Consecutive Patterns: From Permutations to Column-Convex Polyominoes and Back

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Abstract

We expose the ties between the consecutive pattern enumeration problems associated with permutations, compositions, column-convex polyominoes, and words. Our perspective allows powerful methods from the contexts of compositions, column-convex polyominoes, and of words to be applied directly to the enumeration of permutations by consecutive patterns. We deduce a host of new consecutive pattern results, including a solution to the $(2m+1)$-alternating pattern problem on permutations posed by Kitaev.

Keywords ascents, consecutive pattern, column-convex polyomino, descents, levels, maxima, peaks, twin peaks, up-down type, valleys, variation

1 Introduction

The problems of enumerating permutations, compositions, and words by patterns formed by consecutive terms (parts or letters) have been widely studied and, for the most part, their stories are separate and parallel. In contrast, the problem of enumerating column-convex polyominoes (CCPs) by consecutive patterns has received only scant and indirect consideration.

Our primary purpose is to show that these problem sets are in fact intimately related. More precisely, if $\mathcal{PS}$, $\mathcal{PC}$, $\mathcal{PCCP}$, and $\mathcal{PW}$ respectively denote the sets of consecutive pattern enumeration problems on permutations, compositions, column-convex polyominoes, and words, then

$$\mathcal{PS} \subset \mathcal{PC} \subset \mathcal{PCCP} \subset \mathcal{PW}. \tag{1}$$

The significance of (1) is that it allows powerful methods from the larger problem sets to be applied to the smaller problem sets. To illustrate, we will show how various

results on words as well as Bousquet-Mélou's [4] adaptation of Temperley's [37] method for enumerating CCPs may be used to count permutations by consecutive patterns.

In particular, we exploit the perspective of (1) to q-count permutations by (i, d)-peaks, up-down type, uniform m-peak ranges, and (i, m)-maxima. Notably, a specialization of Corollary 4 provides a solution to the $(2m + 1)$-alternating pattern problem on permutations posed by Kitaev [25, Problem 1]. We will also show that the generating function for permutations by a given pattern is deducible from the generating function for a related pattern permutation set; for instance, the generating function for permutations by peaks may be obtained from the one for up-down permutations of odd length.

Our secondary purpose is to initiate the explicit study of CCPs by consecutive (or ridge) patterns. Our introduction of two-column ridge patterns provides a unifying characterization of the common subclasses of CCPs. In subsections 7.1 and 7.2, we use results on words to enumerate directed CCPs by two-column ridge patterns and by valleys. The Temperley method as modified in [4] is employed in subsection 9.3 to count CCPs by peaks.

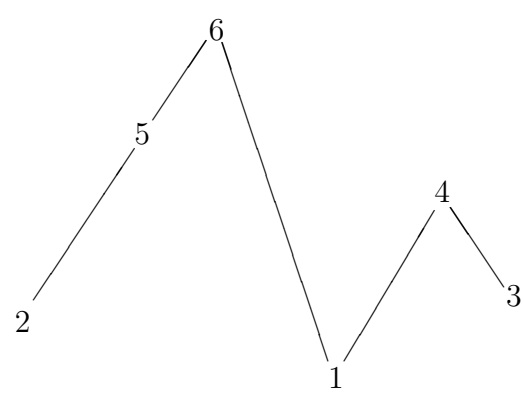
We begin our exposé of (1) with a discussion of $\mathcal{PS}$ and then work our way up the sequence of inclusions.

2 Consecutive patterns in permutations

Let S_n denote the set of permutations of $1, 2, \ldots, n$. When a permutation $\sigma = \sigma_1\sigma_2 \ldots \sigma_n \in S_n$ is sketched in a natural way, patterns take shape. In the sketch of $\sigma = 2\,5\,6\,1\,4\,3 \in S_6$ in Diagram 1, one discerns ascents, descents, peaks, valleys, and other patterns. For $\sigma \in S_n$ and $p \in S_m$ with $m \leqslant n$, a segment $s = \sigma_k\sigma_{k+1} \ldots \sigma_{k+m-1}$ in σ is referred to as a consecutive p-pattern if the relative order of the integers in s agrees with the relative order of the integers in p (that is, σ_{k+j-1} is the p_j^{th} smallest integer in the list $\sigma_k, \sigma_{k+1}, \ldots, \sigma_{k+m-1}$).

Diagram 1

$\sigma = 2\,5\,6\,1\,4\,3 =$



In Diagram 1, the ascent $\sigma_2\sigma_3 = 5\,6$ is a 12-pattern and the segment $\sigma_2\sigma_3\sigma_4 = 5\,6\,1$ is a 231-pattern. A segment that is either a 132-pattern or a 231-pattern is a peak. A peak is therefore a set of patterns.

There are two standard ways of counting the number of times a given set of patterns $P \subseteq \bigcup_{m\geqslant 1} S_m$ occurs consecutively in a permutation $\sigma \in S_n$:

- $P(\sigma)$ = the total number of times elements of P occur in σ and
- $P_{no}(\sigma)$ = the maximum number of non-overlapping times elements of P occur in σ.

When P is of cardinality 1, say $P = \{p\}$, we write $p(\sigma)$ and $p_{no}(\sigma)$ in place of $P(\sigma)$ and $P_{no}(\sigma)$. Relative to Diagram 1, $132(\sigma) = 1 = 132_{no}(\sigma)$. For pic $= \{132, 231\}$, note that $\mathrm{pic}(\sigma) = 2$ whereas $\mathrm{pic}_{no}(\sigma) = 1$ (since the peaks $\sigma_2\sigma_3\sigma_4 = 5\,6\,1$ and $\sigma_4\sigma_5\sigma_6 = 1\,4\,3$ overlap at $\sigma_4 = 1$).

For a pattern set $P \subseteq \bigcup_{m\geqslant 1} S_m$, two primary enumeration questions arise:

- Q1: What is the cardinality of $PS_n = P \cap S_n$? Elements of PS_n are referred to as P-pattern permutations of length n.
- Q2: How many permutations in S_n contain k consecutive P-patterns, counting overlaps?

The variation of Q2 involving the maximal number of non-overlapping patterns will be denoted by $\mathrm{Q2}_{no}$. The problem of counting permutations that contain no P-patterns is known as the avoidance problem. The pattern avoidance cases ($k = 0$) of Q2 and $\mathrm{Q2}_{no}$ are identical as $\{\sigma \in S_n : P(\sigma) = 0\} = \{\sigma \in S_n : P_{no}(\sigma) = 0\}$.

As will be seen, there is a hierarchy between some versions of Q1, Q2, and $\mathrm{Q2}_{no}$; in these cases, solving Q1 solves Q2, which in turn solves $\mathrm{Q2}_{no}$. Our placement of the problem Q1 of enumerating permutations replete with P-patterns at the top of the hierarchy complements and sharply contrasts with the central role played in [23, 29] of the avoidance problem of counting permutations devoid of P in solving $\mathrm{Q2}_{no}$.

2.1 Selected examples

In 1881, André [1] solved what has become the classic example of Q1. For $\mathrm{UD} = \bigcup_{m\geqslant 1}\{p \in S_m : p_1 < p_2 > p_3 < p_4 > \cdots\}$, the elements of $\mathrm{UD}S_n$ are said to be up-down permutations of length n. André showed that

$$\sum_{n\geqslant 0} |\mathrm{UD}S_n| \, \frac{z^n}{n!} = \sec z + \tan z. \tag{2}$$

As an example of Q2, we present the generating function for permutations by peaks obtained by Mendes and Remmel [29]:

$$\sum_{n\geqslant 0} \sum_{\sigma\in S_n} \frac{y^{\mathrm{pic}(\sigma)} z^n}{n!} = \frac{\sqrt{y-1}}{\sqrt{y-1} - \tan(z\sqrt{y-1})}. \tag{3}$$

Prior to [29], Kitaev [25] obtained a different form for the right side of (3). Incidentally, Entringer [13] enumerated "circular" permutations by peaks.

The appearance of the tangent function in both (2) and (3) is no coincidence. A general explanation is provided in section 5, thereby showing that solving Q1 solves Q2.

The q-shifted factorial of an integer $n \geqslant 0$ is $(t;q)_n = \prod_{k=0}^{n-1}(1-tq^k)$. The inversion number of a permutation $\sigma \in S_n$, defined by

$$\operatorname{inv}\sigma = |\{(i,j) : 1 \leqslant i < j \leqslant n \text{ and } \sigma_i > \sigma_j\}|,$$

gives rise to many natural q-analogs. For instance, Gessel [16] and Mendes and Remmel [29] respectively showed that

$$\sum_{n\geqslant 0}\sum_{\sigma\in \mathrm{UD}S_n}\frac{q^{\operatorname{inv}\sigma}z^n}{(q;q)_n} = \sec_q z + \tan_q z \quad \text{and} \tag{4}$$

$$\sum_{n\geqslant 0}\sum_{\sigma\in S_n}\frac{y^{\mathrm{pic}(\sigma)}q^{\operatorname{inv}\sigma}z^n}{(q;q)_n} = \frac{\sqrt{y-1}}{\sqrt{y-1}-\tan_q(z\sqrt{y-1})} \tag{5}$$

with $\cos_q z = \sum_{n\geqslant 0}(-1)^n z^{2n}/(q;q)_{2n}$, $\sin_q z = \sum_{n\geqslant 0}(-1)^n z^{2n+1}/(q;q)_{2n+1}$, $\sec_q z = 1/\cos_q z$, and $\tan_q z = \sin_q z/\cos_q z$. Replacing z by $z(1-q)$ and then letting q approach 1 reduces (4) to (2); hence (4) is a q-analog of (2). Likewise, (5) is a q-analog of (3).

2.2 Solving Q2 solves Q2$_{no}$

In [23], Kitaev made the beautiful observation that Q2$_{no}$ for a single pattern may be reduced to the avoidance problem. Shortly thereafter, Mendes and Remmel [29] extended Kitaev's result by tracking a set of patterns and adding the inversion number to the mix.

Theorem 1 (Mendes and Remmel 2007)**.** *If $P \subseteq S_m$ with $m > 1$, then*

$$\sum_{n\geqslant 0}\sum_{\sigma\in S_n}\frac{q^{\operatorname{inv}\sigma}y^{P_{no}(\sigma)}z^n}{(q;q)_n} = \frac{\mathcal{K}_q(z)}{1-y+y\left(1-z(1-q)^{-1}\right)\mathcal{K}_q(z)}$$

where $\mathcal{K}_q(z) = \sum_{n\geqslant 0}\left(\sum_{\sigma\in S_n} q^{\operatorname{inv}\sigma}0^{P(\sigma)}\right)z^n/(q;q)_n$ is the q-exponential generating function for permutations that avoid P.

Among many consequences of Theorem 1, Mendes and Remmel obtained a solution to Q2$_{no}$ relative to peaks:

$$\sum_{n\geqslant 0}\sum_{\sigma\in S_n}\frac{q^{\operatorname{inv}\sigma}y^{\mathrm{pic}_{no}(\sigma)}z^n}{(q;q)_n} = \left(1-\frac{yz}{1-q}+\sqrt{-1}(1-y)\tan_q(z\sqrt{-1})\right)^{-1}. \tag{6}$$

Theorem 1 provides a bridge from some versions of Q2 to Q2$_{no}$. For instance, setting $y = 0$ in (5) gives the q-exponential generating function for peak-avoiding permutations, which in turn may be plugged into Theorem 1 to get (6). For this reason, our primary focus will be on Q2.

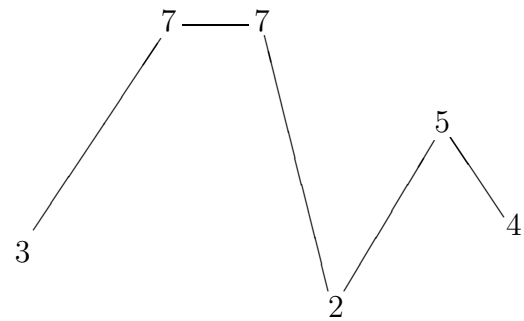
3 Consecutive patterns in compositions

Let $K_n = \{w = w_1 w_2 \ldots w_n : w_1, w_2, \ldots, w_n \text{ are positive integers}\}$. For $w \in K_n$, set $\operatorname{sum} w = w_1 + w_2 + \cdots + w_n$. An element $w \in K_n$ for which $\operatorname{sum} w = m$ is said to be a composition of m into n parts.

As with permutations, a sketch of a composition $w \in K_n$ reveals patterns. When $w = 3\,7\,7\,2\,5\,4 \in K_6$ is sketched as in Diagram 2, one observes ascents, levels, descents, peaks, valleys, and more.

Diagram 2

$w = 3\,7\,7\,2\,5\,4 =$



In particular, we define a peak in a composition w to be a segment $w_i w_{i+1} w_{i+2}$ satisfying $w_i \leqslant w_{i+1} > w_{i+2}$. The number of peaks in w is denoted by $\operatorname{pic}(w)$. In Diagram 2, segment $w_2 w_3 w_4 = 7\,7\,2$ is a peak and $\operatorname{pic}(w) = 2$.

3.1 Two revealing examples

Naturally, Q1 and Q2 have been considered in the context of compositions. Paralleling André [1], a composition w for which $w_1 \leqslant w_2 > w_3 \leqslant w_4 > \cdots$ is said to be up-down. If $\mathrm{UD}K_n$ denotes the set of up-down compositions of length n, then

$$\sum_{n \geqslant 0} \sum_{w \in \mathrm{UD}K_n} q^{\operatorname{sum} w} \, (z/q)^n = \sec_q z + \tan_q z. \tag{7}$$

Carlitz [7] obtained a related result; he used $w_1 \leqslant w_2 \geqslant w_3 \leqslant w_4 \geqslant \cdots$ as the defining property of an up-down composition.

As an example of Q2 for compositions, the generating function for compositions by peaks (see section 5 for a proof) is

$$\sum_{n \geqslant 0} \sum_{w \in K_n} y^{\operatorname{pic}(w)} q^{\operatorname{sum} w} \, (z/q)^n = \frac{\sqrt{y-1}}{\sqrt{y-1} - \tan_q(z\sqrt{y-1})}. \tag{8}$$

Heubach and Mansour [20] obtained the distributions for compositions with parts in an arbitrary alphabet by various three-letter patterns; their result for peaks is more general than (8).

