A characterisation of Newton maps

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Abstract

Conditions are given for a C^k map T to be a Newton map, that is, the map associated with a differentiable real-valued function via Newton's method. For finitely differentiable maps and functions, these conditions are only necessary, but in the smooth case, i.e. for $k = \infty$, they are also sufficient. The characterisation rests upon the structure of the fixed point set of T and the value of the derivative T' there, and it is best possible as is demonstrated through examples.

Contents

1 Introduction

Newton's method (NM) for computing successive approximations of zeros of functions is one of the most widely used methods in all of applied mathematics; variants and generalisations also play a prominent role in numerous other disciplines [2, 3, 8, 10, 11]. Conceptually, NM becomes especially transparent within a dynamical systems context. The purpose of this brief note

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is to characterise, in the simplest possible setting, the local properties of the dynamical systems thus encountered.

Throughout, let $f : I \to \mathbb{R}$ be a differentiable function, defined on some open interval $I \subset \mathbb{R}$, and denote by N_f its associated NM transformation, that is

$$N_f(x) = x - \frac{f(x)}{f'(x)}, \quad \forall x \in I : f'(x) \neq 0; \tag{1}$$

for N_f to be defined for every $x \in I$, set $N_f(x) := x$ whenever $f'(x) = 0$.

NM for finding roots (zeros) of f, i.e., real numbers x^* with $f(x^*) = 0$, amounts to picking an initial point $x_0 \in I$ and iterating N_f, thus generating the sequence

$$x_n = N_f(x_{n-1}) = N_f^n(x_0), \quad \forall n \in \mathbb{N},$$

where, here and throughout, for any map $T : I \to \mathbb{R}$ and any $n \in \mathbb{N}$, $T^n(x) = T\big(T^{n-1}(x)\big)$, provided that $T^{n-1}(x) \in I$, and $T^0(x) = x$. Note that $N_f(x) = x$ precisely if $f(x)f'(x) = 0$; that is, the only fixed points of N_f occur where either f or f' vanish. Thus for $f(x_n)f'(x_n) = 0$, and only then, does NM terminate at x_n: If $f(x_n) = 0$, a root has been found, and otherwise (1) breaks down due to a horizontal tangent to the graph of f at x_n (see Figure 1).

Clearly, if (x_n) converges to x^*, say, and if N_f is continuous at x^*, then $N_f(x^*) = x^*$, i.e., x^* is a fixed point of N_f, and $f(x^*) = 0$. (The trivial alternative $f \equiv$ *const.* is tacitly excluded here, see Lemma 4 below.) It is this correspondence between the roots of f and the fixed points of N_f that suggests that NM be studied as a dynamical system. Under a mild assumption, each (isolated) fixed point x^* is *attracting*, that is, $\lim_{n\to\infty} N_f^n(x_0) = x^*$ for all x_0 sufficiently close to x^*. (For x_0 further away from any root, the sequence (x_n) may exhibit a considerably more complicated long-term behaviour [2, 3, 11].) This aspect of NM is put into perspective by the main result of the present note, Theorem 11 below, which completely characterises the local dynamical properties of N_f.

2 Newton maps

The definition of a Newton map given below entails a relationship between the analytic properties of a function f and the analytic properties of its associated NM transformation N_f. It is a simple fact, rarely alluded to in studies of NM, that in general these properties are quite independent.

Example 1. The function $f(x) = |x|^{3/2}$ is C^1 but not C^2, yet it has a C^∞ NM transformation, namely $N_f(x) = \frac{1}{3}x$.

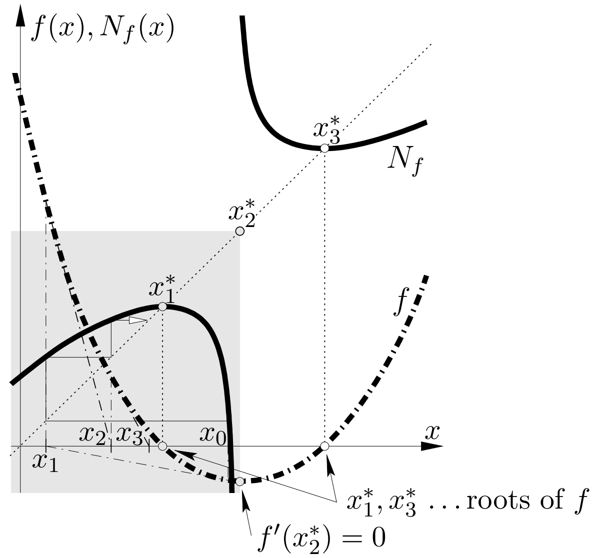

Figure 1: Visualising NM: The first few iterates x_1, x_2, x_3 are found graphically, both by means of tangents to the graph of f (broken line) and via the graph of N_f (solid line). Note how the point x_2^* with $f'(x_2^*) = 0$ causes N_f to have a discontinuity.

