

# Multicolored Simon Newcomb Problems

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Recent progress made by Désarménien and Foata in the area of permutation statistics indirectly points out a certain gap in the theory of sequence enumeration. The remedy of the situation lies in the consideration of some colorful extensions of the Simon Newcomb problem.

## 1. INTRODUCTION

Certain modifications in the classic Simon Newcomb problem (CSNP) are indirectly suggested by the results in [5]. The modified version focused on in this paper will be referred to as the "bicolored" Simon Newcomb problem (BSNP). Multicolored extensions will be briefly discussed in Section 6.

Like the CSNP, the BSNP may be described in terms of a simple card game. A deck of cards is said to be bicolored of specification  $(i_1, i_2, \dots, i_r)$  if each card is either blue or green and if  $i_m$  cards have integer face value  $m$  for  $1 \leq m \leq r$ . The statement of the BSNP goes as follows:

*The BSNP.* A bicolored deck of specification  $(i_1, i_2, \dots, i_r)$  is first to be shuffled and then dealt out into piles; a new pile is begun only with a card which is immediately preceded by

- (0) nothing, that is, the first card in the shuffle begins the first pile,
- or
- (1) a card of strictly greater face value, or
- (2) a green card of the same face value.

An occurrence of (1) or (2) is referred to as a cut in the sequence of cards. The problem then is to determine how many shuffles, when dealt out, result in  $(k + 1)$  piles (or equivalently, have  $k$  cuts).

There are two special cases of the BSNP that are worth singling out. If all of the cards in the deck are blue, then the notion of a cut is none other than the classic one of a descent and the BSNP reduces to the original CSNP as considered in [1–3, 7, 10, 11, 14]. On the other hand, if all of the cards happen to be green, then the notion of a cut corresponds to what is known in the literature as a nonrise. This problem, henceforth referred to as the “reciprocal” Simon Newcomb problem (RSNP), has received much less press than the CSNP. The solution of the RSNP may be implicitly found in [4] and is explicitly equivalent to the generating function for non-falls given in [10, p. 72].

The main purpose of this paper is to provide a “ $q$ -solution” of the BSNP. In order to adequately state the  $q$ -solution, some definitions and notation are needed.

First, for an  $r$ -tuple  $(i_1, i_2, \dots, i_r)$  of non-negative integers, let

$$\begin{aligned} \text{(a)} \quad & i(r) := (i_1, i_2, \dots, i_r) \\ \text{(b)} \quad & n := i_1 + i_2 + \dots + i_r. \end{aligned} \tag{1.1}$$

The symbol  $\mathcal{S}[i(r)]$  will be used to signify the set of bicolored integer sequences  $f$  of the form

$$f = f(1)f(2)\cdots f(n) \tag{1.2}$$

in which the integer  $m$  appears  $i_m$  times for  $1 \leq m \leq r$  and each integer  $f(k)$  in  $f$  is either blue or green. Thus, there are  $2^n \cdot n! / i_1! i_2! \cdots i_r!$  such sequences.

The cut set of  $f \in \mathcal{S}[i(r)]$ , denoted by  $\text{Cut } f$ , is defined to be the union of the two sets

$$\begin{aligned} \text{(a)} \quad & \{k : f(k) > f(k+1), 1 \leq k < n\} \\ \text{(b)} \quad & \{k : f(k) = f(k+1), f(k) \text{ is green}, 1 \leq k < n\}. \end{aligned} \tag{1.3}$$

In considering the BSNP, the relevant statistics on a bicolored sequence  $f$  are defined by

$$\begin{aligned} \text{(a)} \quad & \text{cut } f := \text{cardinality of Cut } f \\ \text{(b)} \quad & \beta(f) := \text{number of blue integers in } f \\ \text{(c)} \quad & \gamma(f) := \text{number of green integers in } f. \end{aligned} \tag{1.4}$$

The sum of the elements of the cut set of  $f$

$$\text{scs } f := \sum_{k \in \text{Cut } f} k \tag{1.5}$$

is another statistic which will be seen to be of interest in the context of the BSNP. In fact, if all of the cards in  $f$  happen to be blue, then the statistic  $\text{scs}$  reduces to the classic statistic known as the major index of MacMahon [11].