Comparison of (4) with (7) and of (5) with (8) strongly suggests that certain problems in $\mathcal{PS}$ and $\mathcal{PC}$ are one-in-the-same. Fédou's [15] insertion-shift bijection provides the connection.

3.2 Fédou's bijection: $\mathcal{PS} \subset \mathcal{PC}$

For $\sigma \in S_n$ and $1 \leqslant i \leqslant n$, set $\operatorname{inv}_i \sigma = |\{k : i < k \leqslant n, \sigma_i > \sigma_k\}|$. Also, let $\Lambda_n = \{w \in K_n : w_1 \leqslant w_2 \leqslant \cdots \leqslant w_n\}$. The inverse of Fédou's [15] insertion-shift bijection $\nabla_n : S_n \times \Lambda_n \to K_n$, as personally communicated by Foata, is given by the rule $\nabla_n(\sigma, \lambda) = w$ where $w_i = \operatorname{inv}_i \sigma + \lambda_{\sigma_i}$. For example,

$$\nabla_6(2\,5\,6\,1\,4\,3,\ 2\,2\,4\,4\,4\,4) = 3\,7\,7\,2\,5\,4. \tag{9}$$

There are two key properties to note. First, if $\nabla_n(\sigma, \lambda) = w$, then

$$\operatorname{inv} \sigma + \operatorname{sum} \lambda = \operatorname{sum} w. \tag{10}$$

Second, ∇_n roughly transfers the overall shape and patterns of σ to the corresponding w. Relative to (9), $\sigma = 2\,5\,6\,1\,4\,3 \in S_6$ and $w = 3\,7\,7\,2\,5\,4 \in K_6$ are of similar shape (see Diagrams 1 and 2). Moreover, the peaks $5\,6\,1$ and $1\,4\,3$ in $\sigma = 2\,5\,6\,1\,4\,3$ coincide with the peaks $7\,7\,2$ and $2\,5\,4$ in $w = 3\,7\,7\,2\,5\,4$. The explanation behind ∇_n's preservation of overall shape lies in the fact that, if $\nabla_n(\sigma, \lambda) = w$ and $1 \leqslant i < m \leqslant n$, then

$$\sigma_i < \sigma_m \ \text{ if and only if } \ w_i \leqslant w_m + |\{j : i < j < m,\ \sigma_i > \sigma_j\}|. \tag{11}$$

In particular, ∇_n preserves peaks: (11) implies that $\sigma_k < \sigma_{k+1} > \sigma_{k+2}$ if and only if $w_k \leqslant w_{k+1} > w_{k+2}$.

Rather than defining consecutive patterns directly on compositions, it is convenient to take an indirect path through ∇_n. For $p \in S_m$, the segment $w_k w_{k+1} \ldots w_{k+m-1}$ is said to be a consecutive p-pattern in w provided the corresponding segment $\sigma_k \sigma_{k+1} \ldots \sigma_{k+m-1}$ is a consecutive p-pattern in the unique permutation σ satisfying $w = \nabla_n(\sigma, \lambda)$. Furthermore, for $P \subseteq \cup_{m \geqslant 1} S_m$ and $w = \nabla_n(\sigma, \lambda)$, we define $P(w) = P(\sigma)$ and $P_{no}(w) = P_{no}(\sigma)$.

The definition of patterns for compositions through ∇_n has at least one shortcoming. For instance, $w_k w_{k+1}$ is a 12-pattern in w if $w_k \leqslant w_{k+1}$. From the perspective of compositions though, distinguishing between the case $w_k < w_{k+1}$ and the case $w_k = w_{k+1}$ may well be of interest. So there are problems in $\mathcal{PC}$ that have no analog in $\mathcal{PS}$. However, $\mathcal{PS} \subset \mathcal{PC}$.

Theorem 2. *If $P \subseteq \bigcup_{m \geqslant 1} S_m$ and if $B_n \subseteq S_n$, then*

$$\sum_{n \geqslant 0} \sum_{\sigma \in B_n} \left(\prod_{p \in P} y_p^{p(\sigma)} \right) \frac{q^{\operatorname{inv} \sigma} z^n}{(q;q)_n} = \sum_{n \geqslant 0} \sum_{w \in \nabla_n(B_n, \Lambda_n)} \left(\prod_{p \in P} y_p^{p(w)} \right) q^{\operatorname{sum} w} (z/q)^n.$$

Moreover, the above equality remains true if $y_p^{p(\sigma)}$ and $y_p^{p(w)}$ are respectively replaced by $y_p^{p_{no}(\sigma)}$ and $y_p^{p_{no}(w)}$ for some or all $p \in P$.

Proof. We prove the first assertion. As is well known, $(q;q)_n^{-1} = \sum_{\lambda \in \Lambda_n} q^{\operatorname{sum} \lambda - n}$. By the properties of ∇_n,

$$\begin{aligned} \sum_{n \geqslant 0} \sum_{\sigma \in B_n} \left(\prod_{p \in P} y_p^{p(\sigma)} \right) \frac{q^{\operatorname{inv} \sigma} z^n}{(q;q)_n} &= \sum_{n \geqslant 0} \sum_{\sigma \in B_n} \sum_{\lambda \in \Lambda_n} \left(\prod_{p \in P} y_p^{p(\sigma)} \right) q^{\operatorname{inv} \sigma + \operatorname{sum} \lambda} (z/q)^n \\ &= \sum_{n \geqslant 0} \sum_{w \in \nabla_n(B_n, \Lambda_n)} \left(\prod_{p \in P} y_p^{p(w)} \right) q^{\operatorname{sum} w} (z/q)^n. \end{aligned}$$

□

There are three immediate applications of Theorem 2. First, Theorem 2 may be used to deduce (8) from Mendes and Remmel's (5). Likewise, (7) follows from Gessel's (4). Finally, Theorem 2 may be used to rewrite Mendes and Remmel's Theorem 1 in the context of compositions.

Corollary 1. *If $P \subseteq S_m$ with $m > 1$, then*

$$\sum_{n\geqslant 0}\sum_{w\in K_n} y^{P_{no}(w)} q^{\operatorname{sum} w} z^n = \frac{L_q(z)}{1-y+y\left(1-zq(1-q)^{-1}\right)L_q(z)}$$

where $L_q(z) = \sum_{n\geqslant 0}\left(\sum_{w\in K_n} q^{\operatorname{sum} w} 0^{P(w)}\right) z^n$ is the generating function for compositions that avoid P.

Corollary 1 is both more and less general than Heubach, Kitaev, and Mansour's [22] Theorem 4.1; for a pattern set of cardinality 1, their result holds for an arbitrary alphabet of positive integers.

3.3 Compositions by two-term patterns and variation

In a composition w, a segment $w_k w_{k+1}$ is said to be an ascent, level, or descent respectively as $w_k < w_{k+1}$, $w_k = w_{k+1}$, or $w_k > w_{k+1}$. The numbers of ascents, levels, and descents in w are denoted by $\operatorname{asc} w$, $\operatorname{lev} w$, and $\operatorname{des} w$. When sketched as in Diagram 2, one of the more compelling features of a composition $w \in K_n$ is its vertical variation defined by

$$\operatorname{var} w = \sum_{k=0}^{n} |w_{k+1} - w_k| \qquad \text{where, by convention, } w_0 = 0 = w_{n+1}.$$

As a consequence of the perspective afforded by (1), we obtain the following joint distribution of (asc, lev, des, var) on compositions from our Corollary 7 on directed column-convex polyominoes recorded in subsection 7.1.

Corollary 2. *The generating function for compositions by ascents, levels, descents, and variation $K(c,z) = \sum_{n\geqslant 0}\sum_{w\in K_n} a^{\operatorname{asc} w} b^{\operatorname{lev} w} d^{\operatorname{des} w} c^{\operatorname{var} w} q^{\operatorname{sum} w} z^n$ is given by*

$$K(c,z) = 1 + \frac{c^2 \displaystyle\sum_{n\geqslant 0} \frac{(qz)^{n+1}}{1-c^2q^{n+1}} \prod_{k=1}^{n}\left(b + \frac{c^2dq^k}{1-c^2q^k} - \frac{a}{1-q^k}\right)}{1 - a\displaystyle\sum_{n\geqslant 1} \frac{(qz)^n}{1-q^n} \prod_{k=1}^{n-1}\left(b + \frac{c^2dq^k}{1-c^2q^k} - \frac{a}{1-q^k}\right)}.$$

Setting $c = 1$ in Corollary 2 and making use of Cauchy's q-binomial theorem gives Carlitz's [6] generating function $K(1,z)$ for compositions by ascents, levels, and descents. Heubach and Mansour [21] recently extended Carlitz's result to an arbitrary alphabet of positive integers.

The distributions of var and of closely related statistics over various combinatorial sets have been considered in [2, 28, 33, 38]. In [38], Tiefenbruck expressed the generating function for compositions with bounded parts by variation as a ratio of coefficients of basic hypergeometric series. Recently, Mansour [28] determined the generating function for the same version of var on compositions as in [2].

4 Factors and consecutive patterns in words

Let X^* be the free monoid generated by a nonempty alphabet X. The number of letters in a word $w \in X^*$ is referred to as its length and is denoted by $\operatorname{len} w$. Set $X^n = \{w \in X^* : \operatorname{len} w = n\}$ and $X^+ = \{w \in X^* : \operatorname{len} w > 0\}$. The kth letter of a word w will be denoted by w_k; so $w = w_1 w_2 \dots w_{\operatorname{len} w}$.

An element $f \in X^+$ is a factor of $w \in X^*$ if $f = w_k w_{k+1} \dots w_{k+\operatorname{len} f-1}$ for some k. The number of times f appears as a factor in w is denoted by $f(w)$.

For a nonempty set $\mathcal{F} \subseteq X^+$, a factor f of a word w is a said to be a consecutive $\mathcal{F}$-pattern in w if $f \in \mathcal{F}$. The number of consecutive $\mathcal{F}$-patterns in w is denoted by $\mathcal{F}(w)$; so $\mathcal{F}(w) = \sum_{f \in \mathcal{F}} f(w)$. We refer to $\mathcal{F}$ as a factor set.

The containment $\mathcal{PC} \subset \mathcal{PW}$ in (1) is now evident: A composition w is just a word with letters selected from the alphabet $N = \{1, 2, 3, \dots\}$. In fact, $N^* = \cup_{n \geqslant 0} K_n$. Also, each pattern p of length m defined on compositions may be naturally matched with the factor set $\mathcal{F}_p = \{f \in N^m : p(f) = 1\}$. For $p = 132$ defined on compositions through Fédou's bijection as in subsection 3.2, $\mathcal{F}_{132} = \{acb \in N^3 : a \leqslant b < c\}$. In general, for a pattern set P on compositions, we define $\mathcal{F}_P = \cup_{p \in P} \mathcal{F}_p$ and note that $P(w) = \mathcal{F}_P(w)$.

As a result, any method for the set $\mathcal{PW}$ may be applied to the set $\mathcal{PC}$ and, via Theorem 2, to $\mathcal{PS}$. In this regard, some modifications of Goulden and Jackson's [17] result for enumerating words by factors are fundamental.

As in Stanley [36, p. 266-267], we state Goulden and Jackson's [17] result in the context of the free monoid. Following Noonan and Zeilberger [31], the stipulation that no element of the factor set $\mathcal{F}$ be a factor of another is dropped. We further drop the requirement that the alphabet be finite, and we consider restrictions on the first and last letters.

For a nonempty set $\mathcal{F} \subset X^+$, an $\mathcal{F}$-cluster is a triple (w, ν, β) in which

$$\begin{aligned} &w = w_1 w_2 \dots w_{\operatorname{len} w} \in X^+, \\ &\nu = (f_{(1)}, f_{(2)}, \dots, f_{(k)}) \text{ for some } k \geqslant 1 \text{ with each } f_{(i)} \in \mathcal{F}, \text{ and} \\ &\beta = (b_1, b_2, \dots, b_k) \text{ with each } b_i \text{ being a positive integer} \end{aligned}$$

where $f_{(i)} = w_{b_i} w_{b_i+1} \dots w_{b_i+\operatorname{len} f_{(i)}-1}$, each $w_i w_{i+1}$ is a factor of some $f_{(j)}$, $b_1 \leqslant b_2 \leqslant \dots \leqslant b_k$, and if $b_i = b_{i+1}$, then $\operatorname{len} f_{(i)} < \operatorname{len} f_{(i+1)}$.

Roughly speaking, the pair (ν, β) is a recipe for covering w with $\mathcal{F}$-factors: β specifies where the factors in ν are to be "placed so as to cover" w. Accordingly, w is said to be $\mathcal{F}$-coverable and the pair (ν, β) is said to be a covering of w. We let $C_{\mathcal{F}}$ denote the set of $\mathcal{F}$-clusters.

For nonempty $A, B \subseteq X^*$, define $AB = \{ab : a \in A, b \in B\}$. The cluster generating function over a subset W of X^* is defined to be the formal series

$$C_{\mathcal{F}}(\mathbf{y}, W) = \sum_{\substack{(w,\nu,\beta) \in C_{\mathcal{F}} \\ w \in W}} \left(\prod_{f \in \mathcal{F}} y_f^{f(\nu)} \right) w$$

where $f(\nu)$ is the number of times f appears as a component in ν. With but trivial changes, Stanley's solution to problem 14(a) in [36, p. 266-267] establishes the following theorem.

