

A Binary Tree Decomposition Space of Permutation Statistics

DON RAWLINGS

Based on the binary tree decomposition of a permutation, a natural vector space of permutation statistics is defined. Besides containing many well known permutation statistics, this space provides the general context for an archetypal recurrence relationship that contains most of the classic combinatorial sequences and some of their known generalizations.

1. INTRODUCTION

Permutation statistics have arisen in connection with various applications of combinatorics to a wide spectrum of mathematics (including differential operators, orthogonal polynomials, hypergeometric functions, probability, and sorting problems). However, in spite of an increased interest in permutation statistics, no systematic study of them has been done. The basic purpose of this paper is to propose a possible framework for such a study.

The approach being advocated for unifying this area consists of the introduction and examination of natural vector spaces of statistics which are defined in terms of permutation decompositions. The central focus here will be on a vector space which arises in connection with the binary tree decomposition of a permutation. Besides containing many well-known permutation statistics (such as the descent number, the inversion number, the trough number, the minimum component number, and others), this binary tree decomposition statistic space provides the setting for an archetypal recurrence relationship that contains many classic combinatorial sequences (including the Catalan numbers, the Fibonacci numbers, the Eulerian polynomials, the Stirling polynomials of both kinds, the Hermite polynomials, and the Bell polynomials).

2. THE BINARY TREE DECOMPOSITION STATISTIC SPACE

A permutation σ of a set D of n integers will be written as a list $\sigma = \sigma_1 \sigma_2 \cdots \sigma_n$ and the symbol $\mathcal{L}[D]$ will denote the set of such lists. For simplicity, $\mathcal{L}[n]$ will signify the set of lists of $\{1, 2, \dots, n\}$.

For $\sigma \in \mathcal{L}[C]$, where C is a non-empty set of $(n+1)$ integers, let $(k+1)$ be the unique index such that σ_{k+1} is equal to the minimum element in C . Further, let

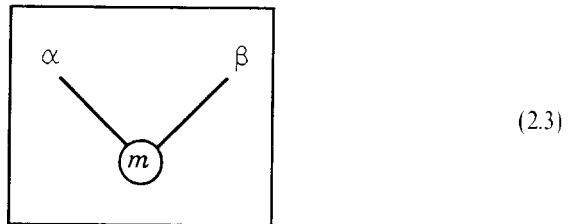
$$\begin{aligned} \text{(i)} \quad A &\equiv \{\sigma_1, \sigma_2, \dots, \sigma_k\} \\ \text{(ii)} \quad B &\equiv \{\sigma_{k+2}, \sigma_{k+3}, \dots, \sigma_{n+1}\}. \end{aligned} \tag{2.1}$$

Then, the "rooted binary planar tree" decomposition of $\sigma \in \mathcal{L}[C]$ is defined to be the unique factorization of σ into the sublists

$$\sigma = \alpha m \beta, \tag{2.2}$$

where $\alpha \equiv \sigma_1 \sigma_2 \cdots \sigma_k \in \mathcal{L}[A]$, $m \equiv$ minimum element of C , and $\beta \equiv \sigma_{k+2} \sigma_{k+3} \cdots \sigma_{n+1} \in \mathcal{L}[B]$.

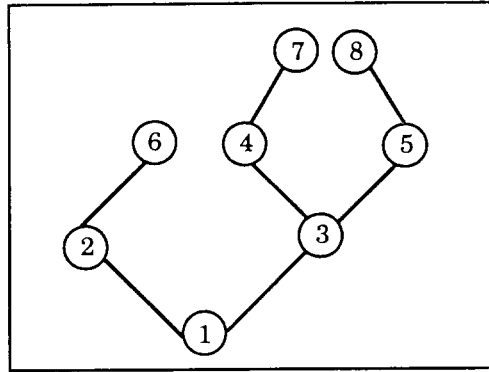
The reason for referring to (2.2) as the rooted binary planar tree decomposition of σ becomes clear if one views $\sigma = \alpha m \beta$ geometrically as



Iteration of (2.3) will produce a unique rooted binary planar tree in which each vertex has a unique label from the set C , such that the labels appear in increasing order as one moves up and away from the root. This unique tree is the so-called "arbre binaire croissant" associated to σ as described by Foata and Schützenberger [FS2]. For example, the rooted binary planar tree associated to

$$\sigma = 26147385 \in \mathcal{L}[8] \tag{2.4}$$

is



(2.5)