As an example of the preceding definitions, consider the bicolored sequence

$$f := 2 \ 2 \ 3 \ 3 \ 3 \ 4 \ 1 \ 1 \ 4 \ 2 \in \mathcal{S}[2, 3, 3, 2], \quad (1.6)$$

where an italicized (resp. bold-faced) integer is to be imagined as being blue (resp. green). For  $f$  in (1.6), one has  $\text{Cut } f = \{4, 6, 9\}$ ,  $\text{cut } f = 3$ ,  $\text{scs } f = 19$ ,  $\beta(f) = 7$ , and  $\gamma(f) = 3$ .

With the conventions that

$$\begin{aligned} \text{(a)} \quad & (u; q)_{m+1} := (1-u)(1-uv) \cdots (1-uv^m) \\ \text{(b)} \quad & X^{i(r)} := \prod_{m=1}^r x_m^{i_m}, \end{aligned} \quad (1.7)$$

a generating function solution to a  $q$ -analog of the BSNP, which will be proven in Section 4, may finally be stated:

**THEOREM 1.1.** *The “exponential” generating function for the polynomial defined by*

$$S(i(r); t, q, b, g) := \sum_{f \in \mathcal{S}[i(r)]} t^{\text{cut } f} q^{\text{scs } f} b^{\beta(f)} g^{\gamma(f)} \quad (1.8)$$

is given by

$$\sum_{i(r) \geq 0} \frac{S(i(r); t, q, b, g) X^{i(r)}}{(t, q)_{n+1}} = \sum_{s \geq 0} t^s \prod_{k=1}^r \frac{(-g x_k; q)_{s+1}}{(b x_k; q)_{s+1}}, \quad (1.9)$$

where  $n := i_1 + i_2 + \cdots + i_r$ , and  $i(r) \geq 0$  means that  $i_m \geq 0$  for  $1 \leq m \leq r$ .

The reason for using the term “ $q$ -solution” should now be apparent: If  $q := 1$ , then Theorem 1.1 gives the generating function for the solution of the BSNP as stated. Explicit formulas for (1.8) and for the distribution of the statistic  $\text{scs}$  will be given in Section 5.

Of course,  $q$ -solutions of both the CSNP and the RSNP may be immediately obtained from Theorem 1.1 as corollaries: If the parameter  $g$  (resp.  $b$ ) is set equal to zero, then only completely blue (resp. green) sequences make non-zero contributions to the sum in (1.8). Respectively, we have

COROLLARY 1.2. *The generating function for the polynomial defined by*

$$C(i(r); t, q) := S(i(r); t, q, 1, 0) \quad (1.10)$$

*is given by*

$$\sum_{i(r) \geq 0} \frac{C(i(r); t, q) X^{i(r)}}{(t; q)_{n+1}} = \sum_{s \geq 0} t^s \prod_{k=1}^r \frac{1}{(x_k; q)_{s+1}}. \quad (1.11)$$

COROLLARY 1.3. *The generating function for the polynomial defined by*

$$R(i(r); t, q) := S(i(r); t, q, 0, 1) \quad (1.12)$$

*is given by*

$$\sum_{i(r) \geq 0} \frac{R(i(r); t, q) X^{i(r)}}{(t; q)_{n+1}} = \sum_{s \geq 0} t^s \prod_{k=1}^r (-x_k; q)_{s+1}. \quad (1.13)$$

Corollary 1.2 for the  $q$ -solution of the CSNP was first derived by MacMahon [11] (also see [9, 13]). In the case when  $q := 1$ , (1.13) is equivalent to the generating function for sequences by nonfalls given on page 72 of [10]. Also, note that taken together, identities (1.11) and (1.13) explain the use of the word “reciprocal” in the context of Corollary (1.3).

Before getting down to the business of proving Theorem (1.1), a brief explanation of the underlying motivation for the BSNP is in order. Recently, Désarménien and Foata [5] made the beautiful observation that some Schur function identities may be used to obtain permutation statistic results. In particular, they derived the three generating functions

$$\begin{aligned} \text{(a)} \quad & \sum_{n \geq 0} \frac{A_n(t_1, t_2, q_1, q_2) u^n}{(t_1; q_1)_{n+1} (t_2; q_2)_{n+1}} = \sum_{r, s \geq 0} t_1^r t_2^s \frac{1}{(u; q_1, q_2)_{r+1, s+1}} \\ \text{(b)} \quad & \sum_{n \geq 0} \frac{B_n(t_1, t_2, q_1, q_2) u^n}{(t_1; q_1)_{n+1} (t_2; q_2)_{n+1}} = \sum_{r, s \geq 0} t_1^r t_2^s (-u; q_1, q_2)_{r+1, s+1} \\ \text{(c)} \quad & \sum_{n \geq 0} \frac{C_n(t_1, t_2, q_1, q_2) u^n}{(t_1; q_1)_{n+1} (t_2; q_2)_{n+1}} = \sum_{r, s \geq 0} t_1^r t_2^s \frac{(-zu; q_1, q_2)_{r+1, s+1}}{(u; q_1, q_2)_{r+1, s+1}} \end{aligned} \quad (1.14)$$

which arise in connection with various statistics defined on permutations. (For details, see [5].)

