A characterisation of Newton maps

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Abstract

Conditions are given for a $C^k$ map $T$ to be a Newton map, that is, the map associated with a differentiable real-valued function via Newton’s method. For finitely differentiable maps and functions, these conditions are only necessary, but in the smooth case, i.e. for $k = \infty$, they are also sufficient. The characterisation rests upon the structure of the fixed point set of $T$ and the value of the derivative $T'$ there, and it is best possible as is demonstrated through examples.

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1 Introduction

Newton’s method (NM) for computing successive approximations of zeros of functions is one of the most widely used methods in all of applied mathematics; variants and generalisations also play a prominent role in numerous other disciplines [2, 3, 8, 10, 11]. Conceptually, NM becomes especially transparent within a dynamical systems context. The purpose of this brief note
is to characterise, in the simplest possible setting, the local properties of the dynamical systems thus encountered.

Throughout, let \( f : I \to \mathbb{R} \) be a differentiable function, defined on some open interval \( I \subset \mathbb{R} \), and denote by \( N_f \) its associated \( \text{Nm} \) transformation, that is
\[
N_f(x) = x - \frac{f(x)}{f'(x)}, \quad \forall x \in I : f'(x) \neq 0;
\]
for \( N_f \) to be defined for every \( x \in I \), set \( N_f(x) := x \) whenever \( f'(x) = 0 \).

\( \text{Nm} \) for finding roots (zeros) of \( f \), i.e., real numbers \( x^* \) with \( f(x^*) = 0 \), amounts to picking an initial point \( x_0 \in I \) and iterating \( N_f \), thus generating the sequence
\[
x_n = N_f(x_{n-1}) = N_f^n(x_0), \quad \forall n \in \mathbb{N},
\]
where, here and throughout, for any map \( T : I \to \mathbb{R} \) and any \( n \in \mathbb{N} \), \( T^n(x) = T(T^{n-1}(x)) \), provided that \( T^{n-1}(x) \in I \), and \( T^0(x) = x \). Note that \( N_f(x) = x \) precisely if \( f(x)f'(x) = 0 \); that is, the only fixed points of \( N_f \) occur where either \( f \) or \( f' \) vanish. Thus for \( f(x_n)f'(x_n) = 0 \), and only then, does \( \text{Nm} \) terminate at \( x_n \): If \( f(x_n) = 0 \), a root has been found, and otherwise (1) breaks down due to a horizontal tangent to the graph of \( f \) at \( x_n \) (see Figure 1).

Clearly, if \( (x_n) \) converges to \( x^* \), say, and if \( N_f \) is continuous at \( x^* \), then \( N_f(x^*) = x^* \), i.e., \( x^* \) is a fixed point of \( N_f \), and \( f(x^*) = 0 \). (The trivial alternative \( f \equiv \text{const.} \) is tacitly excluded here, see Lemma 4 below.) It is this correspondence between the roots of \( f \) and the fixed points of \( N_f \) that suggests that \( \text{Nm} \) be studied as a dynamical system. Under a mild assumption, each (isolated) fixed point \( x^* \) is attracting, that is, \( \lim_{n \to \infty} N_f^n(x_0) = x^* \) for all \( x_0 \) sufficiently close to \( x^* \). (For \( x_0 \) further away from any root, the sequence \( (x_n) \) may exhibit a considerably more complicated long-term behaviour \([2, 3, 11]\).) This aspect of \( \text{Nm} \) is put into perspective by the main result of the present note, Theorem 11 below, which completely characterises the local dynamical properties of \( N_f \).

## 2 Newton maps

The definition of a Newton map given below entails a relationship between the analytic properties of a function \( f \) and the analytic properties of its associated \( \text{Nm} \) transformation \( N_f \). It is a simple fact, rarely alluded to in studies of \( \text{Nm} \), that in general these properties are quite independent.

**Example 1.** The function \( f(x) = |x|^{3/2} \) is \( C^1 \) but not \( C^2 \), yet it has a \( C^\infty \) \( \text{Nm} \) transformation, namely \( N_f(x) = \frac{1}{3} x \).
Figure 1: Visualising \( N_m \): The first few iterates \( x_1, x_2, x_3 \) are found graphically, both by means of tangents to the graph of \( f \) (broken line) and via the graph of \( N_f \) (solid line). Note how the point \( x_2^* \) with \( f'(x_2^*) = 0 \) causes \( N_f \) to have a discontinuity.

