

# Enumeration of Permutations by Descents, Idescents, Imajor Index, and Basic Components

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Multivariable extensions of classic permutation cycle structure results are obtained by counting permutations by descents, idescents, imajor index, and basic components.

## 1. INTRODUCTION

The motivation for this work is that, since enumerating permutations by basic components (for all definitions see Section 2) is equivalent to counting them by cycles (as is demonstrated in Section 3), the new four-variate generating function described by the title provides a multivariable extension of Touchard's [15] exponential formula for permutations by cycles. As corollaries, a number of interesting generalized cycle structure results are presented in Sections 8–11. For instance, recurrence (10.6) of Section 10 simultaneously generalizes the  $q$ -Stirling numbers of the first kind as studied by Gould [11] and the  $q$ -Eulerian numbers of Stanley [14]. Also, extensions of both the derangement problem and the enumeration of involutions are considered in Section 11.

The approach used to derive the four-variate generating function is based on the generalized Worpitzky identity given in [12] and further developed in Section 7 of this paper. Roughly speaking, the Worpitzky identity provides a method for transforming a generating function for finite sequences into what will be referred to as a  $(t, q)$ -generating function for permutations by idescent number and imajor index. In other words, in converting from finite sequences to permutations via the Worpitzky identity, two additional statistics concerning permutations are gained. Consequently, the strategy is to first enumerate finite sequences by descent number and basic components (see

Sections 5 and 6), and then, using the Worpitzky identity, to derive the desired four-variate generating function for permutations (see Section 8).

The inspiration for this paper is to be found in the work of Gessel [9] and [12]. By counting permutations by inversions and basic components, Gessel has obtained a  $q$ -analog of the exponential formula for permutations by cycles. In [12] a slight modification of Foata's bijection [6] was used to demonstrate that, under certain conditions, the  $(t, q)$ -analog obtained by counting idescents and imajor index does in fact generalize the  $q$ -analog that arises when counting inversions. Thus, in this light, it is natural to consider enumerating by idescents and imajor index. As will be seen in Section 9, Gessel's generating function is identical to the one for permutations by imajor index and basic components.

## 2. BASIC DEFINITIONS

Let  $Jr = (j_1, j_2, \dots, j_r)$  be a sequence of nonnegative integers and set  $|Jr| = j_1 + j_2 + \dots + j_r$ . Denote by  $S(Jr)$  the set of sequences  $f = (f(1), f(2), \dots, f(|Jr|))$  of length  $|Jr|$  in which  $k$  occurs  $j_k$  times. It will be convenient to regard a sequence  $f \in S(Jr)$  as the word  $f(1)f(2)\dots f(|Jr|)$  obtained by juxtaposing the letters  $f(1), f(2), \dots, f(|Jr|)$ . Note that when  $|Jr| = n$  and  $j_k = 1$  for  $1 \leq k \leq r$  that  $S(Jr)$  is just the symmetric group on  $\{1, 2, \dots, n\}$  which will be denoted by  $S(n)$ .

The down set of  $f \in S(Jr)$  is defined as

$$\text{down } f = \{i: f(i) > f(i+1), 1 \leq i < |Jr|\}.$$

If  $\#A$  is used to denote the cardinality of a set  $A$ , then the descent number, major index, and inversion number of  $f$  are, respectively,

$$\text{des } f = \# \text{down } f, \quad (2.2a)$$

$$\text{maj } f = \sum_{i \in \text{down } f} i, \quad (2.2b)$$

$$\text{inf } f = \#\{(i, j): 1 \leq i < j \leq |Jr|, f(i) > f(j)\}. \quad (2.2c)$$

Furthermore, in the permutation case, the idescent number and imajor index of  $\sigma \in S(n)$  are defined by

$$\text{ides } \sigma = \text{des } \sigma^{-1}, \quad (2.3a)$$

$$\text{imaj } \sigma = \text{maj } \sigma^{-1}, \quad (2.3b)$$

where  $\sigma^{-1}$  denotes the inverse of  $\sigma$ .