Note the similarity in form between the identities of (1.14) and those respectively of (1.11), (1.13), and (1.9). In fact, there is more than just a mere resemblance between (1.14a) and (1.11). In [12] it was demonstrated that the CSNP of (1.11) implies (1.14a).

Thus, given the relationship between (1.11) and (1.14a), it is clear that the existence of (1.14c) does indeed motivate the combinatorial study of

(1.9). A proof that the BSNP provides an appropriate combinatorial framework for (1.9) is given in the next three sections.

## 2. BICOLORED MATRICES

A certain result on "bicolored" matrices will be employed to derive Theorem (1.1). This key result is now presented.

For the appropriate combinatorial perspective, the rows (resp. columns) of a matrix will be numbered from bottom-to-top (resp. left-to-right). An  $(r \times s)$  matrix  $A = (a_{kl})$  is said to be bicolored if its entries are of the form

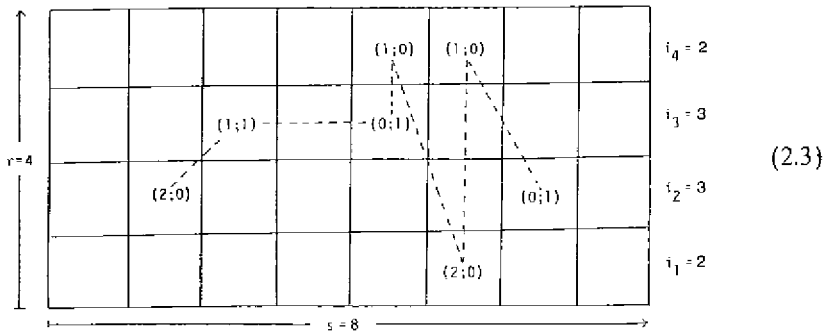
$$a_{kl} := (m; p), \quad (2.1)$$

where  $m$  is any non-negative integer and  $p$  is equal to zero or one. The integer  $m$  (resp.  $p$ ) is to be imagined as being blue (resp. green). The symbol  $\mathcal{M}[i(r); s]$  will be used to denote the set of such  $(r \times s)$  bicolored matrices  $A$  having row vector sum  $i(r) := (i_1, i_2, \dots, i_r)$ ; that is,

$$i_k := \text{sum of all integers in row } k \text{ of } A \quad (2.2)$$

for  $1 \leq k \leq r$ .

Geometrically, a matrix  $A \in \mathcal{M}[i(r); s]$  will be viewed as an array of rectangular cells with entries of the form given in (2.1). Diagram (2.3) illustrates an element of  $\mathcal{M}[i(4); 8]$  in which the trivial entries  $(0; 0)$  have been omitted.



The dotted line segments in (2.3) will be explained in Section 3.

For  $A \in \mathcal{M}[i(r); s]$ , the color statistics and the column vector sum of  $A$  are defined to be

$$\begin{aligned}
\text{(a)} \quad & \beta(A) := \text{sum of all blue integers in } A \\
\text{(b)} \quad & \gamma(A) := \text{sum of all green integers in } A \\
\text{(c)} \quad & \text{col}(A) := (j_1, j_2, \dots, j_s),
\end{aligned} \tag{2.4}$$

where  $j_l := \text{sum of all integers in column } l \text{ of } A \text{ for } 1 \leq l \leq s$ . In diagram (2.3) we have  $\beta(A) = 7$ ,  $\gamma(A) = 3$ , and  $\text{col}(A) = (0, 2, 2, 0, 2, 3, 1, 0)$ .