A map $s: \mathcal{L}[C] \rightarrow \mathbb{R}$ (reals) is said to be an “elementary tree statistic” if for all $\sigma = \alpha m \beta$ we have

$$s(\sigma) = as(\alpha) + bs(\beta) + cI(A, B) + f(|A|, |B|), \quad (2.6)$$

where

- (i) $s(\emptyset) \equiv 0$
- (ii) $a, b, c \in \mathbb{R}$
- (iii) $I(A, B) \equiv |\{i > j : i \in A, j \in B\}|$
- (iv) $f: \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{R}$ ($\mathbb{N} \equiv \{0, 1, 2, \dots\}$).

The number $I(A, B)$ defined in (iii) counts the number of inversions from set A to set B (see page 98 of [GJ]).

Many classic permutation statistics are elementary tree statistics. Some are listed in Table I, where, for convenience, if S symbolizes a statement, then the symbol $\chi(S)$ is equal to 1 if S is true and is equal to 0 otherwise.

The identities of Table I may be readily verified from the combinatorial definitions given in the corresponding references. More radically, one may take these identities as “decomposition based” definitions of the statistics of Table I.

The set of all elementary tree statistics is nearly a vector space over \mathbb{R} . Certainly the zero statistic $\theta(\sigma) \equiv 0$ for all σ is in this set since $\theta(\sigma) = \theta(\alpha) + \theta(\beta)$. Also, if s is an elementary tree statistic, then so is ds for any $d \in \mathbb{R}$. The only property missing is that of additive closure.

Although it is a simple matter to consider closure with respect to finite sums, it is of more interest to consider an infinite additive closure. A

TABLE I
Examples of Elementary Tree Inversion Statistics

Name	References	Identity relative to $\sigma = \alpha m \beta$
1. Length		$\text{len}(\sigma) = \text{len}(\alpha) + \text{len}(\beta) + 1$
2. Descents	[CS, FS1, M]	$\text{des}(\sigma) = \text{des}(\alpha) + \text{des}(\beta) + \chi(A \geq 1)$
3. Rises	[C2]	$\text{ris}(\sigma) = \text{ris}(\alpha) + \text{ris}(\beta) + \chi(A = 0)$
4. Inversions	[C1, F1, M]	$\text{inv}(\sigma) = \text{inv}(\alpha) + \text{inv}(\beta) + I(A, B) + A $
5. 312 patterns	[R2]	$312(\sigma) = 312(\alpha) + 312(\beta) + I(A, B)$
6. 213 patterns	[R2]	$213(\sigma) = 213(\alpha) + 213(\beta) - I(A, B) + A B $
7. Left-to-right minimum components of length i	[C2, G, R1]	$l_i(\sigma) = l_i(\alpha) + \chi(B + 1 = i)$
8. Right-to-left minimum components of length i	[C2, G, R1]	$r_i(\sigma) = r_i(\beta) + \chi(A + 1 = i)$
9. Troughs	[FV]	$\text{tr}(\sigma) = \text{tr}(\alpha) + \text{tr}(\beta) + \chi(A \geq 1) \chi(B \geq 1)$
10. Pics	[FV]	$\text{pc}(\sigma) = \text{pc}(\alpha) + \text{pc}(\beta) + \chi(A = 0) \chi(B = 0)$

sequence $\mathbf{s} \equiv (s_i)_{i \geq 0}$ of elementary tree statistics is said to be a *joint* elementary tree statistic. Furthermore, \mathbf{s} is said to be admissible if for all σ ,

$$|\{i: s_i(\sigma) \neq 0\}| < \infty. \tag{2.7}$$

Then, the closure of the set of elementary tree statistics with respect to sums of admissible joint statistics, that is, the set

$$\left\{ \sum_{i \geq 0} s_i : \mathbf{s} \text{ is admissible} \right\} \tag{2.8}$$

is a vector space over \mathbb{R} . The set of (2.8) will be referred to as the “binary tree decomposition statistic space.” Table II contains some examples of statistics in the binary tree decomposition statistic space written as linear combinations of the statistics of Table I.