As in Gessel [9], a subword  $f(k)f(k+1)\dots f(k+l-1)$  of  $f \in S(Jr)$  is said to be a basic component of length  $l$  of  $f$  if

$$f(i) \leq f(k) \quad \text{for } 1 \leq i \leq k, \quad (2.4a)$$

$$f(k) > f(i) \quad \text{for } k < i < k+l, \quad (2.4b)$$

$$f(k) \leq f(k+l) \quad \text{if } k+l \leq |Jr|. \quad (2.4c)$$

Condition (c) of (2.4) indicates that either  $f(k+l-1)$  is the last letter of  $f$  or else  $f(k+l)$  is the beginning of another basic component. Thus, a word  $f \in S(Jr)$  factorizes uniquely as a product  $f = f_1 f_2 \dots f_i$  of basic components. Moreover, condition (c) implies that the basic components of  $f$  appear in natural order, that is, the first letter of  $f_j$  is less than or equal to the first letter of  $f_{j+1}$ . For example, the word

$$f = 3 \ 1 \ 4 \ 4 \ 3 \ 2 \ 2 \ 3 \in S(1, 2, 3, 2) \quad (2.5)$$

factorizes as  $f = f_1 f_2 f_3$ , where  $f_1 = 3 \ 1$ ,  $f_2 = 4$ , and  $f_3 = 4 \ 3 \ 2 \ 2 \ 3$ . Incidentally, the notion of a basic component is somewhat similar to that of an upper record as studied by Carlitz and Scoville [4].

The number of basic components of length  $l$  of  $f \in S(Jr)$  will be denoted by  $bc_l f$ . The total number of basic components is defined by

$$bc f = \sum_{l \geq 1} bc_l f. \quad (2.6)$$

In example (2.5),  $bc f = bc_1 f + bc_2 f + bc_5 f = 3$ .

## 3. CYCLE STRUCTURE AND THE BASIC COMPONENT FACTORIZATION

To see how cycles correspond to basic components in the permutation case, consider the permutation

$$\sigma = (4 \ 1)(7)(8 \ 5 \ 2 \ 3 \ 6) \in S(8), \quad (3.1)$$

written in cycle notation according to the two conditions

the initial element of each cycle is also the maximal element of the cycle; (3.2a)

the cycles are ordered so that their initial elements are in natural order. (3.2b)

Removal of the cycle brackets from  $\sigma$  in (3.1) yields a permutation

$$\theta = 4 \ 1 \ 7 \ 8 \ 5 \ 2 \ 3 \ 6 \in S(8), \quad (3.3)$$

written as a word with basic component factorization  $\theta = \theta_1 \theta_2 \theta_3$ , where  $\theta_1 = 4\ 1$ ,  $\theta_2 = 7$ , and  $\theta_3 = 8\ 5\ 2\ 3\ 6$ . Observe that the cycle structure of  $\sigma$  corresponds exactly to the basic component factorization of  $\theta$ . Thus, in a sense, factorizing by basic components is equivalent to imposing an order on cycles. It should be noted that other authors have imposed various orders on cycles. For instance, see [5, 8].

Since the notion of basic components is being used to replace that of an imposed order on cycles, the terms derangement and involution need to be redefined. Classically, a permutation  $\sigma \in S(n)$  is said to be a derangement (resp. involution) if  $\sigma$  has no one cycles (resp. no cycles of length greater than 2). As basic components are more generally defined on sequences, in this paper the sets specified by

$$\begin{aligned} \text{(a)} \quad DS(Jr) &= \{f \in S(Jr) : \text{bc}1f = 0\}, \\ \text{(b)} \quad IS(Jr) &= \{f \in S(Jr) : \text{if } l > 2 \text{ then } \text{bc}lf = 0\}, \end{aligned} \quad (3.4)$$

are respectively referred to as the derangements and involutions of  $S(Jr)$ .

#### 4. PRELIMINARY IDENTITIES

The purpose of this section is to present some  $q$ -calculus identities, define certain basic polynomials, and state the solution to the Simon Newcomb problem which are all needed in the course of the study.

