in general*) *polynomials $P \in {}^*\mathscr{P}$ with degree not higher than ν. In particular, ${}^*P * \Delta = {}^*P$ holds for all standard polynomials $P \in \mathscr{P}$.*

(iii) *If f is a continuous function, then ${}^*f * \Delta$ is an extension of f.*

Proof. (i) By the transfer principle, ${}^*T * \Delta \in {}^*\mathscr{E}$ for all T in $\mathscr{D}'$, while ${}^*T * \Delta = 0$ (in ${}^*\mathscr{E}$) implies $T = 0$ (in $\mathscr{D}'$), by the above proposition. The preservation of linear operations follows immediately from the corresponding property of the convolution operator and transfer principle.

(ii) Define the internal set

$$\Omega_\Delta = \left\{ n \in {}^*\mathbb{N} : \int_{{}^*\mathbb{R}^d} \Delta(x) x^\alpha\,dx = 0,\ \ 1 \le |\alpha| \le n,\ \ \alpha \in {}^*\mathbb{N}_0^d \right\}$$

and observe that, by our lemma, $\mathbb{N} \subseteq \Omega_\Delta$. Hence, by overflow, Ω_Δ contains an infinitely large number ν. Suppose now, that $\xi \in {}^*\mathbb{R}^d$. The hyperfinite ($*$-finite) Taylor's expansion of P at ξ gives

$$\begin{aligned}(\Delta * P)(\xi) &\equiv \int_{{}^*\mathbb{R}^d} \Delta(x) P(\xi - x)\,dx \\ &= P(\xi) + \sum_{|\alpha|=1}^{\nu} \frac{(-1)^{|\alpha|}\partial^\alpha P(\xi)}{\alpha!} \int_{{}^*\mathbb{R}^d} \Delta(x) x^\alpha\,dx = P(\xi),\end{aligned}$$

as required. The equality ${}^*P * \Delta = {}^*P$ follows as a particular case since the degree of a standard polynomial is always finite and hence less than ν.

(iii) follows immediately from our lemma for $\varphi = f_x$ and standard $x \in \mathbb{R}^d$, where $f_x(\xi) = f(x - \xi)$. The proof is complete. □

Remark (Multiplication of distributions). Consider ${}^*\mathscr{E}$ as a differential algebra over ${}^*\mathbb{C}$ with respect to pointwise addition, multiplication, and internal partial differentiation. Notice now that the space of Schwartz distributions $\mathscr{D}'$ is isomorphically embedded in ${}^*\mathscr{E}$ through the above injection and hence the

Schwartz distributions can be multiplied within an associative algebra (something impossible in $\mathscr{D}'$ itself). Further, the operations in ${}^*\mathscr{E}$ generalize the usual operations with polynomials in the sense the $\mathscr{P}$ (considered as a subset of $\mathscr{D}'$) is a differential subalgebra of ${}^*\mathscr{E}$ over $\mathbb{C}$. The multiplication in ${}^*\mathscr{E}$ also generalizes the usual multiplication in C^0 (considered as a subset of $\mathscr{D}'$) although in a somewhat weaker sense: if f and g are two continuous functions and ${}^*f*\Delta$ and ${}^*g*\Delta$ are their images in ${}^*\mathscr{E}$, then their product $({}^*f*\Delta)({}^*g*\Delta)$ in ${}^*\mathscr{E}$ is an extension of the usual product fg in C^0. We wish to pay attention to the similarity between the class of nonstandard functions ${}^*\mathscr{E}$ (in the context discussed above) and the classes of generalized functions introduced (in the framework of standard analysis) by Colombeau [1] with the same purpose: multiplication of Schwartz distributions.

References

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