**Example 2.** It is easily seen that the function

\[
f(x) = \begin{cases} 
\exp(-x^{-2} + |x| + \cos(x^{-2})) & \text{if } x \neq 0, \\
0 & \text{if } x = 0,
\end{cases}
\]

is \( C^\infty \), and both \( f \) and \( f' \) vanish only at \( x^* = 0 \). Nevertheless

\[-1 = \lim \inf_{x \to 0} N_f(x) < \lim \sup_{x \to 0} N_f(x) = 1,
\]

hence \( N_f \) is not even *continuous* at \( x^* \).

Since \( N_f \) may fail to be continuous even if \( f \) is \( C^\infty \), in order to ensure the applicability of \( N_m \), some explicit assumption on the smoothness of \( N_f \) has to be imposed. To formulate such conditions concisely, let \( \mathbb{N}_\infty = \mathbb{N} \cup \{\infty\} \) and stipulate that \( \infty^{-1} := 0 \) and \( \infty \pm j = \infty \) for all \( j \in \mathbb{N} \).

In view of (1), for \( N_f \) to be \( C^l \) for some \( l \in \mathbb{N}_\infty \), one might demand that \( f \) be at least \( C^{l+1} \), but Examples 1 and 2 show that this assumption is neither necessary nor sufficient. Simply imposing further conditions on \( N_f \) also seems problematic as long as it is not clear whether any such condition is satisfied for a reasonably large class of functions. Thus it is inevitable to
address the following general inverse problem: Given a $C^l$ map $T$, does there exist a function $f$ such that $T = N_f$?

**Definition 3.** Let $I \subset \mathbb{R}$ be an open interval, and $l \in \mathbb{N}_\infty$. A map $T \in C^l(I)$ is called a *Newton map* (associated with $f$), if $T = N_f$ for some differentiable function $f : I \to \mathbb{R}$.

Clearly, not every $T \in C^l(I)$ is a Newton map, even if $l = \infty$, as the trivial example $T(x) = -x$ shows, for which every $f$ with $N_f = T$ lacks differentiability at $x^* = 0$. As will become clear shortly, most maps are not Newton, but a satisfactory characterisation is not available for finitely differentiable maps. However, in the smooth case, i.e. for $l = \infty$, there is a simple characterisation of Newton maps, as provided by Theorem 11 below.

For any map $T$, denote by $\text{Fix}[T]$ the set of fixed points of $T$, that is, $\text{Fix}[T] := \{x \in I : T(x) = x\}$, and say that $\text{Fix}[T]$ is *attracting* if $\lim_{n \to \infty} T^n(x_0) \in \text{Fix}[T]$ for all $x_0$ sufficiently close to $\text{Fix}[T]$.

**Lemma 4.** Let $f : I \to \mathbb{R}$ be differentiable, and assume that $N_f$ is continuous. Then $\text{Fix}[N_f]$ is either empty or a (possibly one-point) interval; in the latter case,

$$\limsup_{x \to x^*} \frac{N_f(x) - x^*}{x - x^*} = \delta \text{ for some } \delta \in [0, 1] \quad (2)$$

holds for every $x^* \in \text{Fix}[N_f]$.

**Proof.** It will first be shown that both sets $Z_0 := \{x \in I : f(x) = 0\}$ and $Z_1 := \{x \in I : f'(x) = 0\}$ of zeros of $f$ and $f'$, respectively, are (possibly empty or one-point) subintervals of $I$. Moreover, if $Z_1 \neq I$, that is, if $f$ is not constant, then $Z_1 \subset Z_0$; in fact, the two sets coincide unless $Z_0$ contains exactly one point, in which case $Z_1$ may be empty. Since $\text{Fix}[N_f] = Z_0 \cup Z_1$ the first part of the lemma follows immediately from this.

If $Z_1 = I$, then $\text{Fix}[N_f] = I$, so let $Z_1 \neq \emptyset$ be different from $I$. Pick $a \in Z_1$, suppose, by way of contradiction, $f(a) \neq 0$ and, without loss of generality, that $b := \sup\{x \geq a : f(y) = f(a) \text{ for all } y \in [a, x]\}$ belongs to $I$. Clearly, $f(b) = f(a)$ and $f'(b) = 0$, hence $N_f(b) = b$. By the Mean Value Theorem there exists a sequence $b_n \downarrow b$ such that $0 < |f'(b_n)| \leq 1$ for all $n$. But then

$$\liminf_{n \to \infty} |N_f(b_n) - b| \geq \lim_{n \to \infty} |f(b_n)| = |f(b)| = |f(a)| > 0,$$

clearly contradicting the continuity of $N_f$. Therefore $f(a) = 0$, hence $Z_1 \subset Z_0$. If $a_1 < a_2$ both belong to $Z_0$ then, by the previous argument and the
Mean Value Theorem, $Z_0$ contains a point strictly between $a_1$ and $a_2$. Since $Z_0$ is closed, it contains, with any two points, the whole segment joining these points. Thus $Z_0$ is an interval. If $Z_0$ is not a singleton then $Z_0 \subset Z_1$ and therefore $Z_0 = Z_1$. The latter equality also holds if $Z_0$ is one-point because $Z_1 \neq \emptyset$. Finally, if $Z_1$ is empty then clearly $Z_0$ cannot contain more than one point.