Example 2. It is easily seen that the function

$$f(x) = \begin{cases} \exp\bigl(-x^{-2} + |x| + \cos(x^{-2})\bigr) & \text{if } x \neq 0\,, \\ 0 & \text{if } x = 0\,, \end{cases}$$

is C^∞, and both f and f' vanish only at $x^* = 0$. Nevertheless

$$-1 = \liminf_{x\to 0} N_f(x) < \limsup_{x\to 0} N_f(x) = 1,$$

hence N_f is not even *continuous* at x^*.

Since N_f may fail to be continuous even if f is C^∞, in order to ensure the applicability of NM, some explicit assumption on the smoothness of N_f has to be imposed. To formulate such conditions concisely, let $\mathbb{N}_\infty = \mathbb{N} \cup \{\infty\}$ and stipulate that $\infty^{-1} := 0$ and $\infty \pm j = \infty$ for all $j \in \mathbb{N}$.

In view of (1), for N_f to be C^l for some $l \in \mathbb{N}_\infty$, one might demand that f be at least C^{l+1}, but Examples 1 and 2 show that this assumption is neither necessary nor sufficient. Simply imposing further conditions on N_f also seems problematic as long as it is not clear whether any such condition is satisfied for a reasonably large class of functions. Thus it is inevitable to address the following general inverse problem: Given a C^l map T, does there exist a function f such that $T = N_f$?

Definition 3. Let $I \subset \mathbb{R}$ be an open interval, and $l \in \mathbb{N}_\infty$. A map $T \in C^l(I)$ is called a *Newton map* (associated with f), if $T = N_f$ for some differentiable function $f : I \to \mathbb{R}$.

Clearly, not every $T \in C^l(I)$ is a Newton map, even if $l = \infty$, as the trivial example $T(x) = -x$ shows, for which every f with $N_f = T$ lacks differentiability at $x^* = 0$. As will become clear shortly, most maps are not Newton, but a satisfactory characterisation is not available for finitely differentiable maps. However, in the smooth case, i.e. for $l = \infty$, there is a simple characterisation of Newton maps, as provided by Theorem 11 below.

For any map T, denote by $\mathrm{Fix}\,[T]$ the set of fixed points of T, that is, $\mathrm{Fix}\,[T] := \{x \in I : T(x) = x\}$, and say that $\mathrm{Fix}\,[T]$ is *attracting* if $\lim_{n\to\infty} T^n(x_0) \in \mathrm{Fix}\,[T]$ for all x_0 sufficiently close to $\mathrm{Fix}\,[T]$.

Lemma 4. *Let $f : I \to \mathbb{R}$ be differentiable, and assume that N_f is continuous. Then $\mathrm{Fix}\,[N_f]$ is either empty or a (possibly one-point) interval; in the latter case,*

$$\limsup_{x\to x^*} \frac{N_f(x) - x^*}{x - x^*} = \delta \quad \textit{for some } \delta \in [0,1] \tag{2}$$

holds for every $x^* \in \mathrm{Fix}\,[N_f]$.

Proof. It will first be shown that both sets $Z_0 := \{x \in I : f(x) = 0\}$ and $Z_1 := \{x \in I : f'(x) = 0\}$ of zeros of f and f', respectively, are (possibly empty or one-point) subintervals of I. Moreover, if $Z_1 \neq I$, that is, if f is not constant, then $Z_1 \subset Z_0$; in fact, the two sets coincide unless Z_0 contains exactly one point, in which case Z_1 may be empty. Since $\mathrm{Fix}\,[N_f] = Z_0 \cup Z_1$ the first part of the lemma follows immediately from this.

If $Z_1 = I$, then $\mathrm{Fix}\,[N_f] = I$, so let $Z_1 \neq \emptyset$ be different from I. Pick $a \in Z_1$, suppose, by way of contradiction, $f(a) \neq 0$ and, without loss of generality, that $b := \sup\{x \geq a : f(y) = f(a) \text{ for all } y \in [a,x]\}$ belongs to I. Clearly, $f(b) = f(a)$ and $f'(b) = 0$, hence $N_f(b) = b$. By the Mean Value Theorem there exists a sequence $b_n \searrow b$ such that $0 < |f'(b_n)| \leq 1$ for all n. But then

$$\liminf_{n\to\infty} |N_f(b_n) - b| \geq \lim_{n\to\infty} |f(b_n)| = |f(b)| = |f(a)| > 0\,,$$

clearly contradicting the continuity of N_f. Therefore $f(a) = 0$, hence $Z_1 \subset Z_0$. If $a_1 < a_2$ both belong to Z_0 then, by the previous argument and the

Mean Value Theorem, Z_0 contains a point strictly between a_1 and a_2. Since Z_0 is closed, it contains, with any two points, the whole segment joining these points. Thus Z_0 is an interval. If Z_0 is not a singleton then $Z_0 \subset Z_1$ and therefore $Z_0 = Z_1$. The latter equality also holds if Z_0 is one-point because $Z_1 \neq \emptyset$. Finally, if Z_1 is empty then clearly Z_0 cannot contain more than one point.

