

POINTWISE KERNELS OF SCHWARTZ DISTRIBUTIONS

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ABSTRACT. We show that Schwartz distributions have kernels in the class of the pointwise nonstandard functions.

The main purpose of this note is to show that every Schwartz distribution $T \in \mathcal{D}'$ has a kernel $f: {}^*\mathbb{R}^d \rightarrow {}^*\mathbb{C}$ in the class of the pointwise nonstandard functions in the sense that

$$(1) \quad \langle T, \varphi \rangle = \int_{{}^*\mathbb{R}^d} f(x) {}^*\varphi(x) dx$$

for all $\varphi \in \mathcal{D}$, where ${}^*\varphi$ is the nonstandard extension of φ . Recall that, in general, the Schwartz distributions do not have kernels in the class of the standard pointwise functions (Schwartz [2]).

We denote the usual classes of the C^∞ -functions, C^∞ -functions with compact supports, and continuous complex-valued functions defined on \mathbb{R}^d (d is a natural number) by $\mathcal{E} \equiv \mathcal{E}(\mathbb{R}^d) \equiv C^\infty(\mathbb{R}^d)$, $\mathcal{D} \equiv \mathcal{D}(\mathbb{R}^d) \equiv C_0^\infty(\mathbb{R}^d)$, and $C^0 \equiv C^0(\mathbb{R}^d) \equiv C(\mathbb{R}^d)$, respectively, and the class of Schwartz distributions by $\mathcal{D}' \equiv \mathcal{D}'(\mathbb{R}^d)$. Let \mathcal{P} be the ring of standard complex-valued polynomials defined on \mathbb{R}^d . As usual, \mathbb{N} , \mathbb{R} , and \mathbb{C} will be the systems of the natural, real, and complex numbers, respectively, and we use also the notations $\mathbb{N}_0 \equiv \{0\} \cup \mathbb{N}$ and $\check{f}(x) \equiv f(-x)$.

In what follows, we shall work in a nonstandard model with a set of individuals J that contains the complex numbers \mathbb{C} and degree of saturation k larger than 2^κ for $\kappa \equiv \text{card } C^0$. In particular, any polysaturated model of \mathbb{C} will suffice (Stroyan and Luxemburg [3]). If X is a set of complex numbers or a set of (standard) functions, then *X will be its nonstandard extension, and if $f: X \rightarrow Y$ is a (standard) mapping, then ${}^*f: {}^*X \rightarrow {}^*Y$ will be its nonstandard extension. We shall use the same notation, $*$, for the convolution operator $*$: $\mathcal{D}' \times \mathcal{D} \rightarrow \mathcal{E}$ and for its nonstandard extension $*$: ${}^*\mathcal{D}' \times {}^*\mathcal{D} \rightarrow {}^*\mathcal{E}$.

Lemma. *There exists Δ in ${}^*\mathcal{D}$ such that for all φ in C^0 we have*

$$\int_{{}^*\mathbb{R}^d} \Delta(x) {}^*\varphi(x) dx = \int_{{}^*\mathbb{R}^d} \check{\Delta}(x) {}^*\varphi(x) dx = \varphi(0).$$

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For the proof we refer the reader to [4].

Proposition. *If T is a Schwartz distribution, then (1) holds for $f = {}^*T * \Delta$ and all φ in \mathcal{D} .*

Proof. Using the properties of the convolution operator, the transfer principle, and the above lemma (since $T * \check{\varphi}$ is in \mathcal{E}), we obtain

$$\begin{aligned} \int_{\bullet\mathbb{R}^d} ({}^*T * \Delta)(x) {}^*\varphi(x) dx &= \langle {}^*T * \Delta, {}^*\varphi \rangle = \langle {}^*T, {}^*\varphi * \check{\Delta} \rangle \\ &= \langle {}^*T * ({}^*\check{\varphi}), \check{\Delta} \rangle = \int_{\bullet\mathbb{R}^d} \check{\Delta}(x) ({}^*T * \check{\varphi})(x) dx \\ &= (T * \check{\varphi})(0) = \langle T, \varphi \rangle \end{aligned}$$

as required. \square

We shall keep Δ fixed in what follows.