Assertion (2) is trivially true if $x^*$ is an interior point of $\text{Fix} [N_f]$. Without loss of generality therefore assume that $x^*$ is, say, a right boundary point of $\text{Fix} [N_f] = Z_0$. Choose $\delta > 0$ so small that $J := [x^*, x^* + \delta] \subset I$ and, for $0 < t \leq \delta$, let

$$h(t) := \frac{N_f(x^* + t) - x^*}{t};$$

(3)

the function $h$ is continuous on $[0, \delta]$, and $h(t) \neq 1$ for all $t > 0$. Since $x \neq N_f(x)$ for $x \in J$,

$$\frac{f'(x)}{f(x)} = \frac{1}{x - N_f(x)}, \quad \forall x \in J,$$

which after integrating both sides from $x$ to $x^* + \delta$, and using the auxiliary function $h$ defined in (3), can be written as

$$f(x) = f(x^* + \delta) \exp \left( - \int_{x-x^*}^{\delta} \frac{1}{1 - h(t)} \, dt \right), \quad \forall x \in J. \quad (4)$$

Assume $f(x^* + \delta) > 0$ without loss of generality. If $h(t) > 1$ for all $t > 0$, then (4) implies that $f(x^*) \neq 0$, contradicting $x^* \in Z_0$. Thus $h(t) < 1$ for all $t > 0$, and in particular

$$\lim \sup_{t \downarrow 0} h(t) = \lim \sup_{x \neq x^*} \frac{N_f(x) - x^*}{x - x^*} \leq 1.$$ 

Fix $j \in \mathbb{N}$. Dividing (4) by $(x - x^*)^j = \delta^j \exp \left( - j \int_{x-x^*}^{\delta} t^{-1} \, dt \right)$ yields

$$(x - x^*)^{-j} f(x) = f(x^* + \delta) \delta^{-j} \exp \left( \int_{x-x^*}^{\delta} \frac{j - 1 - jh(t)}{1 - h(t)} \, dt \right), \quad \forall x \in J. \quad (5)$$

To bound $\lim \sup_{t \downarrow 0} h(t)$ from below, pick $\varepsilon > 0$ and assume that $h(t) < -\varepsilon$ for all sufficiently small $t > 0$. In this case, (5) with $j = 1$ shows that

$$(x - x^*)^{-1} f(x) \geq f(x^* + \delta) \delta^{-1} \exp \left( (1+\varepsilon)^{-1} (x - x^*)^{-\varepsilon(1+\varepsilon)^{-1}} \right) \to \infty, \quad \text{as } x \searrow x^*,$$

which contradicts the differentiability of $f$ at $x^*$. Since $\varepsilon > 0$ was arbitrary, $\lim \sup_{t \downarrow 0} h(t) \geq 0$. □
Remark 5. (i) Lemma 4 should be contrasted with the simple fact that for every closed set \( A \subset \mathbb{R} \) there exists a \( C^\infty \) map \( T \) with \( T(I) \subset I \) and \( \text{Fix}[T] = A \cap I \).

(ii) Under the conditions of Lemma 4 there is no analogue to (2) for the corresponding \( \liminf \) which, as simple examples show, can be any number between, and including, the trivial bounds \(-\infty\) and \( \delta \).

As pointed out earlier, the applicability of \( \text{Nm} \) rests on the correspondence between the roots of \( f \) and the fixed points of \( N_f \) — and the attractiveness of the latter. Mere continuity of \( N_f \) does not guarantee that \( \text{Fix}[N_f] \) is attracting.