Assertion (2) is trivially true if x^* is an interior point of $\mathrm{Fix}\,[N_f]$. Without loss of generality therefore assume that x^* is, say, a *right* boundary point of $\mathrm{Fix}\,[N_f] = Z_0$. Choose $\delta > 0$ so small that $J := \left]\, x^*, x^* + \delta \,\right] \subset I$ and, for $0 < t \leq \delta$, let

$$h(t) := \frac{N_f(x^* + t) - x^*}{t}\,; \tag{3}$$

the function h is continuous on $]0, \delta\,]$, and $h(t) \neq 1$ for all $t > 0$. Since $x \neq N_f(x)$ for $x \in J$,

$$\frac{f'(x)}{f(x)} = \frac{1}{x - N_f(x)}\,, \quad \forall x \in J\,,$$

which after integrating both sides from x to $x^* + \delta$, and using the auxiliary function h defined in (3), can be written as

$$f(x) = f(x^* + \delta) \exp\left(-\int_{x-x^*}^{\delta} \frac{1}{1 - h(t} \frac{dt}{t}\right), \quad \forall x \in J\,. \tag{4}$$

Assume $f(x^* + \delta) > 0$ without loss of generality. If $h(t) > 1$ for all $t > 0$, then (4) implies that $f(x^*) \neq 0$, contradicting $x^* \in Z_0$. Thus $h(t) < 1$ for all $t > 0$, and in particular

$$\limsup\nolimits_{t \searrow 0} h(t) = \limsup\nolimits_{x \searrow x^*} \frac{N_f(x) - x^*}{x - x^*} \leq 1\,.$$

Fix $j \in \mathbb{N}$. Dividing (4) by $(x - x^*)^j = \delta^j \exp\left(-j \int_{x-x^*}^{\delta} t^{-1} dt\right)$ yields

$$(x - x^*)^{-j} f(x) = f(x^* + \delta)\delta^{-j} \exp\left(\int_{x-x^*}^{\delta} \frac{j - 1 - jh(t)}{1 - h(t)} \frac{dt}{t}\right), \quad \forall\, x \in J\,. \tag{5}$$

To bound $\limsup_{t \searrow 0} h(t)$ from below, pick $\varepsilon > 0$ and assume that $h(t) < -\varepsilon$ for all sufficiently small $t > 0$. In this case, (5) with $j = 1$ shows that

$$(x - x^*)^{-1} f(x) \geq f(x^* + \delta)\delta^{-(1+\varepsilon)^{-1}} (x - x^*)^{-\varepsilon(1+\varepsilon)^{-1}} \to \infty\,, \quad \text{as } x \searrow x^*\,,$$

which contradicts the differentiability of f at x^*. Since $\varepsilon > 0$ was arbitrary, $\limsup_{t \searrow 0} h(t) \geq 0$. $\square$

Remark 5. (i) Lemma 4 should be contrasted with the simple fact that for *every* closed set $A \subset \mathbb{R}$ there exists a C^∞ map T with $T(I) \subset I$ and $\mathrm{Fix}\,[T] = A \cap I$.

(ii) Under the conditions of Lemma 4 there is no analogue to (2) for the corresponding lim inf which, as simple examples show, can be any number between, and including, the trivial bounds $-\infty$ and δ.

As pointed out earlier, the applicability of NM rests on the correspondence between the roots of f and the fixed points of N_f — *and* the attractiveness of the latter. Mere continuity of N_f does not guarantee that $\mathrm{Fix}\,[N_f]$ is attracting.

Example 6. Consider the C^1 function

$$f(x) = \begin{cases} |x|^{3/2} \exp\left(-\int_0^{|x|^{-1}} t^{-1} \sin t \, dt\right) & \text{if } x \neq 0\,, \\ 0 & \text{if } x = 0\,, \end{cases}$$

for which the associated NM transformation

$$N_f(x) = \begin{cases} x \dfrac{1 + 2\sin(|x|^{-1})}{3 + 2\sin(|x|^{-1})} & \text{if } x \neq 0\,, \\ 0 & \text{if } x = 0\,, \end{cases}$$

is continuous yet obviously not C^1. The only fixed point of N_f, and correspondingly the only root of f and f', is $x^* = 0$. Since, for every $j \in \mathbb{N}$, the points $\pm\frac{2}{\pi}(4j-1)^{-1}$ are 2-periodic, $\mathrm{Fix}\,[N_f] = \{0\}$ is *not* attracting.