The key result on bicolored matrices is stated in Lemma 2.1:

LEMMA 2.1. For  $A \in \mathcal{M}[i(r); s]$  with  $\text{col}(A) := (j_1, j_2, \dots, j_s)$ , let

$$s \circ \text{col}(A) = \sum_{l=1}^s (s-l)j_l. \tag{2.5}$$

Then, the polynomial defined by

$$M_s(i(r); q, b, g) := \sum_{A \in \mathcal{M}[i(r); s]} q^{s \circ \text{col}(A)} b^{\beta(A)} g^{\gamma(A)} \tag{2.6}$$

satisfies the identity

$$\sum_{i(r) \geq 0} M_s(i(r); q, b, g) X^{i(r)} = \prod_{k=1}^r (-gx_k; q)_s (bx_k; q)_s^{-1}. \tag{2.7}$$

*Proof.* For an entry  $a_{k,l} := (m; p)$  of  $A \in \mathcal{M}[i(r); s]$ , let

$$\delta(a_{kl}) := (bx_k q^{s-l})^m (gx_k q^{s-l})^p. \tag{2.8}$$

Extending in a ‘‘cell-wise’’ multiplicative fashion, define the weight function  $\delta$  on  $\mathcal{M}[i(r); s]$  by

$$\delta(A) = \prod_{k=1}^r \prod_{l=1}^s \delta(a_{kl}). \tag{2.9}$$

From (2.8) and (2.9) it is not difficult to see that

$$\delta(A) = q^{s \circ \text{col}(A)} b^{\beta(A)} g^{\gamma(A)} X^{i(r)}, \tag{2.10}$$

which in turn implies that the left-hand side of (2.7) is equal to

$$\sum_{i(r) \geq 0} \sum_{A \in \mathcal{M}[i(r); s]} \delta(A). \tag{2.11}$$

On the other hand, since  $\delta$  is multiplicative in the sense of (2.9) and since

the choices of  $m$  and  $p$  are independent, the  $\delta$ -weight generating function of (2.11) on all  $(r \times s)$  bicolored matrices may be written in the form

$$\prod_{k=1}^r \prod_{i=1}^s (1 + gx_k q^r - i)(1 - bx_k q^s - i)^{-1}. \quad (2.12)$$

By re-indexing and by using (1.7a), the expression in (2.12) may be seen to be equal to the right-hand side of (2.7). Thus, the proof is complete.

### 3. THE ENCODING

For converting Lemma 2.1 into Theorem 1.1, an encoding  $\Gamma$  will be used which maps a bicolored matrix  $A \in \mathcal{M}[i(r); s]$  to a pair of the form

$$\begin{pmatrix} h \\ f \end{pmatrix}, \quad (3.1)$$

where  $f \in \mathcal{S}[i(r)]$  and where  $h := h(1)h(2)\cdots h(n)$  is a non-decreasing integer sequence of length  $n := i_1 + i_2 + \cdots + i_r$ , satisfying the conditions that

$$\begin{aligned} \text{(a)} \quad & 1 \leq h(m) \leq s \quad \text{for } 1 \leq m \leq n \\ \text{(b)} \quad & h(m) < h(m+1) \quad \text{if } m \in \text{Cut } f. \end{aligned} \quad (3.2)$$

(The set of such  $h$  will henceforth be denoted by  $\mathcal{H}(s, f)$ .) The map  $\Gamma$  may be best described geometrically in terms of a scanning process:

*The Encoding  $\Gamma$ .* Given a bicolored matrix  $A \in \mathcal{M}[i(r); s]$ , the sequences  $h$  and  $f$  are to be constructed in the following manner. Viewing  $A$  pictorially as in (2.3), scan the cells one at a time in the order outlined below in (3.3):

$$\begin{aligned} & \text{Beginning with the cell of } a_{11}, \text{ completely scan column 1 from} \\ & \text{bottom-to-top. Then, scan column 2 from bottom-to-top.} \\ & \text{Proceed in this way until all cells have been scanned.} \end{aligned} \quad (3.3)$$

If  $a_{kl} := (m; p)$ , then respectively define the sequences  $h_{kl}$  and  $f_{kl}$  each of length  $(m + p)$  by

$$\begin{aligned} \text{(a)} \quad & h_{kl} := ll \cdots l \\ \text{(b)} \quad & f_{kl} := \begin{cases} \ell \ell \cdots \ell & \text{if } p := 0 \\ \ell \ell \cdots \ell \mathbf{k} & \text{if } p := 1. \end{cases} \end{aligned} \quad (3.4)$$

Then, the sequences  $h$  and  $f$  are constructed by respectively juxtaposing the  $h_{kl}$  and  $f_{kl}$  according to the scanning order in (3.3).