Theorem 3 (Modifications of Goulden and Jackson's [17] result)**.** *If, for nonempty $L, R \subseteq X$ and a nonempty $\mathcal{F} \subseteq X^+$, we define*

$$\mathcal{L}(\mathbf{y}) = \sum_{l \in L} l + C_{\mathcal{F}}(\mathbf{y}, LX^*), \quad \mathcal{R}(\mathbf{y}) = \sum_{r \in R} r + C_{\mathcal{F}}(\mathbf{y}, X^*R), \quad \text{and}$$
$$\mathcal{X}(\mathbf{y}) = \sum_{x \in X} x + C_{\mathcal{F}}(\mathbf{y}, X^*)$$

and if the result of replacing each y_f in $\mathbf{y}$ by $y_f - 1$ is denoted by $\mathbf{y} - \mathbf{1}$, then

$$\sum_{w \in X^*} \left(\prod_{f \in \mathcal{F}} y_f^{f(w)} \right) w = (1 - \mathcal{X}(\mathbf{y} - \mathbf{1}))^{-1},$$
$$\sum_{w \in LX^*} \left(\prod_{f \in \mathcal{F}} y_f^{f(w)} \right) w = \mathcal{L}(\mathbf{y} - \mathbf{1})(1 - \mathcal{X}(\mathbf{y} - \mathbf{1}))^{-1},$$
$$\sum_{w \in X^*R} \left(\prod_{f \in \mathcal{F}} y_f^{f(w)} \right) w = (1 - \mathcal{X}(\mathbf{y} - \mathbf{1}))^{-1} \mathcal{R}(\mathbf{y} - \mathbf{1}), \quad \text{and}$$
$$\sum_{w \in LX^*R} \left(\prod_{f \in \mathcal{F}} y_f^{f(w)} \right) w = C_{\mathcal{F}}(\mathbf{y} - \mathbf{1}, LX^*R) + \mathcal{L}(\mathbf{y} - \mathbf{1})(1 - \mathcal{X}(\mathbf{y} - \mathbf{1}))^{-1} \mathcal{R}(\mathbf{y} - \mathbf{1}).$$

5 Application of Theorem 3 to $\mathcal{PS}$ (and $\mathcal{PC}$)

In light of subsection 2.2 (solving Q2 solves Q2$_{no}$), we focus on Q2. We begin with a useful digression into the setting of compositions.

Consider the alphabet $N = \{1, 2, 3, \ldots\}$, let $P \subseteq \bigcup_{m \geqslant 1} S_m$, and put

$$D_P(\mathbf{y}; z) = \sum_{(w,\nu,\beta) \in C_{\mathcal{F}_P}} \left(\prod_{p \in P} y_p^{p(\nu)} \right) q^{\operatorname{sum} w} z^{\operatorname{len} w} \quad \text{where} \quad p(\nu) = \sum_{f \in \mathcal{F}_p} f(\nu).$$

Replacement of w by $q^{\operatorname{sum} w} z^{\operatorname{len} w}$ in the first identity of Theorem 3 yields

$$\sum_{n \geqslant 0} \sum_{w \in K_n} \left(\prod_{p \in P} y_p^{p(\nu)} \right) q^{\operatorname{sum} w} z^n = \left(1 - zq(1-q)^{-1} - D_P(\mathbf{y} - \mathbf{1}; z) \right)^{-1}. \tag{12}$$

Besides being a practical tool for enumerating compositions by patterns, (12) also reveals the fact that solving Q1 solves Q2 for compositions. To illustrate both points, we deduce (8) from (7) and (12). Relative to pic $= \{132, 231\}$, set $y_{132} = y_{231} = y$. As the pic-clusters are in one-to-one correspondence with the up-down compositions of odd length greater than 1,

$$\frac{z}{1-q} + D_P(y; z/q) = \frac{1}{\sqrt{y}} \sum_{n \geqslant 0} \sum_{w \in \mathrm{UD}K_{2n+1}} q^{\operatorname{sum} w} (z\sqrt{y}/q)^{2n+1}.$$

So, (12) with z replaced by z/q and the odd part of (7) imply (8). Thus, counting up-down compositions solves the problem of counting compositions by peaks.

Theorem 2 allows the considerations of the above paragraph to be rephrased in the context of permutations. So, for permutations, solving Q1 solves Q2. Also, Theorem 2 applied to the lefthand side of (12) implies Theorem 4.

Theorem 4. *If $P \subseteq \bigcup_{m \geqslant 1} S_m$, then*

$$\sum_{n \geqslant 0} \sum_{\sigma \in S_n} \left(\prod_{p \in P} y_p^{p(\sigma)} \right) \frac{q^{\operatorname{inv} \sigma} z^n}{(q;q)_n} = \left(1 - z(1-q)^{-1} - D_P(\mathbf{y} - \mathbf{1}; z/q) \right)^{-1}.$$

Theorem 4 strengthens the main result in Rawlings [34] by dropping the restriction that P be permissible (that is, no $p \in P$ occurs as a consecutive pattern in another $r \in P$). The restricted result in [34] was used to extend some permutation results of Elizalde and Noy's [13] as well as to solve a few other problems in $\mathcal{PS}$. The example of subsection 5.3 involves a non-permissible P.

For $P = \{p \in S_m : p_1 *_1 p_2 *_2 \cdots *_{m-1} p_m\}$ where $*_1, *_2, \ldots, *_{m-1} \in \{<, >\}$, there are two common types of problems in $\mathcal{PS}$ to be considered. The first is to track P as a whole and the second involves tracking the patterns in P individually. Relative to Q2, these respective problems are to determine

$$\sum_{\sigma \in S_n} y^{P(\sigma)} q^{\operatorname{inv} \sigma} \quad \text{and} \quad \sum_{\sigma \in S_n} \left(\prod_{p \in P} y_p^{p(\sigma)} \right) q^{\operatorname{inv} \sigma}.$$

To illustrate the use of Theorem 4, we will apply it to deduce four new results. The examples in subsections 5.1 and 5.2 track particular pattern sets as wholes, the example of subsection 5.3 tracks two pattern sets of different lengths, and the example of subsection 5.4 tracks patterns individually. In doing these examples, we must enumerate permutations by up-down type.

5.1 Permutations by (i, d)-peaks and by up-down type

For $i, d \geqslant 2$, let $P_{i,d} = \{p \in S_{i+d-1} : p_1 < p_2 < \cdots < p_i > p_{i+1} > \cdots > p_{i+d-1}\}$. A consecutive occurrence of a $P_{i,d}$-pattern in a permutation σ is said to be an (i, d)-peak. In Diagram 1, $\sigma_1\sigma_2\sigma_3\sigma_4 = 2\,5\,6\,1$ is a $(3, 2)$-peak. Of course, a $(2, 2)$-peak is just a peak

as defined in subsection 2.1. Theorems 3 and 4 may be used to obtain the generating function for permutations by (i,d)-peaks as rational expressions of q-Olivier functions

$$\Phi_{i,k}(z) = \sum_{n\geqslant 0} \frac{z^{in+k}}{(q;q)_{in+k}}.$$

To this end, for $i_1, d_1, \ldots, i_m, d_m \geqslant 2$ and $k \geqslant 1$, let $\mathrm{UD}K_{i_1,d_1;\cdots;i_m,d_m;k}$ denote the set of compositions w that begin with a weakly increasing sequence $w_1 \leqslant w_2 \leqslant \cdots \leqslant w_{i_1}$ of length i_1, then continue with a strictly decreasing sequence $w_{i_1} > w_{i_1+1} > \cdots > w_{i_1+d_1-1}$ of length d_1, followed by a weakly increasing sequence of length i_2, then a strictly decreasing sequence of length d_2, and so on until ending with a weakly increasing sequence of length k.

We let $(j,d)^m$ denote the list $j,d;j,d;\ldots;j,d$ in which j,d appears m times. A composition in $\mathrm{UD}K_{i,d;(j,d)^m;k}$, for any $m \geqslant 0$, is said to be of up-down type $(i,j,d;k)$. Up-down permutations of type $(i,j,d;k)$ are similalry defined.

Corollary 3. *If, for $i,j,d \geqslant 2$, we set $\mu = i+d-2$ and $\xi_m = \sqrt[m]{-1}$, then the generating function for permutations by (i,d)-peaks and inversions is*

$$\sum_{n\geqslant 0}\sum_{\sigma\in S_n} \frac{y^{P_{i,d}(\sigma)} q^{\operatorname{inv}\sigma} z^n}{(q;q)_n} = \left(1 - z(1-q)^{-1} - \frac{K_{i,i,d;1}(\sqrt[\mu]{y-1}\,z)}{\sqrt[\mu]{y-1}}\right)^{-1}$$

where, for $k \geqslant 1$,

$$K_{i,j,d;k}(z) = \sum_{m\geqslant 0} \sum_{w\in K_{i,d;(j,d)^m;k}} q^{\operatorname{sum} w} z^{\operatorname{len} w}.$$

Moreover, $K_{i,j,d;k}(z)$ satisfies, for $d \geqslant 3$ and $\nu = j+d-2$, the recurrence

$$K_{i,j,d;k}(z) = \frac{\xi_\nu^{-\mu} K_{i,j+1,d-1;1}(\xi_\nu z)\big(z^k (q;q)_k^{-1} + \xi_\nu^{-k} K_{j,j+1,d-1;k+1}(\xi_\nu z)\big)}{1 + K_{j,j+1,d-1;1}(\xi_\nu z) - \xi_\nu^{-\mu-k} K_{i,j+1,d-1;k+1}(\xi_\nu z)}$$

with the initial condition $K_{i,j,2;k}(z) = \xi_j^{-i-k}\left[\dfrac{\Phi_{j,i}(\xi_j z)\Phi_{j,k}(\xi_j z)}{\Phi_{j,0}(\xi_j z)} - \Phi_{j,i+k}(\xi_j z)\right]$.

Before providing proof, a few examples are presented. First, the above recurrence provides a straightforward means of determining $K_{i,j,d;k}(z)$ as a rational expression of q-Olivier functions. Therefore, the generating function for permutations by (i,d)-peaks given by Corollary 3 is also a rational expression in q-Olivier functions. For instance,

$$\sum_{n\geqslant 0}\sum_{\sigma\in S_n} \frac{y^{P_{3,3}(\sigma)} q^{\operatorname{inv}\sigma} z^n}{(q;q)_n} = \left(1 - z(1-q)^{-1} - \frac{K_{3,3,3;1}(\sqrt[4]{y-1}\,z)}{\sqrt[4]{y-1}}\right)^{-1}$$

where

$$K_{3,3,3;1}(z) = \frac{-K_{3,4,2;1}(\xi_4 z)\left(z(1-q)^{-1} + \xi_4^{-1} K_{3,4,2;2}(\xi_4 z)\right)}{1 + K_{3,4,2;1}(\xi_4 z)} + \xi_4^{-1} K_{3,4,2;2}(\xi_4 z)$$

with

$$K_{3,4,2;1}(z) = -\frac{\Phi_{4,3}(\xi_4 z)\Phi_{4,1}(\xi_4 z)}{\Phi_{4,0}(\xi_4 z)} + \Phi_{4,4}(\xi_4 z) \quad \text{and}$$

$$K_{3,4,2;2}(z) = \xi_4^{-1}\left[-\frac{\Phi_{4,3}(\xi_4 z)\Phi_{4,2}(\xi_4 z)}{\Phi_{4,0}(\xi_4 z)} + \Phi_{4,5}(\xi_4 z)\right].$$

Second, Corollary 3 and the comments of subsection 2.2 may be used to solve the $\mathrm{Q}2_{no}$ version of counting permutations by (i,d)-peaks. We illustrate by obtaining Mendes and Remmel's [29] result for the case $(i,2)$. First, note that the initial condition at the end of Corollary 3 implies

$$K_{i,i,2;k}(z) \ = \frac{\xi_i^{-k}\Phi_{i,k}(\xi_i z)}{\Phi_{i,0}(\xi_i z)} - \frac{z^k}{(q;q)_k}. \tag{13}$$

Substituting $K_{i,i,2;1}(z)$ into Corollary 3, setting $y=0$, and plugging the result into Theorem 1 gives Mendes and Remmel's result, namely

$$\sum_{n\geqslant 0}\sum_{\sigma\in S_n}\frac{y^{P_{i,2no}(\sigma)}q^{\operatorname{inv}\sigma}z^n}{(q;q)_n} = \left(1 - yz(1-q)^{-1} - \frac{(1-y)\Phi_{i,1}(z)}{\Phi_{i,0}(z)}\right)^{-1}.$$

Third, by definition, $K_{i,j,d;k}(z)$ is the generating function for up-down compositions of type $(i,j,d;k)$. The classic result $z/(1-q)+K_{2,2,2;1}(z) = \tan_q z$ for up-down compositions of odd length is evident in (13). Similarly, $z/(1-q)+K_{3,3,3;1}(z)$ is the generating function for the so-called up-up-down-down compositions. Prodinger and Tshifhumulo [32] gave another recurrence, without obtaining a closed form, for the generating function for up-up-down-down compositions. With "$\geqslant$" in place of "$>$", Carlitz [7] determined the generating function for up-down compositions of type $(i,i,2;k)$.

Finally, we again underscore the value of Theorem 2 in transcribing pattern results between the settings of compositions and permutations. For instance, replacing z in the first part of Corollary 3 with qz and invoking Theorem 2 gives the generating function for compositions by (i,d)-peaks.

Likewise, $K_{i,j,d;k}(z/q)$ transcribes as the generating function for permutations by up-down type $(i,j,d;k)$ and by inversion number. So, $z/(1-q)+K_{3,3,3;1}(z/q)$ is a q-analog of Carlitz and Scoville's [9] result for up-up-down-down permutations. Using another method, Mendes, Remmel, and Riehl [30] obtained generating functions for up-down permutations of type $(i,j,2;k)$ with $k \leqslant j$. For up-down type $(0,j,2;k)$, see Carlitz [5].

Proof of Corollary 3. The relevant cluster generating function is

$$D_{P_{i,d}}(y;z/q) = \sum_{(w,\nu,\beta)\in C_{\mathcal{F}_{P_{i,d}}}} q^{\operatorname{sum} w} y^{P_{i,d}(\nu)}(z/q)^{\operatorname{len} w}.$$

Clearly, a composition is $P_{i,d}$-coverable if and only if it belongs to $\mathrm{UD}K_{(i,d)^m;1}$ for some $m \geqslant 1$. Moreover, each $w \in UDK_{(i,d)^m;1}$ has but one $P_{i,d}$-covering. It follows that

$$D_{P_{i,d}}(y; z/q) = \frac{1}{\sqrt[\mu]{y}} \sum_{m \geqslant 1} \sum_{w \in \mathrm{UD}K_{(i,d)^m;1}} q^{\operatorname{sum} w} (\sqrt[\mu]{y}\, z/q)^{\operatorname{len} w} = \frac{K_{i,i,d;1}(\sqrt[\mu]{y}\, z)}{\sqrt[\mu]{y}}.$$

The above equality and Theorem 4 imply the first part of Corollary 3.