First, the  $q$ -analogue,  $q$ -factorial, and  $q$ -binomial coefficient of a nonnegative integer  $n$  are respectively defined to be

$$[n] = 1 + q + \cdots + q^{n-1}, \quad (4.1a)$$

$$[n]! = [1][2] \cdots [n], \quad (4.1b)$$

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[k]! [n-k]!}. \quad (4.1c)$$

For convenience, set  $(t; q)_{n+1} = (1-t)(1-tq) \cdots (1-tq^n)$ . Note that (see [3, p. 36])

$$(t; q)_{n+1}^{-1} = \sum_{k \geq 0} \begin{bmatrix} n+k \\ k \end{bmatrix} t^k. \quad (4.2)$$

The  $q$ -analogue of the exponential function is defined as

$$e[u] = \sum_{n \geq 0} \frac{u^n}{[n]!}, \quad (4.3)$$

and the infinite product expansion

$$e[u] = \prod_{m \geq 0} (1 - (1-q)uq^m)^{-1} \quad (4.4)$$

for  $|u|, |q| < 1$  may be derived from [3, Corollary (2.2), p. 19]. Finally, the  $q$ -derivative of a function  $f(u)$  as defined by Andrews [2] is

$$Df(u) = \frac{f(u) - f(uq)}{(1-q)u}. \quad (4.5)$$

It is not difficult to show that

$$\text{if } f(u) = \sum_{n \geq 0} \frac{f_n u^n}{[n]!} \quad \text{then } Df(u) = \sum_{n \geq 0} \frac{f_{n+1} u^n}{[n]!}. \quad (4.6)$$

Second, the basic polynomials that arise in the enumeration of sequences and permutations by basic components are defined by

$$A(Jr; s, Z) = \sum_{f \in S(Jr)} s^{\text{des} f} \prod_{l \geq 1} z_l^{\text{bc}l f}, \quad (4.7a)$$

$$A(n; s, t, q, Z) = \sum_{\sigma \in S(n)} s^{\text{des} \sigma} t^{\text{idcs} \sigma} q^{\text{imaj} \sigma} \prod_{l \geq 1} z_l^{\text{bc}l \sigma}, \quad (4.7b)$$

$$D(Jr; s, z) = \sum_{f \in DS(Jr)} a^{\text{des} f} z^{\text{bc} f}, \quad (4.7c)$$

$$D(n; s, t, q, z) = \sum_{\sigma \in DS(n)} s^{\text{des} \sigma} t^{\text{idcs} \sigma} q^{\text{imaj} \sigma} z^{\text{bc} \sigma}, \quad (4.7d)$$

$$I(Jr; s, z) = \sum_{f \in IS(Jr)} s^{\text{des} f} z^{\text{bc}2 f}, \quad (4.7e)$$

$$I(n; s, t, q, z) = \sum_{\sigma \in IS(n)} s^{\text{des} \sigma} t^{\text{idcs} \sigma} q^{\text{imaj} \sigma} z^{\text{bc}2 \sigma}. \quad (4.7f)$$

For convenience, two conventions are made concerning the notation for the basic polynomials. First, in the case  $z_l = z$  for  $l \geq 1$ , the  $Z$  in (4.7a) and (4.7b) will be replaced by  $z$ . For instance,

$$A(Jr; s, z) = \sum_{f \in S(Jr)} s^{\text{des} f} z^{\text{bc} f}. \quad (4.8)$$

Second, a variable that is set equal to one will be entirely removed from an expression. As an example, if  $s = q = 1$  in (4.7b) then

$$A(n; 1, t, 1, Z) = A(n; t, Z). \quad (4.9)$$

In other words, the variables  $s, t, q, z,$  and  $Z$  are respectively associated with the statistics  $\text{des}, \text{ides}, \text{imaj}, \text{bc}$  ( $\text{bc2}$  in the case of involutions), and  $\text{bcl}$ .