Thus while $\mathrm{Fix}\,[N_f]$ is topologically simple whenever N_f is continuous, to make NM practical for approximating zeros, more smoothness is required. Only the case of N_f being at least C^1 will therefore be considered from now on. (For the same reason, the legitimate case $l = 0$ has been excluded from Definition 3.) Also, the properties of N_f', albeit not completely determined by the smoothness of f, do depend on the latter. To describe this dependence, for every $k \in \mathbb{N}_\infty$, define the set

$$\Delta_k := \left\{0, \tfrac{1}{2}, \tfrac{2}{3}, \ldots, 1 - k^{-1}\right\} \cup \left]1 - k^{-1}, 1\right]\,, \tag{6}$$

and note that $[0,1] = \Delta_1 \supset \Delta_2 \supset \ldots \supset \Delta_\infty = \{1 - j^{-1} : j \in \mathbb{N}_\infty\}$.

Lemma 7. *Let $f : I \to \mathbb{R}$ be differentiable, and assume that $N_f \in C^1(I)$. Then $\mathrm{Fix}\,[N_f]$ is either empty or an attracting (possibly one-point) interval. Moreover, if $\mathrm{Fix}\,[N_f] \neq \emptyset$ and $f \in C^k(I)$ with $k \in \mathbb{N}_\infty$ then*

$$N_f'(\mathrm{Fix}\,[N_f]) = \{\delta\} \quad \textit{for some } \delta \in \Delta_k\,. \tag{7}$$

Proof. The assertions are trivially true if f is constant or $\mathrm{Fix}\,[N_f] = \emptyset$. Therefore assume that f is not constant and $\mathrm{Fix}\,[N_f]$ is not empty, hence a subinterval of I, by Lemma 4. If x^* is an interior point of $\mathrm{Fix}\,[N_f]$ then $N_f' \equiv 1$ in a neighbourhood of x^*, and the assertion is again true. Thus assume without loss of generality that x^* is a *right* boundary point of $\mathrm{Fix}\,[N_f]$. By Lemma 4, $N_f'(x^*) \in \Delta_1$, so x^* obviously is attracting from the right, unless perhaps for $N_f'(x^*) = 1$. In the latter case, with the notations introduced in the proof of Lemma 4, the function h defined in (3), supplemented by $h(0) := N_f'(x^*) = 1$, is continuous on $[0,\delta]$ and can be written as $h(t) = 1 - H(t)$, where H is also continuous on $[0,\delta\,]$, and $H(t) \neq 0$ unless $t = 0$. With this, (4) takes the form

$$f(x) = f(x^* + \delta) \exp\left(- \int_{x-x^*}^{\delta} \frac{dt}{tH(t)} \right), \quad \forall\, x \in J\,.$$

Since $f(x^*) = 0$ and $f(x^* + \delta) \neq 0$, the integral $\int_0^\delta \frac{dt}{tH(t)}$ must diverge to $+\infty$. As H is continuous and, except possibly at $t = 0$, does not change sign, $H(t) > 0$ and so $h(t) < 1$ whenever $0 < t \leq \delta$. From $N_f(x^* + t) - x^* = th(t) < t$ and $h(0) = 0$ it follows that $x^* < N_f(x_0) < x_0$ and therefore $N_f^n(x_0) \searrow x^*$ provided that $x_0 \in J$. In other words, x^* is attracting from the right.

It remains to verify (7) for $f \in C^k(I)$. To this end, assume first that $k < \infty$ and $f(x^*) = f'(x^*) = \cdots = f^{(k)}(x^*) = 0$. In this case, since f is C^k, the left-hand side in (5) with $j = k$ tends to a finite limit as $x \searrow x^*$. Consequently,

$$\lim_{\varepsilon \searrow 0} \int_\varepsilon^\delta \frac{k - 1 - kh(t)}{1 - h(t)} \frac{dt}{t} < +\infty\,. \tag{8}$$

If $h(0) < 1 - k^{-1}$, then the integrand in (8) would eventually be positive near $t = 0$, which clearly is impossible. Therefore $h(0) \geq 1 - k^{-1}$. Since $h(0) \leq 1$ by the same argument,

$$N_f'(x^*) = h(0) \in [1 - k^{-1}, 1] \subset \Delta_k\,.$$

If $k = \infty$ and $f^{(j)}(x^*) = 0$ for all $j \in \mathbb{N}$, then similar reasoning shows that $N_f'(x^*) \in \bigcap_{j \in \mathbb{N}} [1 - j^{-1}, 1] = \{1\} \subset \Delta_\infty$.