There are several theoretical frameworks (including the Pattern Algebra of Goulden and Jackson [18] described in section 8) for determining $K_{i,j,d;k}(z)$. We will use Theorem 3; in this approach, up-down compositions having strictly descending runs of length d are exchanged for "straighter" up-down compositions having strictly descending runs of length $d-1$.

Let $N_{i,d} = \{w \in N^{i+d-1} : w_1 \leqslant w_2 \leqslant \cdots \leqslant w_i > w_{i+1} > \cdots > w_{i+d-1}\}$. For any word w in N^* or in $N^*_{i,d}$, the symbol $\operatorname{len} w$ is always to be interpreted as the length of w relative to the alphabet N.

Relative to the alphabet $X_{d-1} = N_{k,1} \bigcup \left(\bigcup_{l \geqslant 2} N_{l,d-1}\right)$, let $\mathcal{F}_{d-1}$ denote the set of words of the form uv where $u, v \in X_{d-1}$ and such that the last letter in the factor u is less than or equal to the first letter in v. For a word $w = u_{(1)}u_{(2)} \ldots u_{(n)}$ with each $u_{(m)} \in X_{d-1}$, let $\operatorname{ris} w = \sum_{f \in \mathcal{F}_{d-1}} f(w)$.

As $\mathrm{UD}K_{i,d;(j,d)^m;k} = \{w \in N_{i,d-1} N^*_{j,d-1} N_{k,1} : \operatorname{ris} w = 0\}$, Theorem 3 leads to

$$K_{i,j,d;k}(z) = \sum_{m \geqslant 0} \sum_{w \in N_{i,d-1} N^m_{j,d-1} N_{k,1}} 0^{\operatorname{ris} w} q^{\operatorname{sum} w} z^{\operatorname{len} w} = A_{1,d} + \frac{A_{2,d} A_{3,d}}{1 + A_{4,d}}$$

where

$$A_{1,d} = \sum_{m \geqslant 0} (-1)^{m+1} \sum_{\substack{w \in N_{i,d-1} N^m_{j,d-1} N_{k,1} \\ \operatorname{ris} w = m+1}} q^{\operatorname{sum} w} z^{\operatorname{len} w}, \quad A_{2,d} = \sum_{m \geqslant 0} (-1)^{m} \sum_{\substack{w \in N_{i,d-1} N^m_{j,d-1} \\ \operatorname{ris} w = m}} q^{\operatorname{sum} w} z^{\operatorname{len} w},$$

$$A_{3,d} = \sum_{m \geqslant 0} (-1)^{m} \sum_{\substack{w \in N^m_{j,d-1} N_{k,1} \\ \operatorname{ris} w = m}} q^{\operatorname{sum} w} z^{\operatorname{len} w}, \text{ and } A_{4,d} = \sum_{m \geqslant 1} (-1)^{m} \sum_{\substack{w \in N^m_{j,d-1} \\ \operatorname{ris} w = m-1}} q^{\operatorname{sum} w} z^{\operatorname{len} w}.$$

Completion of the proof is now just a matter of determining the sums $A_{l,d}$. Being of similar nature, only a few are evaluated here.

As an example of the case $d = 2$, note that

$$1 + A_{4,2} = 1 + \sum_{m \geqslant 1} (-1)^m z^{jm} \sum_{0 \leqslant w_1 \leqslant \cdots \leqslant w_{jm}} q^{w_1 + \cdots + w_{jn}} = \sum_{m \geqslant 0} \frac{(-1)^m z^{jm}}{(q;q)_{jm}} = \Phi_{j,0}(\xi_j z).$$

For $A_{1,d}$ with $d \geqslant 3$, note that $\operatorname{len} w = \mu + \nu m + k$ and that $\{w \in N_{i,d-1} N^m_{j,d-1} N_{k,1} : \operatorname{ris} w = m+1\} = \mathrm{UD}K_{i,d-1;(j+1,d-1)^m;k+1}$. Thus, $A_{1,d} = -\xi_\nu^{-\mu-k} K_{i,j+1,d-1;k+1}(\xi_\nu z)$. $\square$

5.2 Uniform range distributions

For $i, d \geqslant 2$ and $m \geqslant 1$, let $P_{(i,d)^m}$ denote the set of patterns $p \in S_{(i+d-2)m+1}$ that begin with an increasing sequence $p_1 < \cdots < p_i$ of length i, continue with a decreasing sequence $p_i > p_{i+1} > \cdots > p_{i+d-1}$ of length d, followed by an increasing sequence $p_{i+d-1} < p_{i+d} < \cdots < p_{2i+d-2}$ of length i, and so on so as to form m consecutive (i,d)-peaks. The consecutive occurrence of a $p \in P_{(i,d)^m}$ in a permutation σ is said to be a uniform m-peak range of type (i,d). The following result extends Corollary 3 to uniform ranges. The coefficient of z^n in a formal power series $F(z)$ is denoted by $F|_{z^n}$.

Corollary 4. *If $i, d \geqslant 2$, $m \geqslant 1$, and $\nu = i + d - 2$, then the generating function for permutations by uniform m-peak ranges and inversions is*

$$\sum_{n \geqslant 0} \sum_{\sigma \in S_n} \frac{y^{P_{(i,d)^m}(\sigma)} q^{\operatorname{inv} \sigma} z^n}{(q;q)_n} = \left(1 - \frac{z}{1-q} - \sum_{n \geqslant m} A_{n,m}(y-1) B_n(q) z^{n\nu+1}\right)^{-1}$$

where

$$A_{n,m}(y) = \left.\frac{y z^m (1-z)}{1 - z - yz(1-z^m)}\right|_{z^n} \quad \text{and} \quad B_n(q) = K_{i,i,d;1}(z)|_{z^{n\nu+1}}$$

with $K_{i,i,d;1}(z)$ as determined in Corollary 3.

The case $i = d = 2$ with $y = 0$ of Corollary 4 provides a solution to the problem posed by Kitaev [25, Problem 1] of counting permutations that avoid $(2m+1)$-reverse-alternating patterns (which, as noted in [25], is the same as the number of permutations that avoid $(2m+1)$-alternating patterns). The case for even-length alternating patterns may be dealt with similarly.

Proof of Corollary 4. First, note that

$$D_{P_{(i,d)^m}}(y; z/q) = \sum_{(w,\nu,\beta) \in C_{\mathcal{F}_{P_{(i,d)^m}}}} q^{\operatorname{sum} w} y^{P_{(i,d)^m}(\nu)} (z/q)^{\operatorname{len} w}.$$

Next, observe that a composition is $P_{(i,d)^m}$-coverable if and only if it belongs to the set of up-down compositions $\bigcup_{n \geqslant m} \mathrm{UD}K_{(i,d)^n;1}$. Moreover, there may be multiple $P_{(i,d)^m}$-coverings (ν, β) for such a composition. For instance, $w = 2\,3\,1\,4\,2\,3\,2\,2\,1 \in \mathrm{UD}K_{(2,2)^4;1}$ is $P_{(2,2)^2}$-covered by $((23142, 23221); (1,5))$ and by $((23142, 14232, 23221); (1,3,5))$.

For $n \geqslant m \geqslant 1$ and $k \geqslant 1$, let $a_{n,m,k}$ denote the number of times that a given $w \in \mathrm{UD}K_{(i,d)^n;1}$ appears in a $P_{(i,d)^m}$-cluster (w, ν, β) with $P_{(i,d)^m}(\nu) = k$. Of course, $a_{n,m,k}$ is independent of the choice of $w \in \mathrm{UD}K_{(i,d)^n;1}$.

For $n \geqslant 1$, let

$$A_{n,m}(y) = \sum_{k \geqslant 1} a_{n,m,k} y^k \quad \text{and} \quad B_n(q) = \sum_{w \in \mathrm{UD}K_{(i,d)^n;1}} q^{\operatorname{sum} w}.$$

Evidently, $B_n(q) = K_{i,i,d;1}(z)|_{z^{n\nu+1}}$. It follows that

$$D_{P_{(i,d)^m}}(y; z/q) = \sum_{n \geqslant m} A_{n,m}(y) B_n(q) z^{n\nu+1}.$$

In view of Theorem 4, we need only establish the formula for $A_{n,m}(y)$. Note that a typical $P_{(i,d)^m}$-cluster that contributes to the count $a_{n,m,k}$ is of the form $(w, \nu, (b_1, b_2, \ldots, b_k))$ with b_2 equaling $i+d-1$, $2(i+d-2)+1$, ..., or $m(i+d-2)+1$. So, for $n \geqslant m \geqslant 1$ and $k \geqslant 2$, $a_{n,m,k} = \sum_{j=1}^{m} a_{n-j,m,k-1}$. Routine computations then lead to the fact that

$$\sum_{n \geqslant m} \sum_{k \geqslant 1} a_{n,m,k} y^k z^n = \frac{yz^m(1-z)}{1-z-yz(1-z^m)}.$$

Thus, $A_{n,m}(y) = yz^m(1-z)\left(1-z-yz(1-z^m)\right)^{-1}|_{z^n}$. □

5.3 Permutations by peaks and twin peaks

Let $\mathrm{tpic} = \{p \in S_5 : p_1 < p_2 > p_3 < p_4 > p_5\}$. A consecutive occurrence of $p \in \mathrm{tpic}$ in a permutation is referred to as a twin peak. The set $P = \mathrm{pic} \cup \mathrm{tpic}$ is not permissible and therefore not within the scope of the theorem in [34]. However, Theorem 4 makes the joint enumeration of permutations by peaks and twin peaks straightforward; we just need to determine

$$D_P(x, y; z/q) = \sum_{(w,\nu,\beta) \in C_{\mathcal{F}_P}} x^{\mathrm{pic}(\nu)} y^{\mathrm{tpic}(\nu)} q^{\mathrm{sum}\, w} (z/q)^{\mathrm{len}\, w}. \tag{14}$$

To this end, first note that the set of P-coverable compositions corresponds to the set of up-down compositions $\bigcup_{n \geqslant 1} \mathrm{UD}K_{2n+1}$.

For $w \in \mathrm{UD}K_{2n+1}$, let $a_{n,l,k}$ denote the number of P-coverings (ν, β) of w by l peaks and k twin peaks. Set $A_n(x, y) = \sum_{l,k \geqslant 0} a_{n,l,k} x^l y^k$. From the easily deduced recurrence relationship

$$a_{n,l,k} = a_{n-1,l-1,k} + a_{n-1,l-1,k-1} + a_{n-2,l-1,k-1} + a_{n-1,l,k-1} + a_{n-2,l,k-1},$$

it follows that $A_n(x, y) = (xz + yz^2 + xyz^2)(1 - xz - xyz - xyz^2 - yz - yz^2)^{-1}|_{z^n}$.

Let $B_n(q) = \sum_{w \in \mathrm{UD}K_{2n+1}} q^{\mathrm{sum}\, w} = \tan_q z|_{z^{2n+1}}$. In view of (14), we have

$$D_P(x, y; z/q) = \sum_{n \geqslant 1} A_n(x, y) B_n(q) (z/q)^{2n+1}.$$

Finally, the last equality and Theorem 4 imply

$$\sum_{n \geqslant 0} \sum_{\sigma \in S_n} \frac{q^{\mathrm{inv}\, \sigma} x^{\mathrm{pic}(\sigma)} y^{\mathrm{tpic}(\sigma)} z^n}{(q;q)_n} = \left(1 - \frac{z}{1-q} - \sum_{n \geqslant 1} A_n(x-1, y-1) B_n(q) (z/q)^{2n+1}\right)^{-1}.$$

Another solution to the joint peak and twin peak $\mathcal{PS}$ is given in subsection 8.4.

5.4 Permutations and up-down permutations by (i,m)-maxima

For $i \geqslant 2$ and $1 \leqslant m \leqslant i$, let $p_{(m)}$ denote the unique permutation in S_{i+1} with $p_{(m)1} < p_{(m)2} < \cdots < p_{(m)i}$ and $p_{(m)i+1} = i+1-m$. Also, let $P_i = \{p_{(1)}, p_{(2)}, \ldots, p_{(i)}\}$. A consecutive occurrence of $p_{(m)} \in P_i$ in a permutation σ is said to be an (i,m)-maximum. Carlitz and Scoville [8] refer to $(2,1)$-maxima and $(2,2)$-maxima respectively as rising and falling maxima; they expressed the joint distribution of $\{12, 132, 231, 21\}$ over $0S_n0 = \{0\sigma0 : \sigma \in S_n\}$ in terms of a second order differential equation.