Example 6. Consider the \( C^1 \) function
\[
f(x) = \begin{cases} |x|^{3/2} \exp\left(-\int_0^{|x|-1} t^{-1} \sin t \, dt\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0,
\end{cases}
\]
for which the associated \( \text{Nm} \) transformation
\[
N_f(x) = \begin{cases} \frac{1 + 2 \sin(|x|^{-1})}{3 + 2 \sin(|x|^{-1})} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0,
\end{cases}
\]
is continuous yet obviously not \( C^1 \). The only fixed point of \( N_f \), and correspondingly the only root of \( f \) and \( f' \), is \( x^* = 0 \). Since, for every \( j \in \mathbb{N} \), the points \( \pm \frac{2}{\pi}(4j - 1)^{-1} \) are 2-periodic, \( \text{Fix}[N_f] = \{0\} \) is not attracting.

Thus while \( \text{Fix}[N_f] \) is topologically simple whenever \( N_f \) is continuous, to make \( \text{Nm} \) practical for approximating zeros, more smoothness is required. Only the case of \( N_f \) being at least \( C^1 \) will therefore be considered from now on. (For the same reason, the legitimate case \( l = 0 \) has been excluded from Definition 3.) Also, the properties of \( N_f' \), albeit not completely determined by the smoothness of \( f \), do depend on the latter. To describe this dependence, for every \( k \in \mathbb{N}_\infty \), define the set
\[
\Delta_k := \left\{ 0, \frac{1}{2}, \frac{2}{3}, \ldots, 1 - k^{-1} \right\} \cup \left[ 1 - k^{-1}, 1 \right],
\]
and note that \( [0, 1] = \Delta_1 \supset \Delta_2 \supset \cdots \supset \Delta_\infty = \{1 - j^{-1} : j \in \mathbb{N}_\infty\} \).

Lemma 7. Let \( f : I \to \mathbb{R} \) be differentiable, and assume that \( N_f \in C^1(I) \). Then \( \text{Fix}[N_f] \) is either empty or an attracting (possibly one-point) interval. Moreover, if \( \text{Fix}[N_f] \neq \emptyset \) and \( f \in C^k(I) \) with \( k \in \mathbb{N}_\infty \) then
\[
N_f'\left(\text{Fix}[N_f]\right) = \{\delta\} \quad \text{for some } \delta \in \Delta_k.
\]
Proof. The assertions are trivially true if \( f \) is constant or \( \text{Fix}[N_f] = \emptyset \). Therefore assume that \( f \) is not constant and \( \text{Fix}[N_f] \) is not empty, hence a subinterval of \( I \), by Lemma 4. If \( x^* \) is an interior point of \( \text{Fix}[N_f] \) then \( N'_f( x^* ) \equiv 1 \) in a neighbourhood of \( x^* \), and the assertion is again true. Thus assume without loss of generality that \( x^* \) is a right boundary point of \( \text{Fix}[N_f] \). By Lemma 4, \( N'_f(x^*) \in \Delta_1 \), so \( x^* \) obviously is attracting from the right, unless perhaps for \( N'_f(x^*) = 1 \). In the latter case, with the notations introduced in the proof of Lemma 4, the function \( h \) defined in (3), supplemented by \( h(0) := \left. N'_f(x^*) \right|_{0} = 1 \), is continuous on \([0, \delta] \) and can be written as \( h(t) = 1 - H(t) \), where \( H \) is also continuous on \([0, \delta] \), and \( H(t) \neq 0 \) unless \( t = 0 \). With this, (4) takes the form

\[
f(x) = f(x^* + \delta) \exp \left( - \int_{x-x^*}^{x} \frac{dt}{tH(t)} \right), \quad \forall \; x \in J.
\]

Since \( f(x^*) = 0 \) and \( f(x^* + \delta) \neq 0 \), the integral \( \int_{0}^{\delta} \frac{dt}{tH(t)} \) must diverge to \(+\infty\). As \( H \) is continuous and, except possibly at \( t = 0 \), does not change sign, \( H(t) > 0 \) and so \( h(t) < 1 \) whenever \( 0 < t \leq \delta \). From \( N_f(x^* + t) - x^* = th(t) < t \) and \( h(0) = 0 \) it follows that \( x^* < N_f(x_0) < x_0 \) and therefore \( N'_f(x_0) \setminus x^* \) provided that \( x_0 \in J \). In other words, \( x^* \) is attracting from the right.