Finally assume that $f(x^*) = f'(x^*) = \cdots = f^{(j)}(x^*) = 0$ yet $f^{(j+1)}(x^*) \neq 0$ for some j with $0 \leq j < k$. The same argument as before with k replaced by j shows that $N_f'(x^*) \in [1 - (j+1)^{-1}, 1]$. If $h(0) > 1 - (j+1)^{-1}$, then (5) with j replaced by $j + 1$ would imply that $\lim_{x \searrow x^*} (x - x^*)^{-(j+1)} f(x) = 0$, which contradicts $f^{(j+1)}(x^*) \neq 0$. Thus $N_f'(x^*) = h(0) = 1 - (j+1)^{-1} \in \Delta_\infty \subset \Delta_k$.
□

Example 8. Lemma 7 is best possible in the following sense: For every $k \in \mathbb{N}_\infty$ and $\delta \in \Delta_k$ there exists a C^k function f with $N_f \in C^1$ having a single fixed point x^* such that $N_f'(x^*) = \delta$. For $k \in \mathbb{N}$ and $\delta \in \Delta_k \backslash \{1\}$ let $\gamma = (1-\delta)^{-1}$ and consider the function

$$f(x) = \begin{cases} x^\gamma \left(1 + \frac{1}{2k+4} x^{(1+\gamma)(1+k)} \sin(x^{-\gamma})\right) & \text{if } 0 < |x| < 1\,, \\ 0 & \text{if } x = 0\,, \end{cases}$$

where, for non-integer γ, each argument x has to be replaced by $|x|$. Taking $I =]-1, 1[$, it is readily checked that $f \in C^k(I)$ and $N_f \in C^1(I)$. Moreover, $x^* = 0$ is the only fixed point of N_f in I, and $N_f'(x^*) = 1 - \gamma^{-1} = \delta$. For $\delta = 1$, an example is provided by the C^k function $f(x) = \exp(-|x|^{-1}) + \frac{1}{2}\exp(-(k+4)|x|^{-1}) \sin\bigl(\exp(|x|^{-1})\bigr)$ for which N_f is C^1, has $x^* = 0$ as its only fixed point, and $N_f'(x^*) = 1$. Simple examples in the case $k = \infty$ are $f(x) = x^\gamma$ for $\delta < 1$, and $f(x) = \exp(-|x|^{-1})$ for $\delta = 1$, respectively.

An important special case for which Lemma 7 can be strengthened is the case of a root of finite multiplicity. Recall that $x^* \in I$ is a root of $f \in C^k(I)$ of multiplicity $j \in \mathbb{N}$ if $f(x) = (x - x^*)^j g(x)$ for all $x \in I$, where $g \in C^k(I)$ and $g(x^*) \neq 0$.

Lemma 9. *Let x^* be a root of $f \in C^k(I)$ of finite multiplicity j. Then, for some open interval $J \subset I$ containing x^*, $N_f \in C^{k-1}(J)$, and $N_f'(x^*) = 1 - j^{-1}$; in particular,* $\text{Fix}\,[N_f] \cap J = \{x^*\}$ *is attracting.*

Proof. Since $f(x) = (x - x^*)^j g(x)$ for some $g \in C^k$ with $g(x^*) \neq 0$,

$$N_f(x) - x^* = (x - x^*) \frac{(j-1)g(x) + (x - x^*)g'(x)}{jg(x) + (x - x^*)g'(x)} = (x - x^*)h(x)\,, \qquad (9)$$

where h is C^{k-1} on some open interval $J \subset I$ containing x^*, and $N_f'(x^*) = h(x^*) = 1 - j^{-1}$. Thus, for J chosen sufficiently small, $\text{Fix}\,[N_f] \cap J = \{x^*\}$, and the fixed point x^* clearly is attracting. $\square$

3 Main theorem

Lemma 7 contains necessary conditions for a map to be Newton. In general it is too much to expect that every $T \in C^1(I)$ whose fixed point set is attracting and satisfies (7) would be a Newton map associated with some $f \in C^k(I)$.

Example 10. Let $I =]-1, 1[$ and consider the map

$$T(x) = \begin{cases} \dfrac{x}{\log |x|} & \text{if } 0 < |x| < 1\,, \\ 0 & \text{if } x = 0\,, \end{cases}$$

which has $x^* = 0$ as its only and attracting fixed point and, with $T'(x^*) := 0$, is C^1 on I. Obviously $T'(x^*) \in \Delta_k$ for all $k \in \mathbb{N}_\infty$. Suppose that $N_f = T$ for some $f \in C^k(I)$. Then, with some nonzero constant C,

$$f(x) = Cx(1 - \log x)\,, \quad \forall x : 0 < x < 1\,.$$

Clearly, this function cannot be extended to even a *differentiable* function on I. Thus $N_f \neq T$ for every $f \in C^k(I)$. The fact that in this example T is barely C^1 is not important, as it is easy to find similar examples with T showing any *finite* degree of differentiability: For every $l \in \mathbb{N}$ (and $k \in \mathbb{N}_\infty$) there exist maps $T \in C^l(I)$ such that $T'(\text{Fix}\,[T]) = \{\delta\}$ with $\delta \in \Delta_k$, yet $N_f \neq T$ for all $f \in C^k(I)$.