As an example of how  $\Gamma$  works, consider the matrix  $A$  in (2.3). The dotted line segments, beginning in the cell of  $a_{22}$ , indicate the scanning order of the cells with non-trivial entries. By (3.4),  $h_{16} = 66$ ,  $f_{16} = 11$ , and  $f_{33} = 33$ . Furthermore, juxtaposing the sequences of (3.4) according to the scanning order yields

$$\begin{pmatrix} h \\ f \end{pmatrix} = \begin{pmatrix} 2 & 2 & 3 & 3 & 5 & 5 & 6 & 6 & 6 & 7 \\ 2 & 2 & 3 & 3 & 3 & 4 & 1 & 1 & 4 & 2 \end{pmatrix}. \quad (3.5)$$

Note that the dotted line segments in (2.3) completely illustrate the “up-down” behavior and the cuts of  $f$ :

The bicolored sequence  $f$  has a cut at an index if and only if the corresponding line segment is descending or is level and originates from a cell with an entry of the form  $(m; 1)$ . (3.6)

Define the content vector of a non-decreasing sequence  $h$  in  $\mathcal{H}(s, f)$  by

$$\text{con}(h) := (j_1, j_2, \dots, j_s), \quad (3.7)$$

where  $j_k$  is equal to the number of times that  $k$  appears in  $h$ ,  $1 \leq k \leq s$ . The relevant properties of  $\Gamma$  are listed in Lemma 3.1 below. The proof of Lemma 3.1 is straightforward and is left to the reader.

LEMMA 3.1. *The map  $\Gamma$  is a bijection from  $\mathcal{M}[i(r); s]$  to the set*

$$\left\{ \begin{pmatrix} h \\ f \end{pmatrix} : f \in \mathcal{S}[i(r)] \text{ and } h \in \mathcal{H}(s, f) \right\}. \quad (3.8)$$

Furthermore, if  $\Gamma(A) = \begin{pmatrix} h \\ f \end{pmatrix}$ , then

$$\begin{aligned} \text{(a)} \quad & \text{col}(A) = \text{con}(h) \\ \text{(b)} \quad & \beta(A) = \beta(f) \\ \text{(c)} \quad & \gamma(A) = \gamma(f). \end{aligned} \quad (3.9)$$

One additional lemma, which involves a “ $q$ -counting” of the set  $\mathcal{H}(s, f)$ , is needed. The  $q$ -analog,  $q$ -factorial, and  $q$ -binomial coefficient of a non-negative integer  $n$  are defined by

$$\begin{aligned} \text{(a)} \quad & [n] := 1 + q + q^2 + \dots + q^{n-1} \\ \text{(b)} \quad & [n]! := [1] [2] \cdots [n] \\ \text{(c)} \quad & \begin{bmatrix} n \\ k \end{bmatrix} := [n]! / [k]! [n-k]!, \end{aligned} \quad (3.10)$$



where, by convention,  $[0] := 1$ . For  $h \in \mathcal{H}(s, f)$  of length  $n$ , define  $m_k$ ,  $0 \leq k \leq n$ , by

$$\begin{aligned} \text{(a)} \quad m_0 &:= h(1) - 1 \\ \text{(b)} \quad m_k &:= h(k+1) - h(k) - \mathbb{X}(k \in \text{Cut } f) \quad \text{for } 1 \leq k \leq n-1 \\ \text{(c)} \quad m_n &:= s - h(n) \end{aligned} \tag{3.11}$$

in which, for a statement  $Q$ ,  $\mathbb{X}(Q)$  is equal to 1 if  $Q$  is true and 0 otherwise. Furthermore, let

$$\alpha(h) := \sum_{k=1}^n k m_k. \tag{3.12}$$

Then, the required “ $q$ -counting” lemma, which can be proven in exactly the same way as Lemma 5.1 of [12], may be expressed in the form:

**LEMMA 3.2.** *For  $s$  a positive integer and for  $f \in \mathcal{S}[i(r)]$  of length  $n$ , we have*

$$\sum_{h \in \mathcal{H}(s, f)} q^{\alpha(h)} = \begin{bmatrix} s-1 - \text{cut } f + n \\ n \end{bmatrix}. \tag{3.13}$$