The statement of our next result requires the q-binomial coefficient which, for integers n and k, is defined as

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{cases} (q^{n-k+1};q)_k/(q;q)_k & \text{if } n \geqslant k \geqslant 0 \\ 0 & \text{otherwise.} \end{cases}$$

Corollary 5. *If $i \geqslant 2$ and $1 \leqslant m \leqslant i$, then the generating function for permutations by (i,m)-maxima and inversions is*

$$\sum_{n\geqslant 0}\sum_{\sigma\in S_n}\left(\prod_{m=1}^{i} y_m^{p_{(m)}(\sigma)}\right)\frac{q^{\operatorname{inv}\sigma}z^n}{(q;q)_n} = \left(1 - \frac{\Phi_{i,1}(\mathbf{y}-\mathbf{1};\xi_i z)}{\xi_i\Phi_{i,0}(\mathbf{y}-\mathbf{1};\xi_i z)}\right)^{-1}$$

where $\xi_i = \sqrt[i]{-1}$ and

$$\Phi_{i,k}(y_1,\ldots,y_i;z) = \sum_{n\geqslant 0}\frac{z^{in+k}}{(q;q)_{in+k}}\prod_{j=0}^{n-1}\left(y_i + \sum_{m=1}^{i-1}(y_i - y_m)q^m\begin{bmatrix} ij+k+m-1 \\ m \end{bmatrix}\right).$$

For $i = 2$, $y_1 = y$, and $y_2 = 1$, Corollary 5 gives the q-analog obtained in [29, 34] of Elizalde and Noy's [13] result for permutations by $p = 132$:

$$\sum_{n\geqslant 0}\sum_{\sigma\in S_n}\frac{y^{132(\sigma)}q^{\operatorname{inv}\sigma}z^n}{(q;q)_n} = \left(1 - \sum_{n\geqslant 0}\frac{(y-1)^n q^n z^{2n+1}}{(q^2;q^2)_n(1-q^{2n+1})(1-q)^n}\right)^{-1}.$$

Proof of Corollary 5. By Theorem 4, we need to compute

$$D_{P_i}(\mathbf{y};z/q) = \sum_{(w,\nu,\beta)\in C_{\mathcal{F}}}\left(\prod_{m=1}^{i} y_m^{p_{(m)}(\nu)}\right)q^{\operatorname{sum} w}(z/q)^{\operatorname{len} w} \quad \text{where} \quad \mathcal{F} = \bigcup_{p\in P_i}\mathcal{F}_p.$$

Define $\mathcal{L}_i$ to be the set of compositions w of length $in+1$ for any $n \geqslant 0$ such that $w_j > w_{j+1}$ if and only if j is a positive multiple of i. Note that a composition w is P_i-coverable if and only if $w \in \mathcal{L}_i$ and $\operatorname{len} w > 1$. Moreover, such a composition has but one P_i-covering. Thus,

$$\frac{z}{1-q} + D_{P_i}(\mathbf{y};z/q) = \sum_{w\in\mathcal{L}_i}\left(\prod_{p\in P_i} y_p^{p(w)}\right)q^{\operatorname{sum} w}(z/q)^{\operatorname{len} w}.$$

Let $\mathcal{L}_i(y_1, y_2, \ldots, y_i; z)$ denote the right side of the above equality. For $w \in \mathcal{L}_i$ of length $in+1$, observe that $p_{(1)}(w) + p_{(2)}(w) + \cdots + p_{(i)}(w) = n$. So

$$\mathcal{L}_i(y_1, \ldots, y_i; z) = y_i^{(-1/i)} \mathcal{L}_i(y_1/y_i, \ldots, y_{i-1}/y_i, 1; z\sqrt[i]{y_i}).$$

We therefore only need to determine $\mathcal{L}_i(y_1, \ldots, y_{i-1}, 1; z)$.

Recall that $\Lambda_i = \{w \in N^i : w_1 \leqslant w_2 \leqslant \cdots \leqslant w_i\}$. As before, the symbol $\operatorname{len} w$ denotes the length of w relative to the alphabet N.

To determine $\mathcal{L}_i(y_1, \ldots, y_{i-1}, 1; z)$, we appeal to Theorem 3. We work with the factor set $\mathcal{G} = \{uv : u \in \Lambda_i, v \in N\}$. For each $uv \in \mathcal{G}$, define

$$y_{uv} = \begin{cases} t & \text{if } u_i \leqslant v \\ y_m & \text{if } u_{i+1-m} > v \geqslant u_{i-m} \text{ for } 1 \leqslant m \leqslant i-1 \\ 1 & \text{if } u_1 > v. \end{cases}$$

For $w = u_{(1)}u_{(2)} \ldots u_{(n)}$ with $u_{(1)}, \ldots, u_{(n-1)} \in \Lambda_i$ and $u_{(n)} \in N$, let $\operatorname{ris} w$ be the number of indices j such that the last letter of $u_{(j)}$ is less than or equal to the first letter of $u_{(j+1)}$. Note that $\prod_{g \in \mathcal{G}} y_g^{g(w)} = t^{\operatorname{ris} w} \prod_{m=1}^{i-1} y_m^{p_{(m)}(w)}$.

As $\mathcal{L}_i(y_1, \ldots, y_{i-1}, 1; z) = \sum_{w \in \Lambda_i^* N} 0^{\operatorname{ris} w} q^{\operatorname{sum} w} (z/q)^{\operatorname{len} w} \prod_{m=1}^{i-1} y_m^{p_{(m)}(w)}$, Theorem 3 implies

$$\mathcal{L}_i(y_1, \ldots, y_{i-1}, 1; z) = \frac{\displaystyle\sum_{n \geqslant 0} (-1)^n z^{in+1} \sum_{0 \leqslant \operatorname{sum} \alpha \leqslant n} \prod_{m=1}^{i-1} (1-y_m)^{\alpha_m} \sum_{w \in \mathcal{C}_{i,1,n;\alpha}} q^{\operatorname{sum} w - in - 1}}{1 + \displaystyle\sum_{n \geqslant 0} (-1)^n z^{in+i} \sum_{0 \leqslant \operatorname{sum} \alpha \leqslant n} \prod_{m=1}^{i-1} (1-y_m)^{\alpha_m} \sum_{w \in \mathcal{C}_{i,i,n;\alpha}} q^{\operatorname{sum} w - in - i}}, \tag{15}$$

where the sums on the right are over $\alpha \in \{0, 1, 2, \ldots\}^{i-1}$ and w in $\mathcal{C}_{i,k,n;\alpha} = \{w \in \Lambda_i^n \Lambda_k : p_{(i)}(w) = 0 \text{ and } p_{(m)}(w) = \alpha_m \text{ for } 1 \leqslant m \leqslant i-1\}$.

The proof is completed by showing that the numerator and denominator in (15) are respectively $\Phi_{i,1}(y_1, \ldots, y_{i-1}, 1; \xi_i z)/\xi_i$ and $\Phi_{i,0}(y_1, \ldots, y_{i-1}, 1; \xi_i z)$. □

Another generating function for permutations by (i, m)-maxima is derived in subsection 9.2. Notably, Theorem 2 applied to the lefthand side of (15) yields the generating function for the set $\mathrm{UD}S_{i,i,2;1}$ of up-down permutations of type $(i, i, 2; 1)$ by (i, m)-maxima. Setting $y_1 = y_2 = \ldots = y_i = 1$, replacing z by $(1-q)z$, and letting $q \to 1$ in Corollary 6 gives a result of Carlitz's [5].

Corollary 6. *For $i \geqslant 2$, the generating function for permutations of up-down type $(i, i, 2; 1)$ by (i, m)-maxima is given by*

$$\frac{z}{1-q} + \sum_{\sigma \in \mathrm{UD}S_{i,i,2;1}} \left(\prod_{m=1}^{i} y_m^{p_{(m)}} \right) \frac{q^{\operatorname{inv} \sigma} z^{\operatorname{len} \sigma}}{(q;q)_{\operatorname{len} \sigma}} = \frac{\Phi_{i,1}(y_1, \ldots, y_{i-1}, 1; \xi_i z)}{\xi_i \Phi_{i,0}(y_1, \ldots, y_{i-1}, 1; \xi_i z)}.$$

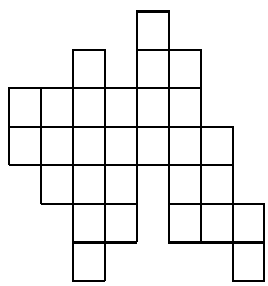
6 Ridge patterns in CCPs

A column-convex polyomino is constructed by successively gluing a finite sequence of columns, each consisting of a finite number of unit square cells, together in the xy-plane so that (i) the lower left vertex of the leftmost column has coordinates (0,0), (ii) each pair of adjacent columns share an edge of positive integer length, and (iii) all cell vertices have integer coordinates.

The area, perimeter, and number of columns of a column-convex polyomino Q are denoted by $\operatorname{area} Q$, $\operatorname{per} Q$, and $\operatorname{col} Q$. In Diagram 3, $\operatorname{area} Q = 29$, $\operatorname{per} Q = 38$, and $\operatorname{col} Q = 8$. The kth column of Q will be denoted by Q_k. We sometimes write $Q = Q_1 Q_2 \cdots Q_{\operatorname{col} Q}$.

The enumeration of CCPs and of subclasses of CCPs by various statistics has been widely studied. Polyomino enumeration is surveyed in Delest [10], Guttmann [19], Rensburg [35], and Viennot [39]. Our purpose here is to essentially initiate the study of CCPs by consecutive (or ridge) patterns.

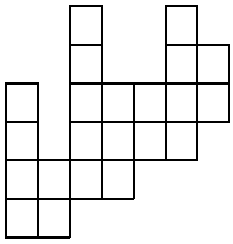
Diagram 3: A Column-Convex Polyomino

$Q =$ 

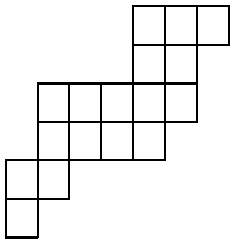
The simplest ridge patterns are formed between two adjacent columns. For a column-convex polyomino Q, we say that an upper ascent (respectively upper level, upper descent) occurs at index k if the top cell in Q_k is lower than (respectively level with, higher than) the top cell in Q_{k+1}. Lower ascents, lower levels, and lower descents are similarly defined along the lower ridge. In Diagram 3, Q has lower descents at indices 1,2,5, and 7. The numbers of upper ascents, upper levels, upper descents, lower ascents, lower levels, and lower descents in Q are respectively denoted by $\operatorname{uasc} Q$, $\operatorname{ulev} Q$, $\operatorname{udes} Q$, $\operatorname{lasc} Q$, $\operatorname{llev} Q$, and $\operatorname{ldes} Q$. In Diagram 3, $\operatorname{uasc} Q = 2$ and $\operatorname{llev} Q = 1$.

As displayed in Diagram 4, the two-column ridge patterns may be used to characterize many of the common subclasses of CCPs.

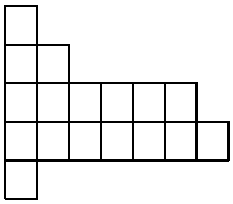
Diagram 4: Common Classes of CCPs.



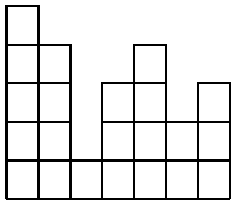
Directed Column–Convex Polyomino (DCCP): No lower descents



Parallelogram Polyomino: No lower or upper descents



Stack Polyomino: No upper ascents and no lower descents



Wall Polyomino: No lower ascents and no lower descents

More complex consecutive patterns along either the lower or upper ridges are formed by segments of 3 or more columns.

The relative height of a CCP Q, denoted by $\operatorname{relh} Q$, is defined to be the y-ordinate of the top edge in the rightmost column of Q. In Diagram 3, $\operatorname{relh} Q = -1$. The relative height of a parallelogram polyomino is known as its row number.

6.1 Verification that $\mathcal{PC} \subset \mathcal{PCCP} \subset \mathcal{PW}$

Let WP_n be the set of wall polyominoes with n columns. The map $\gamma_n : K_n \to \mathrm{WP}_n$ defined by $\gamma_n(w) = Q$ where Q_k has w_k cells is a bijection such that

$$\operatorname{area} Q = \operatorname{sum} w \quad \text{and} \quad \operatorname{per} Q - 2 \operatorname{col} Q = \operatorname{var} w. \tag{16}$$

For example, γ_7 maps the composition $w = 5\,4\,1\,3\,4\,2\,3 \in K_7$ to the wall polyomino displayed in Diagram 4. Interestingly, the second part of (16) relates the variation of a composition to the perimeter of a wall polyomino, and (10) together with the first part of (16) provides a connection between the inversion number of a permutation and the area of a wall polyomino.

Through γ_n, a consecutive p-pattern in a composition w induces an upper ridge p-pattern in the associated wall polyomino Q. For instance, $Q_kQ_{k+1}Q_{k+2}$ is deemed a 132-pattern in Q if $w_kw_{k+1}w_{k+2}$ is a 132-pattern in the associated w; that is, $Q_kQ_{k+1}Q_{k+2}$ is a 132-pattern if $Q_{k+1}Q_{k+2}$ is an upper descent and if the top cell in Q_{k+2} is level with or above the top cell in Q_k. The number of times an upper ridge pattern p occurs in Q is denoted by $p(Q)$.

The bijection γ_n immediately implies $\mathcal{PC} \subset \mathcal{PCCP}$: If $P \subseteq \cup_{m\geqslant 1} S_m$ and if $B_n \subseteq K_n$, then

$$\sum_{n\geqslant 0}\sum_{w\in B_n} c^{\operatorname{var} w} q^{\operatorname{sum} w}\left(\prod_{p\in P} y_p^{p(w)}\right) z^n = \sum_{n\geqslant 0}\sum_{Q\in\gamma_n(B_n)} c^{\operatorname{per} Q} q^{\operatorname{area} Q}\left(\prod_{p\in P} y_p^{p(Q)}\right)\frac{z^n}{c^{2n}}. \tag{17}$$

Of course, (17) also holds for maximal numbers of non-overlapping patterns.

To see that $\mathcal{PCCP} \subset \mathcal{PW}$, consider the alphabet of biletters $X = \left\{\binom{j}{m} : j, m \in N\right\}$ and let

$$\mathcal{Y} = \bigcup_{n\geqslant 0}\left\{\begin{pmatrix} j_1 & j_2 & \dots & j_n \\ m_1 & m_2 & \dots & m_n \end{pmatrix} \in X^n : m_n = 1 \text{ and } j_k + j_{k+1} > m_k \text{ for } 1 \leqslant k < n\right\}.$$

For a column-convex polyomino Q with n columns, define

$$\delta(Q) = \begin{pmatrix} j_1 & j_2 & \dots & j_n \\ m_1 & m_2 & \dots & m_n \end{pmatrix} \tag{18}$$

where j_k is the number of cells in Q_k, $m_n = 1$, and, for $1 \leqslant k < n$, m_k is the change in the y-ordinate from the bottom edge of Q_{k+1} to the top edge of Q_k. For Q in Diagram 3, $\delta(Q) = \binom{2\,3\,6\,4\,4\,5\,3\,2}{3\,5\,5\,2\,6\,5\,4\,1}$.

The map δ is a bijection from CCP to $\mathcal{Y}$. As such, δ allows CCPs to be viewed as words. Such a viewpoint is implicit in Temperley [37] and explicit in Bousquet-Mélou and Viennot [3]. Thus, a problem in $\mathcal{PCCP}$ may readily be converted into a problem in $\mathcal{PW}$; so $\mathcal{PCCP} \subset \mathcal{PW}$.