Example 10 shows that there is no hope for a converse of Lemma 7 to hold, even if N_f is assumed to be more regular than C^1. However, the situation is much clearer for smooth maps, that is, for $l = \infty$. In this case, the converse of Lemma 7 does actually hold, i.e., the stated conditions are also sufficient.

Theorem 11. *Let $k \in \mathbb{N}_\infty$, and suppose $T \in C^\infty(I)$. Then T is a Newton map, associated with $f \in C^k(I)$, if and only if* Fix $[T]$ *either is empty or an attracting (possibly one-point) interval, and*

$$T'(\text{Fix}\,[T]) = \{\delta\}\,, \quad \textit{for some } \delta \in \Delta_k\,. \tag{10}$$

Moreover, the function f is uniquely determined up to a multiplicative constant if either $\delta \in \{0, \frac{1}{2}, \frac{1}{3}, \ldots, 1 - k^{-1}\} \backslash \{1\}$ or the set $I \backslash \text{Fix}\,[T]$ is connected.

Proof. If T is a Newton map then, by Lemma 7, Fix $[T]$ is an attracting interval (which may be empty or one-point), and (10) holds. Thus only the converse statement and the uniqueness assertion have yet to be proved. To this end, three cases will be distinguished; throughout let $g(x) := x - T(x)$.

Case 1. Assume that Fix $[T] = \emptyset$. Then g is nonvanishing and C^∞ on I, and so is

$$f(x) = \exp\left(\int_\xi^x \frac{dt}{g(t)}\right)\,, \quad \forall x \in I\,,$$

where ξ is any point in I. Since g is C^∞ and does not vanish on I, the solution f of the first-order ODE $f'/f = 1/g$, or equivalently, $N_f = T$, is unique up to multiplication by a constant.

Case 2. Assume that $x^* \in \mathrm{Fix}\,[T]$ and $T'(x^*) = \delta$ with $\delta \in \Delta_k \backslash \{1\}$. Clearly this implies that $\mathrm{Fix}\,[T] = \{x^*\}$, and T can be written as

$$T(x) = x^* + \delta(x - x^*) + (1 - \delta)(x - x^*)^2 h(x)\,,$$

with a uniquely determined $h \in C^\infty$. Note that $(x - x^*)h(x) \neq 1$ for all $x \in I$. Let $\gamma = (1 - \delta)^{-1}$, pick points $x^-, x^+ \in I$ with $x^- < x^* < x^+$, and define $f : I \to \mathbb{R}$ by

$$f(x) := \begin{cases} c^+(x^+ - x^*)^\gamma \exp\left(-\int_x^{x^+} \frac{dt}{g(t)}\right) & \text{if } x > x^*\,, \\ 0 & \text{if } x = x^*\,, \\ c^-(x^* - x^-)^\gamma \exp\left(\int_{x^-}^{x} \frac{dt}{g(t)}\right) & \text{if } x < x^*\,; \end{cases} \tag{11}$$

here c^+, c^- are nonzero real constants. Since x^* is the only fixed point of T in I it follows that $f \in C^\infty(I \backslash \{x^*\})$, and $N_f = T$. By using the identity

$$(x - x^*)^\gamma = (x^+ - x^*)^\gamma \exp\left(-\gamma \int_x^{x^+} \frac{dt}{t - x^*}\right), \quad \forall\, x > x^*\,, \tag{12}$$

a short computation yields

$$(x - x^*)^{-\gamma} f(x) = c^+ \exp\left(-\gamma \int_x^{x^+} \frac{h(t)dt}{1 - (t - x^*)h(t)}\right), \quad \forall\, x > x^*\,.$$

An analogous computation for $x < x^*$ yields

$$(x^* - x)^{-\gamma} f(x) = c^- \exp\left(\gamma \int_{x^-}^{x} \frac{h(t)dt}{1 - (t - x^*)h(t)}\right), \quad \forall\, x < x^*\,.$$

Since the integrand $\dfrac{h(t)}{1 - (t - x^*)h(t)}$ is C^∞ on I, both one-sided limits for $|x - x^*|^{-\gamma} f(x)$, as x approaches x^*, are finite and nonzero. If $\delta = 1 - j^{-1}$ for some $1 \leq j \leq k$ then, for f to be C^j on I, these two one-sided limits have to be equal or, equivalently,

$$c^- = (-1)^j c^+ \exp\left(-j \int_{x^-}^{x^+} \frac{h(t)dt}{1 - (t - x^*)h(t)}\right)$$

must hold. In the latter case, for all $x \in I$,

$$f(x) = c^+(x - x^*)^j \exp\left(-j \int_x^{x^+} \frac{h(t)dt}{1 - (t - x^*)h(t)}\right),$$

which shows $f \in C^k(I)$. Since the two-parameter family defined in (11) contains all solutions of $N_f = T$ on $x < x^*$ and $x > x^*$ separately, the solution of $N_f = T$ is unique up to multiplication by a nonzero constant if $\delta \in \{0, \frac{1}{2}, \frac{1}{3}, \ldots, 1 - k^{-1}\}\backslash\{1\}$.