TABLE II
Some Statistics Written as Linear Combinations

Name	References	In terms of Table I statistics
1. Length		$\text{len}(\sigma) = \text{des}(\sigma) + \text{ris}(\sigma)$
2. Left-to-right min. comp.	[C2, G, R1]	$l(\sigma) = \sum_{i \geq 0} l_i(\sigma)$
3. Right-to-left min. comp.	[C2, G, R1]	$r(\sigma) = \sum_{i \geq 0} r_i(\sigma)$
4. Double descents	[FV]	$\text{ddes}(\sigma) = \text{len}(\sigma) - \text{pc}(\sigma) - \text{ris}(\sigma)$
5. Double rises	[FV]	$\text{dris}(\sigma) = \text{ris}(\sigma) - \text{tr}(\sigma)$

3. AN ARCHETYPAL RECURRENCE RELATIONSHIP

Besides serving as a common setting for many classic statistics, the binary tree decomposition statistic space also provides a framework for unification. The possibility for unification is exemplified by the problem of determining the distribution of a binary tree decomposition statistic space statistic over $\mathcal{L}[n]$. In order to present a recursive solution to this problem, the stage needs to be further set with a few more comments and definitions.

First, to find the distribution over $\mathcal{L}[n]$ of a binary tree statistic $s \equiv \sum_{i \geq 0} s_i$, where $\mathbf{s} \equiv (s_i)_{i \geq 0}$ is an admissible sequence of elementary tree statistics, it clearly suffices to determine the joint distribution of $\mathbf{s} \equiv (s_i)_{i \geq 0}$.

Second, let $\mathbf{p} \equiv (p_i)_{i \geq 0}$ be a sequence of indeterminates and, given a sequence of real valued constants $\mathbf{d} \equiv (d_i)_{i \geq 0}$, define

$$\begin{aligned} \text{(i)} \quad \mathbf{p}_d &\equiv (p_i^{d_i})_{i \geq 0} \\ \text{(ii)} \quad \mathbf{p}^d &\equiv \prod_{i \geq 0} p_i^{d_i}. \end{aligned} \tag{3.1}$$

Third, for an indeterminate q , the q -analog, q -fractional, and q -binomial coefficient of a non-negative integer n are respectively defined to be

$$\begin{aligned} \text{(i)} \quad (n)_q &\equiv 1 + q + q^2 + \cdots + q^{n-1} \\ \text{(ii)} \quad (n)_q! &\equiv (1)_q (2)_q \cdots (n)_q \\ \text{(iii)} \quad \binom{n}{k}_q &\equiv \frac{(n)_q!}{(k)_q! (n-k)_q!}, \end{aligned} \tag{3.2}$$

where, by convention, $(0)_q! \equiv 1$.

Furthermore, suppose that $\mathbf{s} \equiv (s_i)_{i \geq 0}$ is an admissible sequence of elementary tree statistics with

$$s_i(\sigma) = a_i s_i(\alpha) + b_i s_i(\beta) + c_i I(A, B) + f_i(|A|, |B|) \tag{3.3}$$

relative to the decomposition $\sigma = \alpha m \beta$ for all σ . Finally, for a set D of positive integers with $|D| = n$ and a sequence of indeterminates $\mathbf{p} \equiv (p_i)_{i \geq 0}$, the generating polynomial over $\mathcal{L}[D]$ for the admissible joint statistics $\mathbf{s} \equiv (s_i)_{i \geq 0}$ is defined to be

$$\mathcal{L}_n(\mathbf{p}) \equiv \sum_{\sigma \in \mathcal{L}[D]} \mathbf{p}^{\mathbf{s}(\sigma)}. \tag{3.4}$$

(The notation $\mathcal{L}_n(\mathbf{p})$ is adequate, since the sum in (3.4) depends only on the cardinality of D .)