Finally, the solution to the classic Simon Newcomb problem [1, 3, 13], that is, the generating function for sequences by decents is

$$\sum_{Jr \geq 0} A(Jr; s) X^{Jr} = (1-s) \left[ -s + \prod_{i=1}^r (1 - (1-s)x_i) \right]^{-1}, \quad (4.10)$$

where  $Jr \geq 0$  means that  $j_k \geq 0$  for  $1 \leq k \leq r$  and, as throughout the paper,

$$X^{Jr} = \prod_{m=1}^r x_m^{j_m}. \quad (4.11)$$

## 5. THE MAIN RESULT

The generating function for sequences by descents and basic components is given by

$$\begin{aligned} \sum_{Jr \geq 0} A(Jr; s, Z) X^{Jr} & \quad (5.1) \\ &= \prod_{m=1}^r \left[ 1 - x_m \left( z_1 + s \sum_{J(m-1) > 0} A(J(m-1); s) X^{J(m-1)} z_{|J(m-1)|+1} \right) \right]^{-1}. \end{aligned}$$

As a rough sketch of the proof, first, observe that if a sequence  $f \in S(Jr)$  has the basic component factorization  $f = f_1 f_2 \cdots f_i$  with

$$l(k) = \text{length of } f_k, \quad (5.2a)$$

$$d(k) = \text{des } f_k, \quad (5.2b)$$

$$X(f) = x_{f(1)} x_{f(2)} \cdots \quad (5.2c)$$

then

$$s^{\text{des } f} X(f) \prod_{i \geq 1} z_i^{\text{bcl } f} = \prod_{k=1}^i s^{d(k)} X(f_k) z_{l(k)}. \quad (5.3)$$

In other words, the contribution of a sequence  $f$  to the left-hand side of (5.1) is equal to the product of the contributions of the basic components of  $f$ . For example, if  $f$  is the sequence given in (2.5) then (5.3) becomes

$$s^3 x_1 x_2^2 x_3^3 x_4^2 z_1 z_2 z_5 = (s x_1 x_3 z_2)(x_4 z_1)(s^2 x_2^2 x_3^2 x_4 z_5). \quad (5.4)$$

Consequently, the strategy will be to first consider the contribution of basic components with initial letter  $m$  and then to multiply the contributions for  $1 \leq m \leq r$ .

A basic component that begins with  $m$  must be of one of the two forms

$$m, \quad (5.5a)$$

$$mf, \quad (5.5b)$$

where  $mf$  is the juxtaposition product of the letter  $m$  with a nonempty sequence  $f$  whose letters are less than  $m$ . The contribution of a basic component of form (a) is

$$x_m z_1. \quad (5.6)$$

The total contribution of the basic components of form (5.5b) is

$$x_m s \sum_{J(m-1) > 0} A(J(m-1); s) X^{J(m-1)} z_{|J(m-1)|+1}, \quad (5.7)$$

since the sequences  $f$  are nonempty and there is a descent between  $m$  and  $f$ . In view of (5.6) and (5.7), the total contribution of basic components that begin with  $m$  is

$$\begin{aligned} \sum_{k \geq 0} \left[ x_m \left( z_1 + s \sum_{J(m-1) > 0} A(J(m-1); s) X^{J(m-1)} z_{|J(m-1)|+1} \right) \right]^k \\ = \left[ 1 - x_m \left( z_1 + s \sum_{J(m-1) > 0} A(J(m-1); s) X^{J(m-1)} z_{|J(m-1)|+1} \right) \right]^{-1}. \end{aligned} \quad (5.8)$$

As the basic components of a sequence in  $S(Jr)$  may have initial letter  $m$  only for  $1 \leq m \leq r$ , multiplying the contributions of (5.8) for  $1 \leq m \leq r$  yields the right-hand side of (5.1)

## 6. COROLLARIES OF IDENTITY (5.1)

The four corollaries of (5.1) that are derived here will be used, instead of using (5.1) directly, to obtain  $(t, q)$ -generating functions for permutations in Section 8.

First, if  $s = 1$  then  $A(J(m-1); s)$  is equal to the multinomial coefficient

$$\binom{|J(m-1)|}{j_1 j_2 \cdots j_{m-1}}, \quad (6.1)$$

and it follows that (5.1) reduces to

$$\sum_{Jr \geq 0} A(Jr; Z) X^{Jr} = \prod_{m=1}^r \left[ 1 - x_m \sum_{k \geq 0} (x_1 + x_2 + \cdots + x_{m-1})^k z_{k+1} \right]^{-1}, \quad (6.2)$$