If, on the other hand, $\delta > 1 - k^{-1}$, and correspondingly $\gamma > k$, then $f \in C^k(I)$ for any choice of the constants c^+, c^-, and $f(x^*) = f'(x^*) = \cdots = f^{(k)}(x^*) = 0$.

Case 3. Assume that $T'(\mathrm{Fix}\,[T]) = \{1\}$. If $\mathrm{Fix}\,[T] = I$, then trivially T is the Newton map associated with $f \equiv 1$. Without loss of generality, therefore, assume that x^* is the right boundary point of $\mathrm{Fix}\,[T]$. In this case

$$T(x) = x - (x - x^*)^2 h(x)\,,$$

where $h \in C^\infty(I)$ and $h(x) > 0$ whenever $x > x^*$, and $h(x) = 0$ for all $x \in \mathrm{Fix}\,[T]$; in particular, therefore, $h(x^*) = 0$. As before, pick $x^+ \in I$ with $x^+ > x^*$ and, analogously to (11), let

$$f^+(x) := \begin{cases} \exp\left(-\int_x^{x^+} \frac{dt}{g(t)}\right) & \text{if } x > x^*\,, \\ 0 & \text{if } x \le x^*\,. \end{cases}$$

Using (12), with γ replaced by j, and recalling that $g(t) = (t - x^*)^2 h(t)$, it follows that $\lim_{x \searrow x^*} (x - x^*)^{-j} f^+(x) = 0$ for all $j \in \mathbb{N}$. Thus $f^+ \in C^\infty(I)$ and $N_{f^+}(x) = T(x)$ whenever $x > x^*$ or $x \in \mathrm{Fix}\,[T]$. If $\mathrm{Fix}\,[T]$ has a left boundary point in I as well, then define f^- in a "mirrored" manner and let $f = c^+ f^+ + c^- f^-$ with nonzero constants c^+, c^-. Clearly, $f \in C^\infty(I)$ and $N_f = T$ for any choice of c^+, c^-.

The assertion concerning uniqueness up to multiplication by a constant is now obvious from the three cases detailed above. □

Corollary 12. *Suppose $T \in C^\infty(I)$. Then T is a Newton map, associated with $f \in C^\infty(I)$, if and only if* $\mathrm{Fix}\,[T]$ *is either empty or an attracting (possibly one-point) interval, and*

$$T'(\mathrm{Fix}\,[T]) = \{1 - j^{-1}\}\,, \quad \textit{for some } j \in \mathbb{N}_\infty\,. \tag{13}$$

Moreover, f is uniquely determined up to a multiplicative constant unless $j = \infty$ in (13) *and the set* $I\backslash\mathrm{Fix}\,[T]$ *is not connected.*

The next corollary requires T to be not only C^∞ but even real-analytic. Recall that a map is *real-analytic* if it can be represented by its Taylor's series in a neighbourhood of every point in its domain. Real-analytic Newton

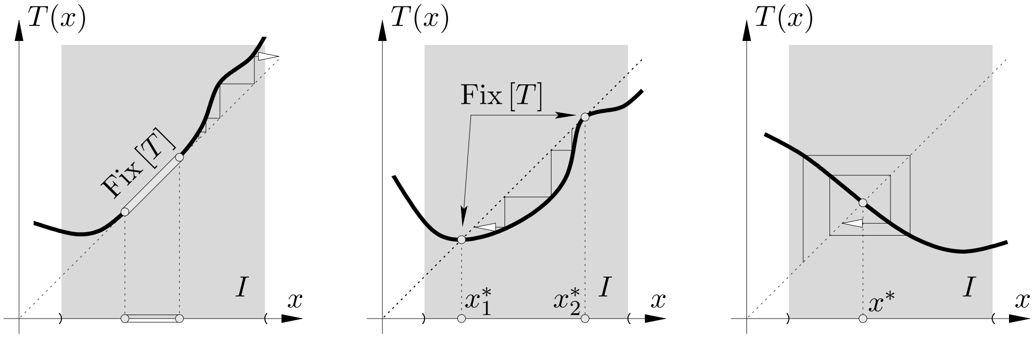

Figure 2: Three C^∞ maps T which are not Newton maps associated with any C^k function on the interval I because Fix $[T]$ is not attracting (left), Fix $[T]$ is not an interval (middle), and $T'(x^*) \notin \Delta_k$ for any $k \in \mathbb{N}_\infty$, respectively.

maps are especially easy to characterise. Although analyticity is a strong assumption indeed, the class of real-analytic functions is of great historical [7, 11] and practical relevance, as it contains, for example, all rational and trigonometric functions and compositions thereof [1, 6]. If f is real-analytic then so is N_f, provided the latter map is continuous [1, 2].