With respect to the setup of the preceding paragraph and due to the fact (see page 98 of [GJ]) that

$$\sum q^{I(A,B)} = \binom{n}{k}_q, \quad (3.5)$$

where the sum is over all ordered pairs of sets (A, B) such that $A \cup B = \{2, 3, \dots, n+1\}$ and $A \cap B = \emptyset$, it follows quite readily that the sequence of polynomials $\mathcal{L}_n(\mathbf{p})$ recursively satisfies the “archetypal” identity

$$\mathcal{L}_{n+1}(\mathbf{p}) = \sum_{k=0}^n \mathbf{p}^{f(k,n-k)} \binom{n}{k}_{\mathbf{p}^c} \mathcal{L}_k(\mathbf{p}_a) \mathcal{L}_{n-k}(\mathbf{p}_b), \quad (3.6)$$

where $\mathcal{L}_0(\mathbf{p}) \equiv 1$ and the sequences $\mathbf{a} \equiv (a_i)_{i \geq 0}$, $\mathbf{b} \equiv (b_i)_{i \geq 0}$, $\mathbf{c} \equiv (c_i)_{i \geq 0}$, and $\mathbf{f} \equiv (f_i)_{i \geq 0}$ are those of (3.3). The proof of (3.6) goes as follows.

Begin by noting that the binary tree decomposition of (2.2) may be viewed as a bijection from the set of permutations $\mathcal{L}[n+1]$ to the set of 4-tuples

$$\bigcup_{k=0}^n \{(A, B; \alpha, \beta) : |A| = k, A \cup B = \{2, 3, \dots, n+1\}, \\ A \cap B = \emptyset, \alpha \in \mathcal{L}[A], \beta \in \mathcal{L}[B]\}. \quad (3.7)$$

Then, the facts and definitions of this section combine to justify the calculation

$$\begin{aligned} \mathcal{L}_{n+1}(\mathbf{p}) &\equiv \sum_{\sigma \in \mathcal{L}[n+1]} \mathbf{p}^{s(\sigma)} \\ &= \sum_{k=0}^n \sum_{|A|=k} \sum_{\alpha \in \mathcal{L}[A]} \sum_{\beta \in \mathcal{L}[B]} (\mathbf{p}_a)^{s(\alpha)} (\mathbf{p}_b)^{s(\beta)} \mathbf{p}^{cI(A,B)} \mathbf{p}^{f(k,n-k)} \\ &= \sum_{k=0}^n \mathbf{p}^{f(k,n-k)} \binom{n}{k}_{\mathbf{p}^c} \mathcal{L}_k(\mathbf{p}_a) \mathcal{L}_{n-k}(\mathbf{p}_b), \end{aligned} \quad (3.8)$$

which completes the proof of (3.6).

4. SOME CLASSIC CASES OF THE ARCHETYPAL RECURRENCE RELATIONSHIP

Identity (3.6) generalizes many classic recurrence relationships of combinatorics. For example, consider the joint distribution $\mathbf{s} \equiv (312, \theta, \theta, \dots)$ and let $\mathbf{p} \equiv (0, 1, 1, \dots)$. Since

$$\begin{aligned} \text{(i)} \quad 312(\sigma) &= 312(\alpha) + 312(\beta) + I(A, B) \\ \text{(ii)} \quad \theta(\alpha) &= \theta(\alpha) + \theta(\beta) \end{aligned} \quad (4.1)$$

relative to the binary tree decomposition $\sigma = \alpha m \beta$, it follows that

$$\begin{aligned}
 \text{(i)} \quad & \mathbf{a} = (1, 1, \dots), & \mathbf{p}_a &= \mathbf{p} \\
 \text{(ii)} \quad & \mathbf{b} = (1, 1, \dots), & \mathbf{p}_b &= \mathbf{p} \\
 \text{(iii)} \quad & \mathbf{c} = (1, 0, 0, \dots), & \mathbf{p}^c &= 0 \\
 \text{(iv)} \quad & \mathbf{f} = (0, 0, \dots), & \mathbf{p}^{\mathbf{f}(k, n-k)} &= 1.
 \end{aligned} \tag{4.2}$$

Thus, since $\binom{n}{k}_0 = 1$, if one sets $C_n \equiv \mathcal{L}_n(0, 1, 1, 1, \dots)$, then (3.6) immediately reduces to

$$C_{n+1} = \sum_{k=0}^n C_k C_{n-k} \quad \text{with} \quad C_0 \equiv 1, \tag{4.3}$$

which is a fundamental recurrence relationship for the Catalan numbers.