7 Application of Theorem 3 to the set $\mathcal{PCCP}$

The inclusion $\mathcal{PCCP} \subset \mathcal{PW}$ means that Theorem 3 may be applied to solving problems in $\mathcal{PCCP}$. We present two examples on directed column-convex polyominoes.

7.1 DCCPs by two-column ridge patterns

Our first example enumerates DCCPs by the five two-column ridge patterns, perimeter, relative height, area, and column number.

Corollary 7. *The generating function*

$$G = \sum_{Q\in\mathrm{DCCP}} a_u^{\operatorname{uasc} Q} a_l^{\operatorname{lasc} Q} b_u^{\operatorname{ulev} Q} b_l^{\operatorname{llev} Q} c^{\operatorname{per} Q} d^{\operatorname{udes} Q} h^{\operatorname{relh} Q} q^{\operatorname{area} Q} z^{\operatorname{col} Q}$$

is given by

$$G = \frac{c^2 h \displaystyle\sum_{n\geqslant 0}\frac{(c^2qz)^{n+1}}{1-c^2hq^{n+1}}\prod_{k=1}^{n}\left(b_l + \frac{a_l c^2 h q^k}{1-c^2hq^k}\right)\left(b_u + \frac{c^2dq^k}{1-c^2q^k} - \frac{a_u}{1-q^k}\right)}{1 - a_u \displaystyle\sum_{n\geqslant 1}\frac{(c^2qz)^n}{1-q^n}\prod_{k=1}^{n}\left(b_l + \frac{a_l c^2 h q^k}{1-c^2hq^k}\right)\prod_{k=1}^{n-1}\left(b_u + \frac{c^2dq^k}{1-c^2q^k} - \frac{a_u}{1-q^k}\right)}.$$

Proof. Define

$$H(b_u, b_l, d, z) = \sum_{Q \in \mathrm{DCCP}} b_u^{\mathrm{ulev}\, Q} b_l^{\mathrm{llev}\, Q} c^{\mathrm{per}\, Q} d^{\mathrm{udes}\, Q} h^{\mathrm{relh}\, Q} q^{\mathrm{area}\, Q} z^{\mathrm{col}\, Q}. \tag{19}$$

As $\mathrm{uasc}\, Q = \mathrm{col}\, Q - \mathrm{ulev}\, Q - \mathrm{udes}\, Q - 1$ and $\mathrm{lasc}\, Q = \mathrm{col}\, Q - \mathrm{llev}\, Q - 1$, it follows that

$$G = \frac{1}{a_u a_l} H(b_u/a_u, b_l/a_l, d/a_u, a_u a_l z). \tag{20}$$

It then suffices to determine H.

Consider $X = \left\{ \binom{j}{m} : j, m \text{ integers}, j \geqslant m \geqslant 1 \right\}$. Let $R = \left\{ \binom{j}{m} \in X : m = 1 \right\}$ and, for a statement S, let $\chi(S)$ be 1 if S is true and 0 otherwise. An element $\binom{j_1\, j_2 \ldots j_n}{m_1\, m_2 \ldots m_n} \in X^n$ will be abbreviated by $\binom{j}{m}$; so the kth letter in $\binom{j}{m}$ is $\binom{j}{m}_k = \binom{j_k}{m_k}$.

Let $\mathcal{F} = \left\{ \binom{j}{m} \in X^2 : m_1 \geqslant j_2 \right\}$. For $f = \binom{j}{m} \in \mathcal{F}$, set $y_f = c^{2(m_1 - j_2)} d (b_u d^{-1})^{\chi(m_1 = j_2)}$. When restricted to DCCP, the map δ in (18) is a bijection onto $X^* R$. Moreover, if $Q = Q_1 Q_2 \ldots Q_n \in \mathrm{DCCP}$ and $\delta(Q) = \binom{j}{m} \in X^{n-1} R$, then

$$\begin{gathered} \mathrm{area}\, Q = \sum j, \ \mathrm{per}\, Q = 2\left(n + \mathrm{relh}\, Q + S\right), \\ \mathrm{relh}\, Q = \mathrm{sum}\, j - \mathrm{sum}\, m + 1, \text{ and } \ b_u^{\mathrm{ulev}\, Q} d^{\mathrm{udes}\, Q} c^{2S} = \prod_{f \in \mathcal{F}} y_f^{f\binom{j}{m}} \end{gathered} \tag{21}$$

where $S = \sum_{k=1}^{n} (m_k - j_{k+1}) \chi(m_k > j_{k+1})$. The facts in (21) regarding area and relative height were observed by Bousquet-Mélou and Viennot [3].

It follows from (19) and (21) that

$$H = c^2 h \sum_{\binom{j}{m} \in X^* R} q^{\mathrm{sum}\, j} (c^2 h)^{\mathrm{sum}\, j - \mathrm{sum}\, m} (c^2 z)^{\mathrm{len}\binom{j}{m}} \left(\prod_{f \in \mathcal{F}} y_f^{f\binom{j}{m}} \right)^{\mathrm{len}\binom{j}{m} - 1} \prod_{k=1}^{\mathrm{len}\binom{j}{m} - 1} b_l^{\chi(j_k = m_k)}. \tag{22}$$

Note that an $\mathcal{F}$-cluster $\left(\binom{j}{m}, \nu, \beta \right)$ has $\binom{j}{m} \in X^n$, $\nu = \left(\binom{j_1\, j_2}{m_1 m_2}, \binom{j_2\, j_3}{m_2 m_3}, \ldots, \binom{j_{n-1}\, j_n}{m_{n-1} m_n} \right)$, and $\beta = (1, 2, \ldots, n-1)$ for some $n \geqslant 2$. So, application of Theorem 3 to (22) yields

$$H = \frac{c^2 h \sum_{n \geqslant 0} (c^2 z)^{n+1} \sum_T(n)}{1 - \sum_{n \geqslant 1} (c^2 z)^n \sum_B(n)} \tag{23}$$

where $\sum_T(n)$ is

$$\sum_{\binom{j}{m}} (c^2 h)^{\mathrm{sum}\, j - \mathrm{sum}\, m} q^{\mathrm{sum}\, j} \prod_{k=1}^{n} b_l^{\chi(j_k = m_k)} \left(c^{2(m_k - j_{k+1})} d (b_u d^{-1})^{\chi(m_k = j_{k+1})} - 1 \right)$$

summed over $\binom{j}{m}$ satisfying $j_1 \geqslant m_1 \geqslant j_2 \geqslant \ldots \geqslant j_{n+1} \geqslant m_{n+1} = 1$ and $\sum_B(n)$ is

$$\sum_{\binom{j}{m}} (c^2 h)^{\mathrm{sum}\, j - \mathrm{sum}\, m} q^{\mathrm{sum}\, j} b_l^{\chi(j_n = m_n)} \prod_{k=1}^{n-1} b_l^{\chi(j_k = m_k)} \left(c^{2(m_k - j_{k+1})} d (b_u d^{-1})^{\chi(m_k = j_{k+1})} - 1 \right)$$

summed over $\binom{j}{m}$ satisfying $j_1 \geqslant m_1 \geqslant j_2 \geqslant \ldots \geqslant j_n \geqslant m_n \geqslant 1$. Both $\sum_T(n)$ and $\sum_B(n)$ are nested geometric sums. As such, they are easily determined. For instance, $\sum_T(1)$ equals

$$q^2 \sum_{j_2 \geqslant 1} (c^2hq^2)^{j_2-1} \sum_{m_1 \geqslant j_2} q^{m_1-j_2} \left(c^{2(m_1-j_2)} d (b_u d^{-1})^{\chi(m_1=j_2)} - 1\right) \sum_{j_1 \geqslant m_1} b_l^{\chi(j_1=m_1)} (c^2hq)^{j_1-m_1}$$

$$= \frac{q^2}{1-c^2hq^2} \left(b_u + \frac{c^2dq}{1-c^2q} - \frac{1}{1-q}\right)\left(b_l + \frac{c^2hq}{1-c^2hq}\right).$$

In general,

$$\sum_T(n) = \frac{q^{n+1}}{1-c^2hq^{n+1}} \prod_{k=1}^{n} \left(b_l + \frac{c^2hq^k}{1-c^2hq^k}\right)\left(b_u + \frac{c^2dq^k}{1-c^2q^k} - \frac{1}{1-q^k}\right) \text{ and}$$

$$\sum_B(n) = \frac{q^n}{1-q^n} \prod_{k=1}^{n} \left(b_l + \frac{c^2hq^k}{1-c^2hq^k}\right) \prod_{k=1}^{n-1} \left(b_u + \frac{c^2dq^k}{1-c^2q^k} - \frac{1}{1-q^k}\right).$$

The last two equalities for $\sum_T(n)$ and $\sum_B(n)$ together with (23) and (20) complete the proof. □

Corollary 7 (with $a_u = a$, $b_u = b$, $a_l = 0$, $b_l = h = 1$, and z replaced by z/c^2) with (17) implies Corollary 2 of subsection 3.3. Corollary 7 also implies many known results, a few of which are displayed in Table 1.

Table 1

Polyominoes	Distribution	Reference
DCCP	(area, per, relh, udes, col) $a_u, a_l, b_u, b_l = 1$	Rawlings [33]
DCCP	(area, per, relh, col) $a_u, a_l, b_u, b_l, d = 1$	Bousquet–Melou[4]
PP	(area, uasc, lasc, col) $a_u, a_l, c, h = 1;\ d = 0$	Delest, Dubernard, and Dutour [12]
PP	(area, col) $a_u, a_l, b_u, b_l, c, h = 1;\ d = 0$	Delest and Fédou [11]

The noted distribution of Delest, Dubernard, and Dutour [12] also tracked the area of the leftmost column; their notion of corners coincide exactly with upper and lower ascents. Bousquet-Mélou's entry included both the left and right column areas.

7.2 DCCPs by valleys along the upper ridge

A column-segment $Q_kQ_{k+1}Q_{k+2}$ in a column-convex polyomino Q is said to be a valley provided that Q_kQ_{k+1} is an upper descent and $Q_{k+1}Q_{k+2}$ is an upper ascent or an upper

level. The number of valleys in Q is denoted by $\mathrm{val}(Q)$. Furthermore, Q is said to be down-up provided that Q_kQ_{k+1} is an upper descent when k is odd and is an upper ascent or an upper level when k is even. Let DU_n denote the set of down-up directed column-convex polyominoes of length n.

Corollary 8. *The generating function for DCCPs by valleys, area, and column number is*

$$\sum_{Q\in\mathrm{DCCP}} y^{\mathrm{val}(Q)} q^{\mathrm{area}\,Q} z^{\mathrm{col}\,Q} = \frac{\displaystyle\sum_{n\geqslant 0} \frac{(1-y)^n q^{(n+1)(2n+1)} z^{2n+1}}{(q;q)_{2n+1}(q;q)_{2n}}}{\displaystyle\sum_{n\geqslant 0} \frac{(1-y)^n q^{n(2n+1)} z^{2n}}{(q;q)_{2n}^2} - \sum_{n\geqslant 0} \frac{(1-y)^n q^{(n+1)(2n+1)} z^{2n+1}}{(q;q)_{2n+1}^2}}.$$

The proof of Corollary 8 consists of first using Theorem 3 to express the generating function for DCCPs by valleys in terms of down-up DCCPs of odd lengths. Theorem 3 is then applied again in a manner analogous to the second half of the proof of Corollary 3 to show that

$$\sum_{n\geqslant 0} \sum_{Q\in\mathrm{DU}_{2n+1}} q^{\mathrm{area}\,Q}\ z^{2n+1} = \left(\sum_{n\geqslant 0} \frac{(-1)^n q^{(n+1)(2n+1)} z^{2n+1}}{(q;q)_{2n+1}(q;q)_{2n}}\right)\left(\sum_{n\geqslant 0} \frac{(-1)^n q^{n(2n+1)} z^{2n}}{(q;q)_{2n}^2}\right)^{-1}.$$

8 The Pattern Algebra and Q1

Goulden and Jackson's Pattern Algebra [18, section 4.3] is a powerful method for solving the composition version of Q1 for a pattern set, tracked as a whole, of the form $P = \{p \in S_m : p_1 *_1 p_2 *_2 \ldots *_{m-1} p_m\}$ where $*_1, *_2, \cdots, *_{m-1}$ belong to a bipartition of N^2. In this section, we use their Pattern Algebra to obtain a q-analog of Kitaev's [25] Theorem 30 and to deduce a better generating function for permutations by peaks and twin peaks.

The essentials of the Pattern Algebra follow. Let X be an alphabet, $\pi_1 \subset X^2$, and $\pi_2 = X^2 \setminus \pi_1$. Suppose $\alpha = \sum_{w\in X^*} c_w w$ is a formal series where the constants commute with letters of X, and for given $x, y \in X$, let $X_{x,y} = \{w \in X^+ : w_1 = x, w_{\mathrm{len}\,w} = y\}$. Then, the incidence matrix $I(\alpha)$ is a matrix with rows and columns indexed by X such that $I(\alpha)_{x,y} = \sum_{w\in X_{x,y}} c_w(w/y)$, i.e. the restriction of α to words in $X_{x,y}$, except the final y has been removed from each word. For $U \subseteq X^*$, we also define $I(U) = I(\sum_{w\in U} w)$ and note that $I(X) = I$, the identity matrix.

For the remainder of this section, we let $A = I(\pi_1)$, $B = I(\pi_2)$, and $W = I(X^2)$. In particular, $W = A + B$. It is crucial to note that, for formal series α and β, $I(\alpha)I(\beta) = I(\gamma)$, where γ is formed by concatenating words u and v from α and β, respectively, where the last letter of u is the first letter of v, and removing one copy of the repeated letter.

Finally, we define the operator Ψ, which converts an incidence matrix back to a formal series, by $\Psi(I(\sum_{w\in X^*} c_w w)) = \sum_{w\in X^+} c_w w$. The empty word has been removed in the process, as it is not accounted for in the incidence matrix. Note that Ψ is linear and, for incidence matrices F and G, $\Psi(FWG) = \Psi(F)\Psi(G)$.

8.1 A General Strategy

We will consider the problem of enumerating words by pattern sets whose incidence matrices can be written as a rational function of A, W, and B. Such a problem can be solved by the following process, which is distinct, yet equal in scope, to that given by Goulden and Jackson.