Moreover, given  $h \in \mathcal{H}(s, f)$  having content vector  $\text{con}(h) := (j_1, j_2, \dots, j_s)$ ,

$$\text{scs } f + \alpha(h) = \sum_{l=1}^s (s-l) j_l. \tag{3.14}$$

#### 4. THE SOLUTION OF THE BSNP

A proof of Theorem 1.1 may now be given with the aid of the lemmas in Sections 2 and 3. First, the polynomial in (1.8) must be refined by defining

$$S_m(i(r); q, b, g) := \sum q^{\text{scs } f} b^{\beta(f)} g^{r(f)}, \tag{4.1}$$

where the summation is over all  $f \in \mathcal{S}[i(r)]$  with  $\text{cut } f = m$ . Note that

$$S(i(r); t, q, b, g) = \sum_{m \geq 0} t^m S_m(i(r); q, b, g). \tag{4.2}$$

Second, observe that Lemmas 3.1 and 3.2 together justify the calculation in (4.3) which relates the polynomial of (2.6) to the one of (4.1),

$$\begin{aligned}
M_s(i(r); q, b, g) &= \sum_{f \in \mathcal{S}[i(r)]} \sum_{h \in \mathcal{H}(s, f)} q^{\text{scsf} + \alpha(h)} b^{\beta(f)} g^{\gamma(f)} \\
&= \sum_{f \in \mathcal{S}[i(r)]} q^{\text{scsf}} b^{\beta(f)} g^{\gamma(f)} \begin{bmatrix} s-1 - \text{cut } f + n \\ n \end{bmatrix} \\
&= \sum_{m \geq 0} \begin{bmatrix} s-1 - m + n \\ n \end{bmatrix} S_m(i(r); q, b, g). \tag{4.3}
\end{aligned}$$

By shifting the index  $s$  to  $(s+1)$  in (4.3) and by using the first of the two facts (see [2, p. 36]) listed in

$$\begin{aligned}
\text{(a)} \quad (t; q)_{n+1}^{-1} &= \sum_{s \geq 0} \begin{bmatrix} n+s \\ s \end{bmatrix} t^s \\
\text{(b)} \quad (t; q)_{n+1} &= \sum_{s \geq 0} (-1)^s \begin{bmatrix} n+1 \\ s \end{bmatrix} q^{s(s-1)/2} t^s,
\end{aligned} \tag{4.4}$$

it is not difficult to verify that (4.3) is equivalent to

$$\sum_{s \geq 0} t^s M_{s+1}(i(r); q, b, g) = (t; q)_{n+1}^{-1} S(i(r); t, q, b, g). \tag{4.5}$$

Multiplying both sides of (4.5) by  $X^{i(r)}$  and then summing over  $i(r) \geq 0$  leads to

$$\sum_{i(r) \geq 0} \frac{S(i(r); t, q, b, g) X^{i(r)}}{(t; q)_{n+1}} = \sum_{s \geq 0} t^s \sum_{i(r) \geq 0} X^{i(r)} M_{s+1}(i(r); q, b, g). \tag{4.6}$$

Finally, Theorem (1.1) follows immediately from (4.6) together with Lemma 2.1.

## 5. EXPLICIT FORMULAS

As corollaries of Theorem 1.1, explicit formulas for the  $q$ -solution of the BSNP and for the distribution of the scs statistic on  $\mathcal{S}[i(r)]$  may be obtained. To do this, first note that (4.4) can be used to expand the product

$$\prod_{k=1}^r (-gx_k; q)_{s+1} (bx_k; q)_{s+1}^{-1} \tag{5.1}$$

into the form

$$\sum_{i(r) \geq 0} X^{i(r)} \prod_{k=1}^r E(s, i_k; q, b, g) \tag{5.2}$$

where, by definition,

$$E(s, i_k; q, b, g) := \sum_{0 \leq t \leq i_k} \begin{bmatrix} s+1 \\ t \end{bmatrix} \begin{bmatrix} s+i_k-t \\ s \end{bmatrix} q^{t(t-1)/2} b^{i_k-t} g^t. \quad (5.3)$$

By inserting (5.2) into (1.9) and then by extracting the coefficients of  $X^{i(r)}$  from (1.9), one obtains the following corollary:

**COROLLARY 5.1.** *The explicit solution for the BSNP is given by*

$$S(i(r); t, q, b, g) = (t; q)_{n+1} \sum_{s \geq 0} t^s \prod_{k=1}^r E(s, i_k; q, b, g), \quad (5.4)$$

where  $E(s, i_k; q, b, g)$  is defined in (5.3).