As an second example, consider setting $\mathbf{s} \equiv (312, l_1, l_2, \dots)$, $\mathbf{p} \equiv (0, 1, 1, 0, 0, \dots)$, and $F_n \equiv \mathcal{L}_n(\mathbf{p})$. Then, since

$$\begin{aligned}
 \text{(i)} \quad & 312(\sigma) = 312(\alpha) + 312(\beta) + I(A, B) \\
 \text{(ii)} \quad & l_i(\sigma) = l_i(\alpha) + \chi(|B| + 1 = i) \quad \text{for all } i,
 \end{aligned} \tag{4.4}$$

it follows that

$$\begin{aligned}
 \text{(i)} \quad & \mathbf{a} = (1, 1, \dots), & \mathbf{p}_a &= \mathbf{p} = (0, 1, 1, 0, 0, \dots) \\
 \text{(ii)} \quad & \mathbf{b} = (1, 0, 0, \dots), & \mathbf{p}_b &= (0, 1, 1, 1, \dots) \neq \mathbf{p} \\
 \text{(iii)} \quad & \mathbf{c} = (1, 0, 0, \dots), & \mathbf{p}^c &= 0^1 1^0 1^0 0^0 0^0 \dots = 0 \\
 \text{(iv)} \quad & \mathbf{f} = (0, \chi(n-k=0), \chi(n-k=1), \dots), \\
 & & \mathbf{p}^{\mathbf{f}} &= 0^0 1^{\chi(n-k=0)} 1^{\chi(n-k=1)} 0^{\chi(n-k=2)} \dots
 \end{aligned} \tag{4.5}$$

Thus, recurrence relationship (3.6) becomes

$$F_{n+1} = \sum_{k=0}^n 0^0 1^{\chi(n-k=0)} 1^{\chi(n-k=1)} 0^{\chi(n-k=2)} \dots \binom{n}{k}_0 F_k \mathcal{L}_{n-k}(\mathbf{p}_b). \tag{4.6}$$

Since only the terms corresponding to $k = (n-1)$ and $k = n$ do not zero out, (4.6) reduces to

$$F_{n+1} = F_n \mathcal{L}_0(\mathbf{p}_b) + F_{n-1} \mathcal{L}_1(\mathbf{p}_b) = F_n + F_{n-1} \tag{4.7}$$

with initial conditions $F_0 \equiv \mathcal{L}_0(\mathbf{p}) \equiv 1$ and $F_1 \equiv \mathcal{L}_1(\mathbf{p}) = 1$. Of course, F_n defines the classic Fibonacci sequence.

Further identities for classic combinatorial sequences fall out as special cases of (3.6) by making the selections on \mathbf{s} and \mathbf{p} indicated in Table III. The corresponding special cases of (3.6) are listed in Table IV. Although

TABLE III

Some Selections on **s** and **p** That Lead to Classic Recurrence Relationships

Name	s	p	Symbol for $\mathcal{L}_n(\mathbf{p})$
1. Cardinality of $\mathcal{L}[n]$	(θ, θ, \dots)	$(1, 1, \dots)$	\mathcal{L}_n
2. Catalan numbers	$(312, \theta, \theta, \dots)$	$(0, 1, 1, \dots)$	C_n
3. Fibonacci numbers	$(312, l_1, l_2, \dots)$	$(0, 1, 1, 0, 0, \dots)$	F_n
4. Eulerian polynomials	$(\text{des}, \theta, \theta, \dots)$	$(t, 1, 1, \dots)$	$E_n(t)$
5. Number of derangements	(l_1, l_2, \dots)	$(0, 1, 1, \dots)$	D_n
6. First Stirling polynomials	(r_1, r_2, \dots)	(z, z, \dots)	$s_n(z)$
7. Second Stirling polynomials	$(\text{len}, -r, -\text{des}, r_1, r_2, \dots)$	$(0, 0, 0, z, z, \dots)^a$	$S_n(z)$
8. Hermite polynomials	(r_1, r_2, \dots)	$(y, -1, 0, 0, \dots)$	$H_n(y)$
9. Bell polynomials	$(\text{len}, -r, -\text{des}, r_1, r_2, \dots)$	$(0, 0, 0, z_1, z_2, \dots)^a$	$B_n(z_1, z_2, \dots)$

^a All powers of zero in $\mathcal{L}_n(\mathbf{p})$ are to be treated formally and are to be collected before evaluating.