Corollary. (i) *The mapping from \mathcal{D}' into ${}^*\mathcal{E}$ defined by $T \rightarrow {}^*T * \Delta$ is injective and preserves the addition, multiplication by a complex (standard) number, and partial differentiation in \mathcal{D}' .*

(ii) *There exists an infinitely large natural number $\nu \in {}^*\mathbb{N}$ such that $P * \Delta = P$ holds for all (nonstandard, in general) polynomials $P \in {}^*\mathcal{P}$ with degree not higher than ν . In particular, ${}^*P * \Delta = {}^*P$ holds for all standard polynomials $P \in \mathcal{P}$.*

(iii) *If f is a continuous function, then ${}^*f * \Delta$ is an extension of f .*

Proof. (i) By the transfer principle, ${}^*T * \Delta \in {}^*\mathcal{E}$ for all T in \mathcal{D}' , while ${}^*T * \Delta = 0$ (in ${}^*\mathcal{E}$) implies $T = 0$ (in \mathcal{D}'), by the above proposition. The preservation of linear operations follows immediately from the corresponding property of the convolution operator and transfer principle.

(ii) Define the internal set

$$\Omega_\Delta = \left\{ n \in {}^*\mathbb{N} : \int_{\bullet\mathbb{R}^d} \Delta(x) x^\alpha dx = 0, \quad 1 \leq |\alpha| \leq n, \quad \alpha \in {}^*\mathbb{N}_0^d \right\}$$

and observe that, by our lemma, $\mathbb{N} \subseteq \Omega_\Delta$. Hence, by overflow, Ω_Δ contains an infinitely large number ν . Suppose now, that $\xi \in {}^*\mathbb{R}^d$. The hyperfinite ($*$ -finite) Taylor's expansion of P at ξ gives

$$\begin{aligned} (\Delta * P)(\xi) &\equiv \int_{\bullet\mathbb{R}^d} \Delta(x) P(\xi - x) dx \\ &= P(\xi) + \sum_{|\alpha|=1}^{\nu} \frac{(-1)^{|\alpha|} \partial^\alpha P(\xi)}{\alpha!} \int_{\bullet\mathbb{R}^d} \Delta(x) x^\alpha dx = P(\xi), \end{aligned}$$

as required. The equality ${}^*P * \Delta = {}^*P$ follows as a particular case since the degree of a standard polynomial is always finite and hence less than ν .

(iii) follows immediately from our lemma for $\varphi = f_x$ and standard $x \in \mathbb{R}^d$, where $f_x(\xi) = f(x - \xi)$. The proof is complete. \square

Remark (Multiplication of distributions). Consider ${}^*\mathcal{E}$ as a differential algebra over ${}^*\mathbb{C}$ with respect to pointwise addition, multiplication, and internal partial differentiation. Notice now that the space of Schwartz distributions \mathcal{D}' is isomorphically embedded in ${}^*\mathcal{E}$ through the above injection and hence the

Schwartz distributions can be multiplied within an associative algebra (something impossible in \mathcal{D}' itself). Further, the operations in ${}^*\mathcal{E}$ generalize the usual operations with polynomials in the sense the \mathcal{P} (considered as a subset of \mathcal{D}') is a differential subalgebra of ${}^*\mathcal{E}$ over \mathbb{C} . The multiplication in ${}^*\mathcal{E}$ also generalizes the usual multiplication in C^0 (considered as a subset of \mathcal{D}') although in a somewhat weaker sense: if f and g are two continuous functions and ${}^*f*\Delta$ and ${}^*g*\Delta$ are their images in ${}^*\mathcal{E}$, then their product $({}^*f*\Delta)({}^*g*\Delta)$ in ${}^*\mathcal{E}$ is an extension of the usual product fg in C^0 . We wish to pay attention to the similarity between the class of nonstandard functions ${}^*\mathcal{E}$ (in the context discussed above) and the classes of generalized functions introduced (in the framework of standard analysis) by Colombeau [1] with the same purpose: multiplication of Schwartz distributions.

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