It remains to verify (7) for \( f \in C^k(I) \). To this end, assume first that \( k < \infty \) and \( f(x^*) = f'(x^*) = \cdots = f^{(k)}(x^*) = 0 \). In this case, since \( f \) is \( C^k \), the left-hand side in (5) with \( j = k \) tends to a finite limit as \( x \downarrow x^* \). Consequently,

\[
\lim_{x \downarrow x^*} \int_{x}^{\delta} \frac{k - 1 - kh(t)}{1 - h(t)} \frac{dt}{t} < +\infty.
\]

If \( h(0) < 1 - k^{-1} \), then the integrand in (8) would eventually be positive near \( t = 0 \), which clearly is impossible. Therefore \( h(0) \geq 1 - k^{-1} \). Since \( h(0) \leq 1 \) by the same argument,

\[
N'_f(x^*) = h(0) \in [1 - k^{-1}, 1] \subset \Delta_k.
\]

If \( k = \infty \) and \( f^{(j)}(x^*) = 0 \) for all \( j \in \mathbb{N} \), then similar reasoning shows that \( N'_f(x^*) \in \bigcap_{j \in \mathbb{N}} [1 - j^{-1}, 1] = \{1\} \subset \Delta_\infty \).

Finally assume that \( f(x^*) = f'(x^*) = \cdots = f^{(j)}(x^*) = 0 \) yet \( f^{(j+1)}(x^*) \neq 0 \) for some \( j \) with \( 0 \leq j < k \). The same argument as before with \( k \) replaced by \( j \) shows that \( N'_f(x^*) \in [1 - (j+1)^{-1}, 1] \). If \( h(0) > 1 - (j+1)^{-1} \), then (5) with \( j \) replaced by \( j + 1 \) would imply that \( \lim_{x \downarrow x^*} (x - x^*)^{-(j+1)} f(x) = 0 \), which contradicts \( f^{(j+1)}(x^*) \neq 0 \). Thus \( N'_f(x^*) = h(0) = 1 - (j+1)^{-1} \in \Delta_\infty \subset \Delta_k \). \( \square \)
**Example 8.** Lemma 7 is best possible in the following sense: For every $k \in \mathbb{N}_\infty$ and $\delta \in \Delta_k$ there exists a $C^k$ function $f$ with $N_f \in C^1$ having a single fixed point $x^*$ such that $N_f'(x^*) = \delta$. For $k \in \mathbb{N}$ and $\delta \in \Delta_k \backslash \{1\}$ let $\gamma = (1 - \delta)^{-1}$ and consider the function
\[
f(x) = \begin{cases} 
    x^\gamma (1 + \frac{1}{2k+4} x^{(1+\gamma)(1+k)} \sin(x^{-\gamma})) & \text{if } 0 < |x| < 1, \\
    0 & \text{if } x = 0,
\end{cases}
\]
where, for non-integer $\gamma$, each argument $x$ has to be replaced by $|x|$. Taking $I = ]-1, 1[$, it is readily checked that $f \in C^k(I)$ and $N_f \in C^1(I)$. Moreover, $x^* = 0$ is the only fixed point of $N_f$ in $I$, and $N_f'(x^*) = 1 - \gamma^{-1} = \delta$. For $\delta = 1$, an example is provided by the $C^k$ function $f(x) = \exp(-|x|^{-1}) + \frac{1}{2} \exp(-(k+4)|x|^{-1}) \sin(\exp(|x|^{-1}))$ for which $N_f$ is $C^1$, has $x^* = 0$ as its only fixed point, and $N_f'(x^*) = 1$. Simple examples in the case $k = \infty$ are $f(x) = x^\gamma$ for $\delta < 1$, and $f(x) = \exp(-|x|^{-1})$ for $\delta = 1$, respectively.

An important special case for which Lemma 7 can be strengthened is the case of a root of finite multiplicity. Recall that $x^* \in I$ is a root of $f \in C^k(I)$ of multiplicity $j \in \mathbb{N}$ if $f(x) = (x - x^*)^j g(x)$ for all $x \in I$, where $g \in C^k(I)$ and $g(x^*) \neq 0$.

**Lemma 9.** Let $x^*$ be a root of $f \in C^k(I)$ of finite multiplicity $j$. Then, for some open interval $J \subset I$ containing $x^*$, $N_f \in C^{k-1}(J)$, and $N_f'(x^*) = 1 - j^{-1}$; in particular, $\text{Fix}[N_f] \cap J = \{x^*\}$ is attracting.