TABLE IV

Classic Recurrences of Combinatorics Contained in (3.6)

Name	References	Recurrence
1. Cardinality of $\mathcal{L}[n]$		$\mathcal{L}_{n+1} = \sum_{k=0}^n \binom{n}{k} \mathcal{L}_k \mathcal{L}_{n-k}$
2. Catalan numbers	[Co]	$C_{n+1} = \sum_{k=0}^n C_k C_{n-k}$
3. Fibonacci numbers	[Co]	$F_{n+1} = F_n + F_{n-1}$
4. Eulerian polynomials	[Co, FS1]	$E_{n+1}(t) = E_n(t) + t \sum_{k=1}^n \binom{n}{k} E_k(t) E_{n-k}(t)$
5. Number of derangements	[Co]	$D_{n+1} = \sum_{k=0}^{n-1} \binom{n}{k} (n-k)! D_k$
6. First Stirling polynomials ^a	[Co]	$s_{n+1}(z) = z \sum_{k=0}^n \binom{n}{k} k! s_{n-k}(z)$
7. Second Stirling polynomials ^b	[Co]	$S_{n+1}(z) = z \sum_{k=0}^n \binom{n}{k} S_{n-k}(z)$
8. Hermite polynomials	[Co, F2]	$H_{n+1}(y) = H_n(y) - nH_{n-1}(y)$
9. Bell polynomials	[Co]	$B_{n+1}(z_1, z_2, \dots) = \sum_{k=0}^n z_{k+1} \binom{n}{k} B_{n-k}(z_1, z_2, \dots)$

^a The coefficient of z^k in $s_n(z)$ is the k th Stirling number of the first type.

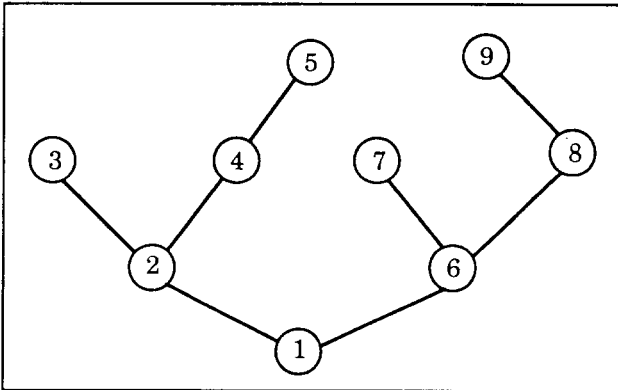
^b The coefficient of z^k in $S_n(z)$ is the k th Stirling number of the second type.

omitted, the respective initial conditions may be calculated from definition (3.4) of $\mathcal{L}_n(\mathbf{p})$.

At first glance it may seem somewhat odd that all of the recurrence relationships of Table IV can be obtained within the same combinatorial context. The explanation for this apparent paradox lies in the fact that setting parameters equal to zero is in a sense equivalent to selecting out subsets of trees. For instance, in the case of the Catalan numbers, setting the parameter associated with 312 patterns to zero means that the only binary trees that will make a non-zero contribution to $\mathcal{L}_n(\mathbf{p})$ have no 312 patterns. This subset of binary trees may be characterized as follows:

For a vertex v of a binary tree T , define A_v (resp. B_v) to be the set of vertices in the upper left (resp. right) subtree attached to v . Then, T has no 312 patterns iff $I(A_v, B_v) = 0$ for all vertices v in T . (4.8)

An example of a tree with no 312 patterns is sketched below:



It is interesting to note that each T satisfying the condition $I(A_v, B_v) = 0$ for all v is uniquely labelled by what is commonly known as prefix order. Thus, the act of removing the labels is a bijection from the set of trees with no 312 patterns to the set of unlabeled binary rooted planar trees. The latter set is a well-known combinatorial model of the Catalan numbers.

5. EXOTIC AND Q -SEQUENCES

Identity (3.6) contains, as special cases, a “vector space” of recurrence relationships, some of which define new exotic sequences and some of which define both known and new q -analogs of many classic sequences.