## 7. THE WORPITZKY IDENTITY

which is the generating function for sequences by basic component structure. As a second corollary, when  $z_l = z$  for  $l \geq 1$  the solution to the Simon Newcomb problem (4.10) may be used to show that the right-hand side of (5.1) becomes

$$\prod_{m=1}^r \left[ 1 - x_m z \left( 1 - s + s(1-s) \left[ -s + \prod_{i=1}^{m-1} (1 - (1-s)x_i) \right]^{-1} \right) \right]^{-1}. \quad (6.3)$$

By manipulating the expression in (6.3), it may be verified that (5.1) reduces to

$$\sum_{Jr \geq 0} A(Jr; s, z) X^{Jr} = \prod_{m=1}^r \left[ 1 - \frac{(1-s)x_m z}{1-s \prod(m)} \right]^{-1}, \quad (6.4)$$

where

$$\prod(m) = \prod_{i=1}^{m-1} (1 - (1-s)x_i)^{-1}. \quad (6.5)$$

In order to obtain a generating function for derangements there are two restrictions that must be placed on the right-hand side of (5.1). First, as no basic components of a derangement have length one,  $z_1 = 0$ . Second, because no basic component of a derangement can begin with a one, the product must be restricted to  $2 \leq m \leq r$ . Then, setting  $z_l = z$  for  $l \geq 2$ , identity (5.1) reduces in the derangement case to

$$\sum_{Jr \geq 0} D(Jr; s, z) X^{Jr} = \prod_{m=2}^r \left[ 1 - x_m s z \sum_{J(m-1) > 0} A(J(m-1); s) X^{J(m-1)} \right]^{-1}. \quad (6.6)$$

With the aid of (4.10), identity (6.6) may be rewritten as

$$\sum_{Jr \geq 0} D(Jr; s, z) X^{Jr} = \prod_{m=2}^r \left[ 1 - x_m s z \frac{\prod(m) - 1}{1 - s \prod(m)} \right]^{-1}, \quad (6.7)$$

where  $\prod(m)$  is defined in (6.5).

Finally, to obtain the generating function for involutions from (5.1) first set  $z_1 = 1$ ,  $z_2 = z$ , and  $z_l = 0$  for  $l \geq 3$ . Then restricting (5.1) to involutions yields

$$\sum_{Jr \geq 0} I(Jr; s, z) X^{Jr} = \prod_{m=1}^r \left[ 1 - x_m \left( 1 + s z \sum_{i=1}^{m-1} x_i \right) \right]^{-1}. \quad (6.8)$$

As mentioned in the introduction, the method for transforming the generating functions for sequences of Section 6 into  $(t, q)$ -generating functions for permutations is based on a generalization of the identity

$$r^n = \sum_{k=0}^n \binom{r-1-k+n}{n} A(n, k), \quad (7.1)$$

which is known as the Worpitzky identity for the Eulerian numbers  $A(n, k) = \#\{\sigma \in S(n) : \text{idcs } \sigma = k\}$ . The statement of the generalization of (7.1) necessitates a brief discussion of the combinatorial proof of the Worpitzky identity.

As described in [12], the bijective proof of (7.1) consists of, first, interpreting  $r^n$  as the cardinality of the set defined by

$$(r)^n = \bigcup_{|Jr|=n} S(Jr) \quad (7.2)$$

and, then, associating to each  $f \in (r)^n$  an ordered pair  $\Gamma(f) = (g, \sigma)$  where  $g$  is the nondecreasing rearrangement of  $f$  and  $\sigma \in S(n)$  tells how to reconstruct  $f$  from  $g$ . The pairs  $(g, \sigma)$  give rise to the right-hand side of (7.1):  $\binom{r-1-k+n}{n}$  is the number of nondecreasing sequences  $g$  that may be paired with a permutation of the set  $\{\sigma \in S(n) : \text{idcs } \sigma = k\}$ . The two features of  $\Gamma$  that are essential for the present discussion are

the map  $\Gamma$  is a bijection of  $(r)^n$  onto the set  $\{(g, \sigma) : \sigma \in S(n), g(i) < g(i+1) \text{ whenever } i \in \text{down } \sigma^{-1}\}$ , (7.3a)

if  $\Gamma(f) = (g, \sigma)$  and  $k < l$ , then  $f(k) > f(l)$  if and only if  $\sigma(k) > \sigma(l)$ . (7.3b)

For a complete description of  $\Gamma$  and details on (7.3) see [12].