**Proof.** Since $f(x) = (x - x^*)^j g(x)$ for some $g \in C^k$ with $g(x^*) \neq 0$,
\[
N_f(x) - x^* = (x - x^*) \frac{(j - 1)g(x) + (x - x^*)g'(x)}{jg(x) + (x - x^*)g'(x)} = (x - x^*)h(x),
\]
where $h$ is $C^{k-1}$ on some open interval $J \subset I$ containing $x^*$, and $N_f'(x^*) = h(x^*) = 1 - j^{-1}$. Thus, for $J$ chosen sufficiently small, $\text{Fix}[N_f] \cap J = \{x^*\}$, and the fixed point $x^*$ clearly is attracting. $\square$

3 Main theorem

Lemma 7 contains necessary conditions for a map to be Newton. In general it is too much to expect that every $T \in C^1(I)$ whose fixed point set is attracting and satisfies (7) would be a Newton map associated with some $f \in C^k(I)$.
Example 10. Let $I = ]−1, 1[$ and consider the map 

$$T(x) = \begin{cases} 
\frac{x}{\log|x|} & \text{if } 0 < |x| < 1, \\
0 & \text{if } x = 0, 
\end{cases}$$

which has $x^* = 0$ as its only and attracting fixed point and, with $T'(x^*) := 0$, is $C^1$ on $I$. Obviously $T'(x^*) \in \Delta_k$ for all $k \in \mathbb{N}_\infty$. Suppose that $N_f = T$ for some $f \in C^k(I)$. Then, with some nonzero constant $C$,

$$f(x) = C x (1 - \log x), \quad \forall x : 0 < x < 1.$$ 

Clearly, this function cannot be extended to even a differentiable function on $I$. Thus $N_f \neq T$ for every $f \in C^k(I)$. The fact that in this example $T$ is barely $C^1$ is not important, as it is easy to find similar examples with $T$ showing any finite degree of differentiability: For every $l \in \mathbb{N}$ (and $k \in \mathbb{N}_\infty$) there exist maps $T \in C^l(I)$ such that $T'(\text{Fix}[T]) = \{\delta\}$ with $\delta \in \Delta_k$, yet $N_f \neq T$ for all $f \in C^k(I)$.

Example 10 shows that there is no hope for a converse of Lemma 7 to hold, even if $N_f$ is assumed to be more regular than $C^1$. However, the situation is much clearer for smooth maps, that is, for $l = \infty$. In this case, the converse of Lemma 7 does actually hold, i.e., the stated conditions are also sufficient.

Theorem 11. Let $k \in \mathbb{N}_\infty$, and suppose $T \in C^\infty(I)$. Then $T$ is a Newton map, associated with $f \in C^k(I)$, if and only if $\text{Fix}[T]$ either is empty or an attracting (possibly one-point) interval, and

$$T'(\text{Fix}[T]) = \{\delta\}, \quad \text{for some } \delta \in \Delta_k. \quad (10)$$

Moreover, the function $f$ is uniquely determined up to a multiplicative constant if either $\delta \in \{0, \frac{1}{2}, \frac{1}{3}, \ldots, 1 - k^{-1}\}\setminus\{1\}$ or the set $I \setminus \text{Fix}[T]$ is connected.

Proof. If $T$ is a Newton map then, by Lemma 7, $\text{Fix}[T]$ is an attracting interval (which may be empty or one-point), and (10) holds. Thus only the converse statement and the uniqueness assertion have yet to be proved. To this end, three cases will be distinguished; throughout let $g(x) := x - T(x)$.

Case 1. Assume that $\text{Fix}[T] = \emptyset$. Then $g$ is nonvanishing and $C^\infty$ on $I$, and so is

$$f(x) = \exp\left(\int_{\xi}^{x} \frac{dt}{g(t)}\right), \quad \forall x \in I,$$

where $\xi$ is any point in $I$. Since $g$ is $C^\infty$ and does not vanish on $I$, the solution $f$ of the first-order ODE $f'/f = 1/g$, or equivalently, $N_f = T$, is unique up to multiplication by a constant.
**Case 2.** Assume that \( x^* \in \text{Fix}[T] \) and \( T'(x^*) = \delta \) with \( \delta \in \Delta_k \setminus \{1\} \). Clearly this implies that \( \text{Fix}[T] = \{x^*\} \), and \( T \) can be written as

\[
T(x) = x^* + \delta(x - x^*) + (1 - \delta)(x - x^*)^2 h(x),
\]

with a uniquely determined \( h \in C^\infty \). Note that \((x - x^*)h(x) \neq 1\) for all \( x \in I \). Let \( \gamma = (1 - \delta)^{-1} \), pick points \( x^-, x^+ \in I \) with \( x^- < x^* < x^+ \), and define \( f : I \rightarrow \mathbb{R} \) by

\[
f(x) := \begin{cases} 
  c^+(x^+ - x^*)^\gamma, & \text{if } x > x^*, \\
  0, & \text{if } x = x^*, \\
  c^-(x^* - x^-)^\gamma, & \text{if } x < x^*;
\end{cases}
\]