On the exotic side, consider making the selection $\mathbf{s} \equiv (l_1, r_1, l_2, r_2, l_3, r_3, \dots)$ and $\mathbf{p} \equiv (0, z, 1, z, 1, z, \dots)$. Furthermore, let $sd_n(z) \equiv \mathcal{L}_n(\mathbf{p})$. Note that

$$\begin{aligned}
 \text{(i)} \quad \mathbf{a} &= (1, 0, 1, 0, 1, 0, \dots) & \mathbf{p}_a &= (0, 1, 1, 1, 1, 1, \dots) \\
 \text{(ii)} \quad \mathbf{b} &= (0, 1, 0, 1, 0, 1, \dots) & \mathbf{p}_b &= (1, z, 1, z, 1, z, \dots) \\
 \text{(iii)} \quad \mathbf{c} &= (0, 0, 0, 0, 0, 0, \dots) & \mathbf{p}^c &= 1 \\
 \text{(iv)} \quad \mathbf{f} &= (f_1, g_1, f_2, g_2, f_3, g_3, \dots) & \mathbf{p}^f &= 0^{f_1 z^{g_1} 1^{f_2 z^{g_2} 1^{f_3 z^{g_3} \dots}}
 \end{aligned} \tag{5.1}$$

where $f_i(k, n-k) \equiv \chi(n-k+1=i)$ and $g_i(k, n-k) \equiv \chi(k+1=i)$. Therefore, identity (3.6) reduces to

$$sd_{n+1}(z) = z \sum_{k=0}^{n-1} \binom{n}{k} D_k s_{n-k}(z), \tag{5.2}$$

where $sd_0 = 1$, $sd_1 = 0$, and both D_n and $s_n(z)$ are defined in Table IV. The sequence $sd_n(z)$ is thus a curious cross between derangements and first Stirling polynomials.

There are also heretofore never considered statistics in the binary tree decomposition space which lead to new sequences. For instance, relative to the permutation factorization $\sigma = \alpha m \beta$, consider the elementary tree statistic w defined by

$$w(\sigma) = w(\alpha) + w(\beta) + \chi(|A| \geq 2), \tag{5.3}$$

which is a slight variation on *des*. Since (5.3) is equivalent to

$$w(\sigma) = |\{i: 1 < i < n \text{ and both } \sigma_{i-1}, \sigma_i > \sigma_{i+1}\}|, \tag{5.4}$$

the statistic w may be thought of as counting the “waves of length greater than or equal to 2” in σ . Now, if one takes $\mathbf{s} \equiv (w, \theta, \theta, \dots)$, $\mathbf{p} \equiv (t, 1, 1, \dots)$, and $W_n(t) \equiv \mathcal{L}_n(\mathbf{p})$, then (3.6) reduces to

$$W_{n+1}(t) = W_n(t) + nW_{n-1}(t) + t \sum_{k=2}^n \binom{n}{k} W_k(t) W_{n-k}(t) \tag{5.5}$$

with initial conditions $W_0(t) = W_1(t) \equiv 1$.

With regards to q -analogs, any statistic for which the constant c in (2.6) is non-zero will yield a q -analog of the particular sequence under consideration. For instance, to obtain the q -Eulerian numbers due to Stanley [S], let $\mathbf{s} \equiv (\text{des}, \text{inv}, \theta, \theta, \dots)$ and $E_n(t, q) \equiv \mathcal{L}_n(t, q, 1, 1, \dots)$. Recurrence (3.6) then reduces to

$$E_{n+1}(t, q) = E_n(t, q) + t \sum_{k=1}^n q^k \binom{n}{k}_q E_k(t, q) E_{n-k}(t, q) \tag{5.6}$$

with initial conditions $E_0(t, q) \equiv \mathcal{L}_0(\mathbf{p}) = 1$ and $E_1(t, q) \equiv \mathcal{L}_1(\mathbf{p}) = 1$. Further examples may be obtained by selecting \mathbf{s} and \mathbf{p} as in Table V.

The corresponding cases of (3.6) are listed in Table VI. In doing the calculations to obtain entries 2 and 7, it will be necessary to treat powers of zero formally and collect all such powers before evaluating. Also, the respective initial conditions for all entries can be derived from definition (3.4) of $\mathcal{L}_n(\mathbf{p})$.