In order to express the formula for the distribution of scs in a compact form, let

$$\begin{bmatrix} n \\ i(r) \end{bmatrix} := [n]! / [i_1]! \cdots [i_r]!. \quad (5.5)$$

To derive the formula of this distribution, first note that setting  $t := 1$  in a series of the form

$$F(t) := (1-t) \sum_{s \geq 0} f_s t^s \quad (5.6)$$

gives, provided that  $f_m$  converges,  $F(1) = \lim f_s$ . Then, by restricting  $q$  so that  $0 < |q| < 1$  and using the identity

$$(q; q)_n = (1-q)^n [n]!, \quad (5.7)$$

one is led through some tedious calculations to the fact that

**COROLLARY 5.2.** *The distribution of scs on  $\mathcal{S}[i(r)]$  is given by*

$$S(i(r); 1, q, b, g) = \begin{bmatrix} n \\ i(r) \end{bmatrix} \prod_{k=1}^r \sum_{0 \leq t \leq i_k} \begin{bmatrix} i_k \\ t \end{bmatrix} q^{t(t-1)/2} b^{i_k-t} g^t. \quad (5.8)$$

Of course, formulas for the distribution of scs in the special cases corresponding to the CSNP and the RSNP may be obtained immediately from (5.8). Respectively, we have

$$\begin{aligned} \text{(a)} \quad C(i(r); 1, q) &= \begin{bmatrix} n \\ i(r) \end{bmatrix} \\ \text{(b)} \quad R(i(r); 1, q) &= \begin{bmatrix} n \\ i(r) \end{bmatrix} \prod_{k=1}^r q^{i_k(i_k-1)/2}, \end{aligned} \quad (5.9)$$

where  $C(i(r); t, q)$  and  $R(i(r); t, q)$  are defined in (1.10) and (1.12). As previously mentioned, in the case of the CSNP the statistic  $\text{scs}$  reduces to the classic statistic known as the major index which was first introduced by MacMahon [11]. In fact, identity (5.9a) is due to MacMahon.

It is worth noting, as suggested by the referee, that (5.9b) may be easily derived from (5.9a) in a bijective manner. To a blue sequence  $f = f(1)f(2)\cdots f(n)$  associate the sequence  $h = h(1)h(2)\cdots h(n)$ , where each  $h(k)$  is green and, numerically,  $h(k) = r + 1 - f(k)$ . The obvious fact that  $\text{Cut } h = \{1, 2, \dots, n-1\} \setminus \text{Cut } f$  implies that

$$\begin{aligned} R(i(r); 1, q) &:= \sum_h q^{\text{scs } h} = q^{\binom{n}{2}} \sum_f (q^{-1})^{\text{scs } f} \\ &:= q^{\binom{n}{2}} C(i(r); 1, q^{-1}). \end{aligned} \quad (5.10)$$

Identity (5.9b) then follows from (5.9a), (5.10), and a few simple calculations.

## 6. FURTHER VARIATIONS OF THE CSNP

The method developed in Sections 2, 3, and 4 may be extended to deal with a variety of Simon Newcomb-type problems. The first of two examples given in this section will be referred to as the “multicolored” Simon Newcomb problem (MSNP) and can be stated as follows:

*The MSNP.* Suppose each card in a deck of specification  $i(r)$  is either one of  $u$  shades of blue  $\{b_1, b_2, \dots, b_u\}$  or one of  $v$  shades of green  $\{g_1, g_2, \dots, g_v\}$ . The deck is to be shuffled and then dealt out into piles; a new pile is begun only with

- (0) the first card in the shuffle, or
- (1) a card which is immediately preceded by one of greater face value, or
- (2) a card of color  $b_j$  which is immediately preceded by a card with equal face value of color  $b_m$  where  $1 \leq j < m \leq u$ , or
- (3) a card of color  $b_j$  or  $g_l$  which is immediately preceded by a card with equal face value of color  $g_k$ , where  $1 \leq j \leq u$  and  $1 \leq l \leq k \leq v$ .

An occurrence of (1), (2), or (3) is referred to as a cut in the sequence of cards. The problem is to determine how many shuffles have a given number of cuts.

Note that if  $u := 1$  and  $v := 1$ , then the MSNP reduces down to the BSNP. Thus, there is no inconsistency in the way the term “cut” is used in the two contexts. As already indicated, the MSNP may be solved by

generalizing the method used for the BSNP. Only a very rough sketch will be given here.