1. Define a variable to be the incidence matrix for the desired formal series, and then devise a system of linear equations to describe it. The design of this system should mimic that of a regular grammar in that each variable will be multiplied by at most one of A, W, and B and always on the same side. We will name our variables F_i and use right multiplication in this section.

2. Substitute either $A = W - B$ or $B = W - A$ and solve the system, treating F_iW terms as constants. Using $B = W - A$, we obtain a system of the form $F_i = f_i(A) + \sum_j F_jWf_{ij}(A)$, where the fs are rational functions.

3. Finally, apply Ψ to the entire system, noting that $\Psi(FWG) = \Psi(F)\Psi(G)$, and solve for $\Psi(F_i)$.

We will typically compute $\Psi(f(A))$ by expanding $f(A)$ as a power series in A and using the linearity of Ψ to get a formal sum of the $\Psi(A^n)$. We then may apply homomorphisms to the solution to obtain various generating functions. We compute the image of $\Psi(A^n)$ under some common homomorphisms in the next subsection.

For DCCPs, we must also compute $\Psi(F_iZ)$, where Z is an incidence matrix that restricts the last letter of each word. We achieve this by multiplying the equation for F_i by Z before the final step and using known values of $\Psi(F_i)$. We will then also need to compute the image of $\Psi(A^nZ)$.

8.2 Key formulas

As $\Psi(A^n)$ and $\Psi(B^n)$ show up frequently, it is prudent to give their values under a few common homomorphisms.

Recall that $N = \{1, 2, 3, \ldots\}$. For $\pi_1 = \{ij : i \leqslant j\}$ and ϕ_N defined on N^* by $\phi_N(w) = q^{\operatorname{sum} w}(z/q)^{\operatorname{len} w}$, well-known partition identities imply that

$$\phi_N(\Psi(A^{n-1})) = z^n/(q;q)_n \text{ and } \phi_N(\Psi(B^{n-1})) = z^n q^{\binom{n}{2}}/(q;q)_n. \tag{24}$$

Next, consider the alphabets $X = \left\{\binom{j}{m} : j, m \text{ integers}, j \geqslant m \geqslant 1\right\}$ and $R = \{\binom{j}{m} \in X : m = 1\}$. Let $\pi_1 = \{\binom{j_1 j_2}{m_1 m_2} : j_2 \geqslant m_1\}$. Define ϕ_X to be the homomorphism that maps $\binom{j}{m}$, where $j, m \in N^*$, to $q^{\operatorname{sum} j} z^{\operatorname{len} j}$. Then $\phi_X(\Psi(B^{n-1}))$ is the sum of $q^{\operatorname{sum} j} z^n$ over $\binom{j}{m} \in X^n$ where $0 < m_n \leqslant j_n < m_{n-1} \leqslant j_{n-1} < \cdots < m_1 \leqslant j_1$. $\phi_X(\Psi(B^{n-1}I(R)))$ is the same sum as $\phi_X(\Psi(B^{n-1}))$, except $m_n = 1$. Thus,

$$\phi_X(\Psi(B^{n-1})) = \frac{q^{\binom{n+1}{2}}}{(q;q)_n^2} \text{ and } \phi_X(\Psi(B^{n-1}I(R))) = \frac{q^{\binom{n+1}{2}}}{(q;q)_n(q;q)_{n-1}}. \tag{25}$$

The sums $\phi_X(\Psi(A^{n-1}))$ and $\phi_X(\Psi(A^{n-1}I(R)))$ are more difficult to compute. Theorem 5 addresses this issue.

Theorem 5. *Given* $A(x) = 1+\Psi(A^0)x+\Psi(A^1)x^2+\cdots$, $B(x) = 1+\Psi(B^0)x+\Psi(B^1)x^2+\cdots$, $AZ(x) = \Psi(Z) + \Psi(AZ)x + \Psi(A^2Z)x^2 + \cdots$, *and* $BZ(x) = \Psi(Z) + \Psi(BZ)x + \Psi(B^2Z)x^2 + \cdots$, *then* $A(x) = (B(-x))^{-1}$ *and* $AZ(x) = A(x)BZ(-x)$.

Proof. Let $F = x(I - xA)^{-1}$, so that $A(x) = 1 + \Psi(F)$. Then, $F - xFA = xI$. Set $A = W - B$ and solve for F, treating FW as a constant. It follows that

$$F = x(I + xB)^{-1} + xFW(I + xB)^{-1}.$$

Applying Ψ and using the fact that $\Psi(FWG) = \Psi(F)\Psi(G)$ yields

$$\Psi(F) = x\Psi((I + xB)^{-1}) + x\Psi(F)\Psi((I + xB)^{-1}).$$

Factor, add 1 to both sides, and solve for $1 + \Psi(F)$ to obtain

$$1 + \Psi(F) = (1 - x\Psi((I + xB)^{-1}))^{-1} = (B(-x))^{-1}.$$

Now, let $G = (I - xA)^{-1}Z$, so that $AZ(x) = \Psi(G)$. Then, $G - xAG = Z$. Set $A = W - B$ and solve for G, treating WG as a constant. It follows that

$$G = (I + xB)^{-1}Z + x(I + xB)^{-1}WG.$$

Applying Ψ and using the fact that $\Psi(FWG) = \Psi(F)\Psi(G)$ yields

$$\Psi(G) = \Psi((I + xB)^{-1}Z) + x\Psi((I + xB)^{-1})\Psi(G).$$

Finally, solve for $\Psi(G)$ and note that $(1 - x\Psi((I + xB)^{-1}))^{-1} = (B(-x))^{-1} = A(x)$ and $\Psi((I + xB)^{-1}Z) = BZ(-x)$ to complete the result. $\square$

Applying Theorem 5, we find that

$$\phi_X(\Psi(A^{n-1})) = \left(\sum_{k\geqslant 0} \frac{(-x)^k q^{\binom{k+1}{2}}}{(q;q)_k^2}\right)^{-1}\Bigg|_{x^n} \quad \text{and that}$$

$$\phi_X(\Psi(A^{n-1}I(R))) = \left(\sum_{k\geqslant 0} \frac{(-x)^k q^{\binom{k+2}{2}}}{(q;q)_{k+1}(q;q)_k}\right)\left(\sum_{k\geqslant 0} \frac{(-x)^k q^{\binom{k+1}{2}}}{(q;q)_k^2}\right)^{-1}\Bigg|_{x^{n-1}}. \tag{26}$$

Note that (26) counts the number of parallelogram polyominoes with n columns by area, matching the result of Delest and Fédou [11].

8.3 A q-analog of a distribution due to Kitaev

In this section, we use the Pattern Algebra to obtain a q-analog of Kitaev's Theorem 30 in [25] that enumerates permutations that avoid $P = \{4312, 4213, 4123, 3214, 3124, 2134\}$. We begin by deducing the relevant generating function on words.

Theorem 6. *If $\mathcal{F} \subset X^4$ such that $I(\mathcal{F}) = AWB$, then*

$$\sum_{w \in X^*} y^{\mathcal{F}(w)} w = (1 - \alpha\beta^{-1})^{-1}$$

where $\mathcal{F}(w) = \sum_{f \in \mathcal{F}} f(w)$ and

$$\begin{aligned}
\alpha &= \Psi(U) - \Psi(BU)\gamma \\
\beta &= 1 + (y-1)^2\Psi(B^3U) - \Psi(U)\gamma \\
\gamma &= (y-1)(1 + (y-1)\Psi(B^2U + (y-1)B^3U))^{-1}\Psi(BU + (y-1)B^2U) \\
U &= (I - (y-1)^2B^4)^{-1}.
\end{aligned}$$

Proof. For all $f \in \mathcal{F}$, let $y_f = y$, and let $F_0 = I(X) + I(C_{\mathcal{F}}(\mathbf{y} - \mathbf{1}, X^*))$, where $C_{\mathcal{F}}$ is the cluster generating function as defined in section 4. By Theorem 3, we then seek to compute $(1 - \Psi(F_0))^{-1}$. Following our general strategy in subsection 8.1, we construct the system:

$$\begin{aligned}
F_0 &= I + (y-1)F_2B + (y-1)F_3B & \qquad F_2 &= F_1A \\
F_1 &= F_0A & F_3 &= F_1B + (y-1)F_2B.
\end{aligned}$$

Next, we substitute $A = W - B$ and, since there were no F_2A or F_3A terms in the system, immediately eliminate F_2 and F_3, giving

$$F_0 = I + (y-1)F_1W(B + (y-1)B^2) - (y-1)^2F_1B^3 \text{ and } F_1 = F_0W - F_0B.$$

Treating F_0W and F_1W as constants, solve for F_0 and F_1 to get

$$\begin{aligned}
F_0 &= (I + (y-1)F_1W(B + (y-1)B^2) - (y-1)^2F_0WB^3)U \text{ and} \\
F_1 &= (F_0W - B - (y-1)F_1W(B^2 + (y-1)B^3))U.
\end{aligned}$$

Using the property $\Psi(FWG) = \Psi(F)\Psi(G)$, take Ψ of both sides to obtain

$$\begin{aligned}
\Psi(F_0) &= \Psi(U) + (y-1)\Psi(F_1)\Psi(BU + (y-1)B^2U) - (y-1)^2\Psi(F_0)\Psi(B^3U) \\
\Psi(F_1) &= \Psi(F_0)\Psi(U) - \Psi(BU) - (y-1)\Psi(F_1)\Psi(B^2U + (y-1)B^3U).
\end{aligned}$$

Finally, solve the system for $\Psi(F_0)$ and compute $(1 - \Psi(F_0))^{-1}$. □

Recall $\sin_q z$ and $\cos_q z$ of subsection 2.1. Let $e_q(z) = \sum_{n \geqslant 0} z^n/(q;q)_n$. The generating function $\mathcal{K}$ in the following result gives a q-analog of Theorem 30 in [25].

Corollary 9. *If* $P = \{1243, 1342, 1432, 2341, 2431, 3421\}$, *then*

$$\sum_{n\geqslant 0}\sum_{\sigma\in S_n}\frac{y^{P_{no}(\sigma)}q^{\mathrm{inv}\,\sigma}z^n}{(q;q)_n} = \frac{\mathcal{K}}{1-y+y(1-\frac{z}{1-q})\mathcal{K}}$$

where

$$\mathcal{K} = \frac{e_q(z)e_q(-z)+\cos_q^2 z+\sin_q^2 z+2e_q(-z)\cos_q z+(e_q(z)+e_q(-z))\sin_q z}{4e_q(-z)\cos_q z}.$$

Proof. Let $T = \{4312, 4213, 4123, 3214, 3124, 2134\}$. Then, the "complement-reversal" bijection $c_n : S_n \to S_n$ given by $c_n(\sigma_1\sigma_2\ldots\sigma_n) = (n+1-\sigma_n)(n+1-\sigma_{n-1})\ldots(n+1-\sigma_1)$ shows that $\sum_{\sigma\in S_n} y^{P(\sigma)}q^{\mathrm{inv}\,\sigma} = \sum_{\sigma\in S_n} y^{T(\sigma)}q^{\mathrm{inv}\,\sigma}$. By Theorem 2,

$$\sum_{n\geqslant 0}\sum_{\sigma\in S_n}\frac{y^{T(\sigma)}q^{\mathrm{inv}\,\sigma}z^n}{(q;q)_n} = \sum_{n\geqslant 0}\sum_{w\in K_n} y^{T(w)}q^{\mathrm{sum}\,w}(z/q)^n. \tag{27}$$

Using $\pi_1 = \{ij : i > j\}$, note that B^n is equivalent to what A^n is when $\pi_1 = \{ij : i \leqslant j\}$. The right-hand side of (27) may then be evaluated by applying (24) to the result in Theorem 6. Finally, set $y = 0$ and apply Corollary 1. □

8.4 The twin peak problem revisited

Define a peak to be a single word from $\Psi(AB)$ and a twin peak to be one from $\Psi(ABAB)$. For $w \in X^*$, let $\mathrm{pic}(w)$ be the number of peaks in w and $\mathrm{tpic}(w)$ be the number of twin peaks. Theorem 7 solves Q2 on words for peaks and twin peaks; as corollaries, we obtain corresponding results on permutations and DCCPs.

Theorem 7. *For any nonempty alphabet* X,

$$\sum_{w\in X^*} x^{\mathrm{pic}(w)}y^{\mathrm{tpic}(w)}w = (1-\alpha\beta^{-1})^{-1}$$

where

$$\begin{aligned}
\alpha &= \Psi(U) + x\Psi(B^2U)(y-1-\gamma),\\
\beta &= 1 + x(y-1)\Psi(B^3U) - \Psi(BU)(x-1+x\gamma),\\
\gamma &= x(y-1)(1-x(y-1)\Psi(BU-B^3U))^{-1}\Psi(BU),\\
U &= (I+(xy-1)B^2 - x(y-1)B^4)^{-1}.
\end{aligned}$$

Proof. Let $\mathcal{F}$ be the set of peaks and twin peaks, $y_f = x$ if $f \in \mathcal{F}$ is a peak and $y_f = y$ if f is a twin peak. Then, let $F_0 = I(X) + I(C_{\mathcal{F}}(\mathbf{y}-\mathbf{1}, X^*))$, where $C_{\mathcal{F}}$ is the cluster generating function of section 4. By Theorem 3, we then seek to compute $(1-\Psi(F_0))^{-1}$. Following our general strategy in subsection 8.1, we deduce a system of equations:

$$\begin{aligned}
F_0 &= I + (x-1)F_1B + x(y-1)F_3B & F_2 &= xF_1B + x(y-1)F_3B\\
F_1 &= F_0A & F_3 &= F_2A.
\end{aligned}$$

Substitute $A = W - B$ and eliminate F_1 and F_3 to obtain

$$\begin{aligned} F_0 &= I + (x-1)F_0WB - (x-1)F_0B^2 + x(y-1)F_2WB - x(y-1)F_2B^2 \\ F_2 &= xF_0WB - xF_0B^2 + x(y-1)F_2WB - x(y-1)F_2B^2. \end{aligned}$$

Treating F_0W and F_2W as constants, solve for F_0 and F_2:

$$\begin{aligned} F_0 &= (I + x(y-1)B^2 + F_0W((x-1)B - x(y-1)B^3) + x(y-1)F_2WB)U \\ F_2 &= x(-B^2 + F_0WB + (y-1)F_2W(B - B^3))U. \end{aligned} \tag{28}$$

Apply Ψ, recall that $\Psi(FWG) = \Psi(F)\Psi(G)$, and solve for $\Psi(F_0)$. □

When deriving a result for permutations, we generally prefer to set $B = W - A$ instead of $A = W - B$, since $\phi_N(\Psi(A^n))$ is nicer than $\phi_N(\Psi(B^n))$. However, as the following paragraph shows, when a homomorphism ϕ whose image is commutative is applied to Theorem 7, A and B may be switched without affecting the result.