Unfortunately, this “vector space” of q -analogs does not include some of the classic ones that arise in connection with the statistic known as the major index (maj). This is because, relative to the binary tree decomposition $\sigma = \alpha m \beta$, the major index satisfies the relationship (see [R1])

$$\text{maj}(\sigma) = \text{maj}(\alpha) + \text{maj}(\beta) + |A| + (|A| + 1) \text{des}(\beta). \quad (5.7)$$

Therefore, maj is not an element of the binary tree decomposition space under consideration.

One could however, at the expense of complicating matters somewhat, expand the space of consideration in a way so as to include the major index. This direction of inquiry will not be pursued here. For those interested, a recurrence relationship analogous to (3.6) of the joint distribution $\mathbf{s} \equiv (\text{des}, \text{maj}, \text{inv}, 312, 213, l_1, r_1, l_2, r_2, \dots)$ over $\mathcal{L}[n]$ is considered in some detail in [R2].

TABLE VI
Some q -Sequences Derived from (3.6)

Name	References	Recurrence
1. q -Catalan numbers I	[FH]	$c_{n+1}(q) = \sum_{k=0}^n q^k c_k(q) c_{n-k}(q)$
2. q -Catalan numbers II	[FH]	$C_{n+1}(q) = \sum_{k=0}^n q^{(k+1)(n-k)} C_k(q) C_{n-k}(q)$
3. q -derangements	[R1]	$D_{n+1}(q) = \sum_{k=0}^n q^k \binom{n}{k}_q (n-k)_q! D_k(q)$
4. First q -Stirling polynomials	[Go]	$s_{n+1}(q; z) = z \sum_{k=0}^n q^k \binom{n}{k}_q (k)_q! s_{n-k}(q; z)$
5. q -Hermite polynomials I	[Ci, D, ISV]	$h_{n+1}(q; y) = y h_n(q; y) - (n)_q h_{n-1}(q; y)$
6. q -Hermite polynomials II	[Ci, D, ISV]	$H_{n+1}(q; y) = y H_n(q; y) - q^n (n)_q H_{n-1}(q; y)$
7. q -Bell polynomials		$B_{n+1}(q; z_1, z_2, \dots)$ $= \sum_{k=0}^n q^k z_{k+1} \binom{n}{k}_q B_{n-k}(q; z_1, z_2, \dots)$

TABLE V

Some Selections of \mathbf{s} and \mathbf{p} That Lead to q -Analogues

Name	\mathbf{s}	\mathbf{p}	Symbol for $\mathcal{L}_n(\mathbf{p})$
1. q -Catalan numbers I	(312, inv, θ , θ , ...)	(0, q , 1, 1, ...)	$c_n(q)$
2. q -Catalan numbers II	(213, inv, θ , θ , ...)	(0, q , 1, 1, ...)	$C_n(q)$
3. q -derangements	(inv, l_1 , l_2 , ...)	(q , 0, 1, 1, ...)	$D_n(q)$
4. First q -Stirling polynomials	(inv, r_1 , r_2 , ...)	(q , z , z , ...)	$s_n(q; z)$
5. q -Hermite polynomials I	(312, r_1 , r_2 , ...)	(q , y , -1, 0, 0, ...)	$h_n(q; y)$
6. q -Hermite polynomials II	(inv, 213, 312, r_1 , r_2 , ...)	(q , q , q , y , -1, 0, 0, ...)	$H_n(q; y)$
7. q -Bell polynomials	(len, $-r$, $-\text{des}$, inv, r_1 , r_2 , ...)	(0, 0, 0, q , z_1 , z_2 , ...)	$B_n(q; z_1, z_2, \dots)$

6. CONCLUDING REMARKS

The idea of examining decomposition-based statistic spaces is new and there are many possible directions of inquiry to be pursued. For one, a general study of decomposition statistic spaces would be of interest. Such a study would include consideration of spaces based on other decompositions, a classification of known statistics according to decomposition, and an examination of the structural aspects of these spaces.

A second possible direction involves the study of the generalizations of the classic sequences that arise. For instance, the Bell polynomials of Table III play a key role in functional composition and Lagrange inversion (see [Co]). A natural line of inquiry would be to determine the subspace of the binary tree decomposition statistic space for which the Lagrange inversion formula remains valid.

Finally, although recurrence relationship (3.6) exists, it does not appear possible to determine a generating function for a general joint statistic s . It may, however, be fruitful to consider isolating classes or subspaces of statistics according to "generating function type."

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