Let  $M\mathcal{S}[i(r)]$  denote the set of multicolored sequences  $f$  of the form

$$f := f(1)f(2)\cdots f(n), \quad (6.1)$$

where  $n := i_1 + i_2 + \cdots + i_r$ ,  $k$  appears  $i_k$  times in  $f$  for  $1 \leq k \leq r$ , and each letter  $f(k)$  is either a shade of blue  $b_j$  or a shade of green  $g_l$ . After appropriately extending definitions (1.3), (1.4), and (1.5), let

$$M\mathcal{S}(i(r); t, q, B, G) := \sum_{f \in M\mathcal{S}[i(r)]} t^{\text{cut } f} q^{\text{scs } f} B^{\beta(f)} G^{\gamma(f)} \quad (6.2)$$

where

$$B^{\beta(f)} := \prod_{j=1}^u b_j^{\beta_j(f)}$$

$$G^{\gamma(f)} := \prod_{l=1}^v g_l^{\gamma_l(f)}$$

and  $\beta_j(f)$  (resp.  $\gamma_l(f)$ ) is equal to the number of integers of shade  $b_j$  (resp.  $g_l$ ) in  $f$ .

The corresponding set of  $(r \times s)$  “multicolored” matrices  $A = (a_{kl})$  with row sum vector  $i(r)$  denoted by  $M\mathcal{M}[i(r); s]$  has entries of the form

$$a_{kl} := (m_1, m_2, \dots, m_u; p_1, p_2, \dots, p_v), \quad (6.3)$$

where  $m_j$  for  $1 \leq j \leq u$  may be any non-negative integer and  $p_l$  for  $1 \leq l \leq v$  may be either 0 or 1. The entry  $m_j$  (resp.  $p_l$ ) can be thought of as being the color  $b_j$  (resp.  $g_l$ ).

The encoding  $I$  in the context of the MSNP will be the same except for one alteration: If  $a_{kl}$  is of the form given in (6.3), then in the scanning process we associate to  $a_{kl}$  the sequences  $h_{kl}$  and  $f_{kl}$  each of length  $(m_1 + m_2 + \cdots + m_u + p_1 + p_2 + \cdots + p_v)$  respectively defined by

$$(a) \quad h_{kl} := ll \cdots l$$

$$(b) \quad f_{kl} := \ell_1^{m_1} \ell_2^{m_2} \cdots \ell_u^{m_u} \mathbf{k}_1^{p_1} \mathbf{k}_2^{p_2} \cdots \mathbf{k}_v^{p_v}, \quad (6.4)$$

where a symbol of the form  $\alpha^j$  denotes  $j$  copies of  $\alpha$  and the notation  $\ell_j$  (resp.  $\mathbf{k}_l$ ) signifies an integer  $k$  of color  $b_j$  (resp.  $g_l$ ).

It is left as an exercise to properly extend the lemmas in Sections 2 and 3 and then to prove that the  $q$ -solution of the MSNP is given by the following theorem:

THEOREM 6.1. *The exponential generating function for the polynomial defined in (6.2) is*

$$\sum_{i(r) \geq 0} \frac{M\mathcal{S}(i(r); t, q, B, G) X^{i(r)}}{(t; q)_{n+1}} = \sum_{s \geq 0} t^s \prod_{k=1}^r \frac{(-g_1 x_k; q)_{s+1} (-g_2 x_k; q)_{s+1} \cdots (-g_v x_k; q)_{s+1}}{(b_1 x_k; q)_{s+1} (b_2 x_k; q)_{s+1} \cdots (b_u x_k; q)_{s+1}}, \quad (6.5)$$

where  $n := i_1 + i_2 + \cdots + i_r$ .

The second Simon Newcomb-type problem considered in this section arises when the definition of the cut set is altered in such a way as to bound the entries in the corresponding matrix by fixed positive integers. This problem, referred to as the ASNP, will only be described for a single color. The extension to the multicolored setting is not difficult.

*The ASNP.* Let  $m$  be a fixed positive integer. A deck of specification  $i(r)$  consisting of white cards is to be shuffled and then dealt out into piles; a new pile is begun only with a card  $C$  which is immediately preceded by

- (0) nothing, or
- (1) a card of greater face value, or
- (2) a consecutive run of  $m$  cards all having the same face value as  $C$ .

An occurrence of (1) or (2) is called a cut in the shuffle. The problem is to determine how many shuffles have a given number of cuts.

For the ASNP, the entries in the corresponding matrices are non-negative integers bounded by  $m$  and, thus, the exponential generating function is equal to

$$\sum_{s \geq 0} t^s \prod_{k=1}^r \prod_{l=1}^s \sum_{j=0}^m (x_k q^{s-l})^j \quad (6.6)$$

in which no parameter for color it utilized (or needed).

It is worth noting that when  $m := 1$  that the ASNP is equivalent to the RSNP and (6.6) reduces to the right-hand side of (1.13). Also, the limiting case as  $m \rightarrow \infty$  of the ASNP is just the CSNP.

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