The interesting fact about  $\Gamma$  is that much more information is preserved between sequences and permutations than is needed to prove (7.1). To make this precise, a real-valued function  $v$  defined on words with integer letters is said to be preserved by  $\Gamma$  if  $\Gamma(f) = (g, \sigma)$  implies  $v(f) = v(\sigma)$ . In particular, (7.3b) implies that

the functions  $\text{des}$ ,  $\text{maj}$ ,  $\text{inv}$ ,  $\text{bc}$ , and  $\text{bcl}$  for  $l \geq 1$  are all preserved by  $\Gamma$ . (7.4)

The generalization of (7.1) as developed in [12] may now be stated. Let  $\{v_1, v_2, v_3, \dots\}$  be a set of functions preserved by  $\Gamma$  and set

$$A(Jr; Y) = \sum_{f \in S(Jr)} \prod_{l \geq 1} y_l^{v_l(f)}, \quad (7.5a)$$

$$A(n; t, q, Y) = \sum_{\sigma \in S(n)} t^{\text{idess}\sigma} q^{\text{imaj}\sigma} \prod_{l \geq 1} y_l^{v_l(\sigma)}, \quad (7.5b)$$

$$A(n, k; q, Y) = \sum_{\sigma} q^{\text{imaj}\sigma} \prod_{l \geq 1} y_l^{v_l(\sigma)}, \quad (7.5c)$$

where the summation in part (c) is over  $\{\sigma \in S(n): \text{idess}\sigma = k\}$ . Then, the generalized Worpitzky identity is

$$\sum_{k=0}^n \begin{bmatrix} r-1-k+n \\ n \end{bmatrix} A(n, k; q, Y) = \sum_{|Jr|=n} q^{r \cdot Jr} A(Jr; Y), \quad (7.6)$$

where  $r \cdot Jr = (r-1)j_1 + (r-2)j_2 + \dots + j_{r-1}$ .

For practical use, with the aid of identity (4.2) and by replacing  $r$  with  $r+1$ , Eq. (7.6) is rewritten in the form

$$\sum_{n \geq 0} \frac{A(n; t, q, Y) u^n}{(t; q)_{n+1}} = \sum_{r \geq 0} t^r \sum_{J(r+1) \geq 0} A(J(r+1); Y) \prod_{m=1}^{r+1} (uq^{r-m+1})^{j_m}. \quad (7.7)$$

Furthermore, because  $\Gamma$  preserve basic components, (7.7) holds as well in both the derangement and involution cases.

## 8. APPLICATIONS TO PERMUTATION ENUMERATION

Equation (7.7) is now used to convert the corollaries of Section 6 into  $(t, q)$ -generating functions for permutations. Replacing  $r$  by  $r+1$  and  $x_m$  by  $uq^{r-m+1}$  in identities (6.2), (6.4), (6.7), and (6.8) and then applying (7.7), respectively, yields

$$\sum_{n \geq 0} \frac{A(n; t, q, Z) u^n}{(t; q)_{n+1}} = \sum_{r \geq 0} t^r \prod_{m=0}^r \left[ 1 - uq^m \sum_{k \geq 0} [r-m]^k (uq^{m+1})^k z_{k+1} \right]^{-1}, \quad (8.1a)$$

$$\sum_{n \geq 0} \frac{A(n; s, t, q, z) u^n}{(t; q)_{n+1}} = \sum_{r \geq 0} t^r \prod_{m=0}^r \left[ 1 - \frac{(1-s)uq^m z}{1-s \prod(m, r)} \right]^{-1} \quad (8.1b)$$

$$\sum_{n \geq 0} \frac{D(n; s, t, q, z) u^n}{(t; q)_{n+1}} = \sum_{r \geq 0} t^r \prod_{m=0}^{r-1} \left[ 1 - uq^m s z \left( \frac{\prod(m, r) - 1}{1-s \prod(m, r)} \right) \right]^{-1}, \quad (8.1c)$$

$$\sum_{n \geq 0} \frac{I(n; s, t, q, z) u^n}{(t; q)_{n+1}} = \sum_{r \geq 0} t^r \prod_{m=0}^r [1 - uq^m (1 + szuq^{m+1} [r-m])]^{-1}, \quad (8.1d)$$

where  $[r-m] = 1 + q + \dots + q^{r-m-1}$  and

$$\prod(m, r) = \prod_{i=0}^{r-m-1} (1 - (1-s)uq^{m+1+i})^{-1}. \quad (8.2)$$

Each generating function given in (8.1) deserves further comment. This is the purpose of the remaining three sections.