Let $\mathcal{F}'$ be the set of words from $\Psi(BA + BABA)$, $y_f = x$ if f is from $\Psi(BA)$ and $y_f = y$ if f is from $\Psi(BABA)$. Let (w, ν, β) be an $\mathcal{F}$-cluster with $w \in X^n$. Then, if $w_nw_1 \in \pi_1$, let $w' = w_2w_3 \dots w_nw_1$; otherwise, let $w' = w_nw_1w_2 \dots w_{n-1}$. (w', ν', β) is then an $\mathcal{F}'$-cluster, where ν' contains factors of w' with lengths and positions matching those in ν. Moreover, since no other word of the form $w_i \dots w_nw_1 \dots w_{i-1}$ is $\mathcal{F}'$-coverable, this is a bijection. Then, $\phi(C_{\mathcal{F}}(\mathbf{y}, \mathbf{X}^*)) = \phi(\mathbf{C}_{\mathcal{F}'}(\mathbf{y}, \mathbf{X}^*))$, since w' is a rearrangement of the letters of w.

Using the above fact, we obtain the following alternative solution to the problem considered in subsection 5.3.

Corollary 10. *The generating function for permutations by peaks, twin peaks, and inversions is*

$$\sum_{n \geqslant 0} \sum_{\sigma \in S_n} \frac{q^{\operatorname{inv}\sigma} x^{\operatorname{pic}(\sigma)} y^{\operatorname{tpic}(\sigma)} z^n}{(q;q)_n} = \left(1 - \frac{s_+ \sin_q(z\sqrt{r_+})}{2\sqrt{r_+}\cos_q(z\sqrt{r_+})} - \frac{s_- \sin_q(z\sqrt{r_-})}{2\sqrt{r_-}\cos_q(z\sqrt{r_-})}\right)^{-1}$$

where $r_\pm = (xy - 1 \pm \sqrt{D})/2$, $s_\pm = 1 \pm (2x - xy - 1)/\sqrt{D}$, *and* $D = (xy+1)^2 - 4x$.

Proof. Apply ϕ_N to Theorem 7, replace each B with an A, expand U as a series, then apply (24) and simplify. □

A peak in a CCP Q is a sequence of columns $Q_iQ_{i+1}Q_{i+2}$ such that Q_iQ_{i+1} is an upper ascent or level and $Q_{i+1}Q_{i+2}$ is an upper descent. A twin peak is a sequence $Q_iQ_{i+1}Q_{i+2}Q_{i+3}Q_{i+4}$ such that $Q_iQ_{i+1}Q_{i+2}$ and $Q_{i+2}Q_{i+3}Q_{i+4}$ are peaks. Let $\operatorname{pic}(Q)$ and $\operatorname{tpic}(Q)$ respectively denote the numbers of peaks and twin peaks in Q. Corollary 11 applies Theorem 7 to DCCPs.

Corollary 11. *The generating function for DCCPs by peaks, twin peaks, area, and column number is*

$$\sum_{Q \in \mathrm{DCCP}} x^{\operatorname{pic}(Q)} y^{\operatorname{tpic}(Q)} q^{\operatorname{area} Q} z^{\operatorname{col} Q} = \frac{\phi_X(\Psi(F_0Z))}{1 - \phi_X(\Psi(F_0))}$$

where $Z = I(R)$, F_0 and U are as in Theorem 7, and

$$\begin{aligned}\Psi(F_0Z) =& \Psi(UZ + x(y-1)B^2UZ) + \Psi(F_0)\Psi((x-1)BUZ - x(y-1)B^3UZ)\\ &+ x(y-1)\Psi(F_2)\Psi(BUZ),\\ \Psi(F_2) =& x(-\Psi(B^2U) + \Psi(F_0)\Psi(BU))(1 - x(y-1)\Psi(BU - B^3U))^{-1}.\end{aligned}$$

Proof. The bijection δ of (18), ϕ_X and R of subsection 8.2, and Theorem 3 imply that we seek $\phi_X((1-\Psi(F_0))^{-1}\Psi(F_0Z))$. To evaluate $\Psi(F_0Z)$, right-multiply both sides of the first equality in (28) by Z, apply Ψ, and solve. □

9 The Temperley method

In short, the Temperley [37] method involves introducing a variable to track the size of the last letter in a word, deriving a recurrence for the generating function using that variable, and then solving the recurrence. By streamlining Temperley's method, Bousquet-Mélou [4] developed a powerful tool for enumerating CCPs by practically any desired set of statistics. In this final section, we illustrate the use of the Temperley method as modified in [4] in solving a few select consecutive pattern problems on permutations, compositions, and CCPs. Theorems 3 and 4 also play a role.

9.1 Compositions by mesas

A mesa in a composition w is a segment $w_iw_{i+1}w_{i+2}w_{i+3}$ satisfying $w_i < w_{i+1} = w_{i+2} > w_{i+3}$. Let $\mathrm{mes}(w)$ be the number of mesas in w.

Corollary 12. *The generating function for compositions by mesas is*

$$\sum_{n\geqslant 0}\sum_{w\in K_n} y^{\mathrm{mes}(w)}q^{\mathrm{sum}\,w}(z/q)^n = \left(1 - \frac{\displaystyle\sum_{n\geqslant 0}\frac{(-1)^n(y-1)^n z^{3n+1}q^{3n(n+1)/2}}{(q;q^3)_{n+1}(q^3;q^3)_n}}{\displaystyle\sum_{n\geqslant 0}\frac{(-1)^n(y-1)^n z^{3n}q^{n(3n+1)/2}}{(q^2;q^3)_n(q^3;q^3)_n}}\right)^{-1}.$$

Proof. Let $\mathcal{F}$ be the set of mesas. Set $y_f = y$ for $f \in \mathcal{F}$. Let $\phi(w) = q^{\mathrm{sum}\,w}(z/q)^{\mathrm{len}\,w}b^{w_{\mathrm{len}\,w}-1}$ and let $F_0(b) = \frac{z}{1-qb} + \phi(C_{\mathcal{F}}(\mathbf{y}-\mathbf{1}, N^*))$ where $C_{\mathcal{F}}(\mathbf{y}, N^*)$ is as in section 4. By Theorem 3, we seek $(1-F_0(1))^{-1}$.

As with the Pattern Algebra, we deduce a system of equations. Note that F_0 is the sum of the weights of the one-letter and $\mathcal{F}$-coverable compositions. Let F_1 be the sum over those with an extra ascent at the end and F_2 be the sum over those with an extra ascent and level at the end. Then,

$$\begin{aligned}F_0(b) &= \frac{z}{1-qb} + \frac{(y-1)z}{1-qb}(F_2(1) - F_2(qb)),\\ F_1(b) &= \frac{z}{1-qb}F_0(qb), \quad \text{and} \quad F_2(b) = zF_1(qb).\end{aligned}$$

Elimination of $F_1(b)$ and $F_2(b)$ gives

$$F_0(b) = \frac{z}{1-qb} + \frac{(y-1)z^3q^2}{(1-q^2)(1-qb)}F_0(q^2) - \frac{(y-1)z^3q^3b}{(1-qb)(1-q^3b)}F_0(q^3b).$$

Iteration of this recurrence leads to

$$F_0(b) = \left(z + \frac{(y-1)z^3q^2}{1-q^2}F_0(q^2)\right)\sum_{n\geqslant 0}\frac{(-1)^n(y-1)^nz^{3n}q^{3n(n+1)/2}b^n}{(qb;q^3)_{n+1}(q^3b;q^3)_n}. \tag{29}$$

Substituting $b = q^2$ and then solving yields

$$z + \frac{(y-1)z^3q^2}{1-q^2}F_0(q^2) = \frac{z}{\displaystyle\sum_{n\geqslant 0}\frac{(-1)^n(y-1)^nz^{3n}q^{n(3n+1)/2}}{(q^2;q^3)_n(q^3;q^3)_n}}.$$

Substituting the above into (29) and setting $b = 1$ completes the proof. □

9.2 Permutations by (i,m)-maxima revisited

The modified Temperley method in [4] may be applied to obtain an alternative version of the generating function in Corollary 5.

Corollary 13. *If $i \geqslant 2$ and $1 \leqslant m \leqslant i$, then the generating function for permutations by (i,m)-maxima and inversions is*

$$\sum_{n\geqslant 0}\sum_{\sigma\in S_n}\left(\prod_{m=1}^{i} y_m^{p_{(m)}(\sigma)}\right)\frac{q^{\mathrm{inv}\,\sigma}z^n}{(q;q)_n} = \left(1 - \frac{\displaystyle\sum_{n\geqslant 0}\frac{z^{in+1}}{1-q^{in+1}}\prod_{k=0}^{n-1}T(q^{ik})}{1 - \dfrac{y_i-1}{(q;q)_{i-1}}\displaystyle\sum_{n\geqslant 1}\frac{z^{in}}{1-q^{in}}\prod_{k=1}^{n-1}T(q^{ik-1})}\right)^{-1}$$

where

$$T(b) = \sum_{m=1}^{i-1}\frac{(y_m-1)q^m}{(q;q)_m(q^{m+1}b;q)_{i-m}} - \frac{y_i-1}{(q;q)_{i-1}(1-qb)}.$$

Proof. Define $F(b) = \frac{z}{1-qb} + \phi(C_{\mathcal{F}}(\mathbf{y}-\mathbf{1}, N^*))$, where $\mathcal{F} = \cup_{p\in P_i}\mathcal{F}_p$ and ϕ is as in Corollary 12. By Theorem 4, we seek $(1-F(1))^{-1}$. As usual, we derive a recurrence for $F(b)$. Note that (i,m)-maxima can never overlap non-trivially, so the only way to extend an $\mathcal{F}$-cluster is to append an (i,m)-maximum. For $1 \leqslant m < i$, the generating function for words from $F(b)$ with an (i,m)-maximum appended is

$$\frac{(y_m-1)q^mz^i}{(q;q)_m(q^{m+1}b;q)_{i-m}}F(q^ib). \tag{30}$$

The generating function for words from $F(b)$ with an extra (i,i)-maximum appended is

$$\frac{(y_i-1)z^i}{(q;q)_{i-1}(1-qb)}(F(q^{i-1})-F(q^ib)). \tag{31}$$

Summing (30) over m and adding in (31) and the one letter words, we obtain a recurrence for $F(b)$:

$$F(b)=\frac{z}{1-qb}+\frac{(y_i-1)z^i}{(q;q)_{i-1}(1-qb)}F(q^{i-1})+z^iT(b)F(q^ib).$$

Iteration of the recurrence leads to

$$F(b)=\left(z+\frac{(y_i-1)z^i}{(q;q)_{i-1}}F(q^{i-1})\right)\sum_{n\geqslant 0}\frac{z^{ni}}{1-q^{1+ni}b}\prod_{k=0}^{n-1}T(q^{ki}b). \tag{32}$$

Substitute $b=q^{i-1}$ and solve for $F(q^{i-1})$ to obtain

$$z+\frac{(y_i-1)z^i}{(q;q)_{i-1}}F(q^{i-1})=z\left(1-\frac{y_i-1}{(q;q)_{i-1}}\sum_{n\geqslant 1}\frac{z^{ni}}{1-q^{ni}}\prod_{k=1}^{n-1}T(q^{ki-1})\right)^{-1}.$$

Substitute into (32) and set $b=1$ to obtain the final result. □

9.3 CCPs by peaks

The generating functions for column-convex polyominoes by various upper ridge patterns, area, width, and relative height are always rational functions. Theorem 8 provides an example. Let $\mathrm{pic}(Q)$ be as in Corollary 11.

Theorem 8. *The generating function for CCPs by peaks, area, and column number is*

$$\sum_{Q\in\mathrm{CCP}} y^{\mathrm{pic}(Q)}q^{\mathrm{area}\,Q}z^{\mathrm{col}\,Q}=\frac{(\frac{zq}{1-q}+\frac{2z^2q^3}{(1-q)^3})(1+\frac{2zq}{(1-q)^2})}{(1-\frac{zq^2}{(1-q)^2})(1+\frac{zq}{(1-q)^2})-\frac{2yz^2q^3}{(1-q)^4}}.$$

Proof. Define $F(b)=\sum_{Q\in\mathrm{CCP}}y^{\mathrm{pic}(Q)}q^{\mathrm{area}\,Q}z^{\mathrm{col}\,Q}h^{\mathrm{relh}\,Q}b^{\alpha(Q)}$, where $\alpha(Q)$ is the area of the last column in Q. Let $F_0(b)$ be the restriction of $F(b)$ to Q such that $\mathrm{col}\,Q\geqslant 2$ and the last pair of columns form an upper level or ascent, and let $F_1(b)$ be the restriction to the rest. Then, we seek $F(1)=F_0(1)+F_1(1)$, with $h=1$. As usual, we derive equations that F_0 and F_1 must satisfy:

$$F_0(b)=\frac{zqb}{(1-qb)(1-qbh)}(F_0(1)+F_1(1)),$$
$$F_1(b)=\frac{zqbh}{1-qbh}+\frac{zqb}{(1-qb)(1-h)}(yhF_0(1/h)-yF_0(1)+hF_1(1/h)-F_1(1)).$$

Substituting $b=1$ and $b=1/h$ into the above system gives four equations in the four unknowns $F_0(1)$, $F_1(1)$, $F_0(1/h)$, and $F_1(1/h)$. Solve the system, compute $F_0(1)+F_1(1)$, and set $h=1$. □

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