(11)

here \( c^+, c^- \) are nonzero real constants. Since \( x^* \) is the only fixed point of \( T \) in \( I \) it follows that \( f \in C^\infty(I \setminus \{x^*\}) \), and \( N_f = T \). By using the identity

\[
(x - x^*)^\gamma = (x^+ - x^*)^\gamma \exp \left(-\gamma \int_{x}^{x^+} \frac{dt}{t - x^*}\right), \quad \forall \ x > x^*, \tag{12}
\]

a short computation yields

\[
(x - x^*)^{-\gamma} f(x) = c^+ \exp \left(-\gamma \int_{x}^{x^+} \frac{h(t) dt}{1 - (t - x^*)h(t)}\right), \quad \forall \ x > x^*.
\]

An analogous computation for \( x < x^* \) yields

\[
(x^* - x)^{-\gamma} f(x) = c^- \exp \left(\gamma \int_{x^*}^{x} \frac{h(t) dt}{1 - (t - x^*)h(t)}\right), \quad \forall \ x < x^*.
\]

Since the integrand \( \frac{h(t)}{1 - (t - x^*)h(t)} \) is \( C^\infty \) on \( I \), both one-sided limits for \( |x - x^*|^{-\gamma} f(x) \), as \( x \) approaches \( x^* \), are finite and nonzero. If \( \delta = 1 - j^{-1} \) for some \( 1 \leq j \leq k \) then, for \( f \) to be \( C^j \) on \( I \), these two one-sided limits have to be equal or, equivalently,

\[
c^- = (-1)^j c^+ \exp \left(-j \int_{x^*}^{x} \frac{h(t) dt}{1 - (t - x^*)h(t)}\right)
\]

must hold. In the latter case, for all \( x \in I \),

\[
f(x) = c^+(x - x^*)^j \exp \left(-j \int_{x}^{x^+} \frac{h(t) dt}{1 - (t - x^*)h(t)}\right),
\]
which shows $f \in C^k(I)$ . Since the two-parameter family defined in (11) contains all solutions of $N_f = T$ on $x < x^*$ and $x > x^*$ separately, the solution of $N_f = T$ is unique up to multiplication by a nonzero constant if $\delta \in \{0, \frac{1}{2}, \frac{1}{3}, \ldots, 1 - k^{-1}\} \backslash \{1\}$.

If, on the other hand, $\delta > 1 - k^{-1}$, and correspondingly $\gamma > k$, then $f \in C^k(I)$ for any choice of the constants $c^+, c^-$, and $f(x^*) = f'(x^*) = \cdots = f^{(k)}(x^*) = 0$.

**Case 3.** Assume that $T'(\text{Fix}[T]) = \{1\}$. If $\text{Fix}[T] = I$, then trivially $T$ is the Newton map associated with $f \equiv 1$. Without loss of generality, therefore, assume that $x^*$ is the right boundary point of $\text{Fix}[T]$. In this case

$$T(x) = x - (x - x^*)^2h(x),$$

where $h \in C^\infty(I)$ and $h(x) > 0$ whenever $x > x^*$, and $h(x) = 0$ for all $x \in \text{Fix}[T]$; in particular, therefore, $h(x^*) = 0$. As before, pick $x^+ \in I$ with $x^+ > x^*$ and, analogously to (11), let

$$f^+(x) := \begin{cases} \exp\left(-\int_{x^*}^x \frac{dt}{g(t)}\right) & \text{if } x > x^*, \\ 0 & \text{if } x \leq x^*. \end{cases}$$

Using (12), with $\gamma$ replaced by $j$, and recalling that $g(t) = (t - x^*)^2h(t)$, it follows that $\lim_{t \searrow x^*}(x - x^*)^{-j}f^+(x) = 0$ for all $j \in \mathbb{N}$. Thus $f^+ \in C^\infty(I)$ and $N_f^+(x) = T(x)$ whenever $x > x^*$ or $x \in \text{Fix}[T]$. If $\text{Fix}[T]$ has a left boundary point in $I$ as well, then define $f^-$ in a “mirrored” manner and let $f = c^+f^+ + c^-f^-$ with nonzero constants $c^+, c^-$. Clearly, $f \in C^\infty(I)$ and $N_f = T$ for any choice of $c^+, c^-$. The assertion concerning uniqueness up to multiplication by a constant is now obvious from the three cases detailed above. □

**Corollary 12.** Suppose $T \in C^\infty(I)$. Then $T$ is a Newton map, associated with $f \in C^\infty(I)$, if and only if $\text{Fix}[T]$ is either empty or an attracting (possibly one-point) interval, and

$$T'(\text{Fix}[T]) = \{1 - j^{-1}\}, \quad \text{for some } j \in \mathbb{N}_\infty. \quad (13)$$

Moreover, $f$ is uniquely determined up to a multiplicative constant unless $j = \infty$ in (13) and the set $I \backslash \text{Fix}[T]$ is not connected.