## 9. COMPARISON WITH GESSEL'S RESULTS

Multiplying Eq. (8.1a) by  $(1-t)$  and then setting  $t=1$  leads to

$$\sum_{n \geq 0} \frac{A(n; q, Z) u^n}{(q; q)_n} = \lim_{r \rightarrow \infty} \prod_{m=0}^r \left[ 1 - uq^m \sum_{k \geq 0} [r-m]^k (uq^{m+1})^k z_{k+1} \right]^{-1}, \quad (9.1)$$

which is the generating function for permutations by basic components and imajor index. Since  $[k] = (1-q^k)(1-q)^{-1}$  and  $[r-m]$  approaches  $(1-q)^{-1}$  as  $r$  goes to  $\infty$ , substitution of  $u$  by  $(1-q)u$  in (9.1) yields

$$\sum_{n \geq 0} \frac{A(n; q, Z) u^n}{[n]!} = \prod_{m \geq 0} \left[ 1 - (1-q)uq^m \sum_{k \geq 0} (uq^{m+1})^k z_{k+1} \right]^{-1}. \quad (9.2)$$

Identity (9.2) is identical to the generating function for permutations by basic components and inversions described in Corollary (5.3) of Gessel's work [9]. Therefore,

$$\sum_{\sigma \in S(n)} q^{\text{imaj}\sigma} \prod_{l \geq 1} z_l^{\text{bcl}\sigma} = \sum_{\sigma \in S(n)} q^{\text{inv}\sigma} \prod_{l \geq 1} z_l^{\text{bcl}\sigma}. \quad (9.3)$$

Actually, there is a much stronger result than (9.3) that holds. Namely, there exists a bijection  $B: S(n) \rightarrow S(n)$  that satisfies

$$\text{imaj}\sigma = \text{inv} B(\sigma) \quad (9.4a)$$

$$\text{down}\sigma = \text{down} B(\sigma) \quad (9.4b)$$

$$\text{bcl}\sigma = \text{bcl} B(\sigma) \quad (9.4c)$$

for all  $\sigma \in S(n)$ . The map  $B$  is described in the proof of Theorem (9.1) of [12]. The verification of properties (9.4), which depends on an analysis of Foata's bijection [6], is left to the interested reader.

10. COROLLARIES OF EQUATION (b) OF (8.1)

A trivariate generating function for  $A(n; s, t, q)$  and a recurrence for  $A(n; s, q, z)$  are now derived.

First, the generating function for  $A(n; s, t, q)$  will provide a partial check of the present work with the paper of Garsia and Gessel [7]. Setting  $z = 1$  and replacing  $u$  by  $u(1-s)^{-1}$  in (8.1b) yields

$$\sum_{n \geq 0} \frac{A(n; s, t, q) u^n}{(1-s)^n (t; q)_{n+1}} = \sum_{r \geq 0} t^r \prod_{m=0}^r \frac{(uq^{m+1}; q)_{r-m-s}}{(uq^m; q)_{r-m+1-s}}. \quad (10.1)$$

Since the product on the right-hand side of (10.1) telescopes down to  $(1-s)[(u; q)_{r+1} - s]^{-1}$ , identity (10.1) simplifies to

$$\sum_{n \geq 0} \frac{A(n; s, t, q) u^n}{(1-s)^{n+1} (t; q)_{n+1}} = \sum_{k, r \geq 0} s^k t^r (u; q)_{r+1}^{-k-1}. \quad (10.2)$$

As a check, the reader may derive (10.2) from Garsia and Gessel's generating function for permutations by descents, idescents, major index, and imajor index.