The next corollary requires $T$ to be not only $C^\infty$ but even real-analytic. Recall that a map is real-analytic if it can be represented by its Taylor’s series in a neighbourhood of every point in its domain. Real-analytic Newton
Figure 2: Three $C^\infty$ maps $T$ which are not Newton maps associated with any $C^k$ function on the interval $I$ because $\text{Fix}[T]$ is not attracting (left), $\text{Fix}[T]$ is not an interval (middle), and $T'(x^*) \not\in \Delta_k$ for any $k \in \mathbb{N}_\infty$, respectively.

maps are especially easy to characterise. Although analyticity is a strong assumption indeed, the class of real-analytic functions is of great historical [7, 11] and practical relevance, as it contains, for example, all rational and trigonometric functions and compositions thereof [1, 6]. If $f$ is real-analytic then so is $N_f$, provided the latter map is continuous [1, 2].

**Corollary 13.** Let $T$ be real-analytic on $I$, and $T(x) \neq x$. Then $T$ is a Newton map, associated with a real-analytic function $f$, if and only if $T$ has at most one fixed point in $I$, and, in case a fixed point $x^*$ exists, $T'(x^*) = 1 - j^{-1}$ for some $j \in \mathbb{N}$. Moreover, $f$ is unique up to multiplication by a constant.

**Example 14.** For $f(x) = \exp(-x)$ and $f_j(x) = x^j$, $j \in \mathbb{N}$, clearly $N_f(x) = x + 1$ and $N_{f_j}(x) = (1 - j^{-1})x$, respectively. Thus all cases contained in Corollary 13 can occur.

**Example 15.** The much-studied logistic map $F_\mu(x) = \mu x (1 - x)$ is a Newton map associated with a real-analytic function on $I = [0, 1[$ if and only if $\mu \in M$, with $M := ]-\infty, 1[ \cup \{1 + j^{-1}: j \in \mathbb{N}\}$. Indeed, $F_\mu = N_{f_\mu}$ with functions

$$f_\mu(x) = \left(\frac{x}{\mu x + 1 - \mu}\right)^{(1-\mu)^{-1}}$$

for $\mu \neq 1$,

and $f_1(x) = \exp(-x^{-1})$. Note that while $f_\mu$ is real-analytic on $I$ for all $\mu \in M$, it is only in the trivial case $\mu = 0$ that $f_\mu$ could be extended to a real-analytic function such that $N_{f_\mu}(x) = F_\mu(x)$ for all $x \in \mathbb{R}$. Consequently, $F_\mu$ is not a Newton map on $\mathbb{R}$ unless $\mu = 0$.

**Example 16.** It must be emphasised that Theorem 11 and Corollaries 12 and 13 do not force the set $\text{Fix}[T]$ of a Newton map $T \in C^\infty(I)$ to attract
all points in $I$. In fact, the map $T$ may at the same time exhibit some stable dynamical feature other than a fixed point. For a simple concrete example consider the (real-analytic) function

$$f(x) = x \frac{3 + x^2}{1 + x^2},$$

for which the associated Newton map

$$N_f(x) = -\frac{4x^3}{3 + x^4}$$

has the stable (in fact, super-attracting) 2-periodic orbit $\{\sqrt{3}, -\sqrt{3}\}$.

**Remark 17.** It is well known that if $f$ is a rational function (i.e., a quotient of two polynomials) then $N_f$ can be extended uniquely to (and studied appropriately as) a smooth function $\overline{N_f}$ on $\mathbb{R}$, the one-point compactification of $\mathbb{R}$. Though finite, $\text{Fix}[\overline{N_f}]$ generally contains more than one point [2, 3]. Corollary 13, however, clearly still applies to $\text{Fix}[\overline{N_f}] \cap I$ for every interval $I$ on which $f$ is real-analytic.

The above results about Newton maps have an immediate bearing on the distribution of the floating-point fractions of the iterates $x_n = N_f^n(x_0)$, that is, on the numerical data generated by NM. (See [9] for an account on the relevance of fraction parts distributions for practical computations.) In particular, this distribution depends significantly on the analytic properties of $N_f$ discussed in this note; the interested reader is referred to [6] for details.

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**References**