The recurrence for  $A(n; s, q, z)$  is derived as follows. Multiplying Eq. (8.1b) by  $(1-t)$ , then setting  $t = 1$ , replacing  $u$  by  $(1-q)u$ , and making use of (4.4) leads to

$$\sum_{n \geq 0} \frac{A(n; s, q, z) u^n}{[n]!} = \prod_{m \geq 0} \frac{1 - se[(1-s)uq^{m+1}]}{1 - (1-s)(1-q)uq^m z - se[(1-s)uq^{m+1}]}. \quad (10.3)$$

Letting  $A(u)$  denote the left-hand side of (10.3), it follows that

$$(1 - (1-s)(1-q)uz - se[(1-s)uq]) A(u) = (1 - se[(1-s)uq]) A(uq). \quad (10.4)$$

Identity (10.4) may be rewritten as

$$(1 - se[(1-s)uq]) \frac{A(u) - A(uq)}{(1-q)u} = (1-s)zA(u). \quad (10.5)$$

Making use of (4.6) and then equating coefficients of  $u^n$  in (10.5) yields the recurrence

$$A(n+1; s, q, z) = zA(n; s, q, z) + sq \sum_{k=1}^n \begin{bmatrix} n \\ k-1 \end{bmatrix} q^{n-k} (1-s)^{n-k} A(k; s, q, z), \quad (10.6)$$

where  $A(0; s, q, z) = 1$ .

There are two interesting special cases of (10.6). In the case  $s = 1$ , recurrence (10.6) becomes

$$A(n+1; q, z) = (z + q[n]) A(n; q, z). \quad (10.7)$$

Iteration of (10.7) yields the  $q$ -Stirling numbers of the first kind

$$A(n+1; q, z) = z \prod_{k=1}^n (z + q[k]) \quad (10.8)$$

as studied by Gould [11]. Of course, in the case  $q = 1$  Eq. (10.8) reduces to the classic identity for the Stirling numbers. As a second special case of (10.6), the generating function for  $A(n; s, q)$  may be obtained as follows. It is not difficult to verify that the  $q$ -derivative of the function

$$F(u) = \frac{(1-s)e[(1-s)u]}{1 - se[(1-s)u]} \quad (10.9)$$

is given by

$$DF(u) = \frac{(1-s)}{1 - se[(1-s)uq]} F(u). \quad (10.10)$$

With  $z = 1$ , one may conclude from (10.5) that the coefficients of  $u^n$  in the power series expansion of  $F(u)$  satisfy recurrence (10.6). Consequently,

$$\sum_{n \geq 0} \frac{A(n; s, q) u^n}{[n]!} = \frac{(1-s)e[(1-s)u]}{1 - se[(1-s)u]}. \quad (10.11)$$

It should be remarked that, in counting permutations by descents and inversions, Stanley [14] obtained a generating function for the  $q$ -Eulerian numbers that is identical to (10.11). This is not surprising in view of (9.4).

11. DERANGEMENTS AND INVOLUTIONS

Both of the recurrences

$$D(n+1; s, q, z) = s \sum_{k=2}^n \begin{bmatrix} n \\ k-1 \end{bmatrix} q^{n-k+1} (1-s)^{n-k} D(k; s, q, z) + sz \sum_{k=0}^{n-1} \begin{bmatrix} n \\ k \end{bmatrix} q^{n-k} (1-s)^{n-k-1} D(k; s, q, z), \quad (11.1a)$$

$$I(n+1; s, q, z) = I(n; s, q, z) + sq[n] I(n-1; s, q, z), \quad (11.1b)$$

where  $D(0; s, q, z) = I(0; s, q, z) = 1$  may be respectively derived from (c) and (d) of (8.1) in exactly the same way (10.6) was obtained from (b) of (8.1). Recurrence (11.1b) already closely resembles the classic recurrence for involutions. To obtain something that looks like the classic recurrence for derangements from (11.1a) set  $s = 1$ . Then (11.1a) reduces to

$$D(n+1; q, z) = q[n] D(n; q, z) + zq[n] D(n-1; q, z). \quad (11.2)$$

Equation (11.2) is somewhat similar to the recurrence for the  $q$ -derangement problem solved by Garsia and Remmel [8]. Although Garsia and Remmel obtained a  $q$ -analog by counting by inversions instead of imajor index, the real difference lies in the fact that they imposed a different order on the cycles of the permutation.

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