Corollary 13. *Let T be real-analytic on I, and $T(x) \not\equiv x$. Then T is a Newton map, associated with a real-analytic function f, if and only if T has at most one fixed point in I, and, in case a fixed point x^* exists, $T'(x^*) = 1 - j^{-1}$ for some $j \in \mathbb{N}$. Moreover, f is unique up to multiplication by a constant.*

Example 14. For $f(x) = \exp(-x)$ and $f_j(x) = x^j$, $j \in \mathbb{N}$, clearly $N_f(x) = x + 1$ and $N_{f_j}(x) = (1 - j^{-1})x$, respectively. Thus all cases contained in Corollary 13 can occur.

Example 15. The much-studied logistic map $F_\mu(x) = \mu x(1-x)$ is a Newton map associated with a real-analytic function on $I =]0, 1[$ if and only if $\mu \in M$, with $M := \,]-\infty, 1] \cup \{1 + j^{-1} \colon j \in \mathbb{N}\}$. Indeed, $F_\mu = N_{f_\mu}$ with functions

$$f_\mu(x) = \left(\frac{x}{\mu x + 1 - \mu}\right)^{(1-\mu)^{-1}} \qquad \text{for } \mu \neq 1,$$

and $f_1(x) = \exp(-x^{-1})$. Note that while f_μ is real-analytic on I for all $\mu \in M$, it is only in the trivial case $\mu = 0$ that f_μ could be extended to a real-analytic function such that $N_{f_\mu}(x) = F_\mu(x)$ for all $x \in \mathbb{R}$. Consequently, F_μ is not a Newton map on $\mathbb{R}$ unless $\mu = 0$.

Example 16. It must be emphasised that Theorem 11 and Corollaries 12 and 13 do not force the set Fix $[T]$ of a Newton map $T \in C^\infty(I)$ to attract *all* points in I. In fact, the map T may at the same time exhibit some *stable* dynamical feature other than a fixed point. For a simple concrete example consider the (real-analytic) function

$$f(x) = x\frac{3+x^2}{1+x^2},$$

for which the associated Newton map

$$N_f(x) = -\frac{4x^3}{3+x^4}$$

has the stable (in fact, super-attracting) 2-periodic orbit $\{\sqrt{3}, -\sqrt{3}\}$.

Remark 17. It is well known that if f is a *rational* function (i.e., a quotient of two polynomials) then N_f can be extended uniquely to (and studied appropriately as) a smooth function $\overline{N_f}$ on $\overline{\mathbb{R}}$, the one-point compactification of $\mathbb{R}$. Though finite, $\mathrm{Fix}\,[\,\overline{N_f}\,]$ generally contains more than one point [2, 3]. Corollary 13, however, clearly still applies to $\mathrm{Fix}\,[\,\overline{N_f}\,] \cap I$ for every interval I on which f is real-analytic.

The above results about Newton maps have an immediate bearing on the distribution of the floating-point fractions of the iterates $x_n = N_f^n(x_0)$, that is, on the numerical data generated by NM. (See [9] for an account on the relevance of fraction parts distributions for practical computations.) In particular, this distribution depends significantly on the analytic properties of N_f discussed in this note; the interested reader is referred to [6] for details.

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References

[1] S. Axler, P. Bourdon and W. Ramey. *Harmonic Function Theory* (1992), Springer, New York.

[2] A. Beardon. *Iteration of Rational Functions* (1991), Springer, New York.

[3] A. Berger. *Chaos and Chance* (2001), deGruyter, Berlin New York.

[4] A. Berger. Multi-dimensional dynamical systems and Benford's law. *Discrete Contin. Dyn. Syst.* **13** (2005), 217–237.

[5] A. Berger, L. Bunimovich and T. Hill. One-dimensional dynamical systems and Benford's law. *Trans. Amer. Math. Soc.* **357** (2005), 197–219.

[6] A. Berger and T. Hill. Newton's method obeys Benford's law, preprint (2005).

[7] P. Colwell. *Solving Kepler's Equation Over Three Centuries* (1993), Willmann-Bell, Richmond.

[8] A. Katok and B. Hasselblatt. *Introduction to the modern theory of dynamical systems* (1995), CUP, Cambridge.

[9] D. Knuth. *The Art of Computer Programming* Vol. 2, 3rd Ed. (1997), Addison-Wesley, Reading, MA.

[10] W. Rheinboldt. *Methods for Solving Systems of Nonlinear Equations*, 2nd Ed. (1998), SIAM, Arrowsmith, Bristol.

[11] S. Smale. Newton's method estimates from data at one point, in *The Merging of Disciplines: New Directions in Pure, Applied and Computational Mathematics* (1986), 185–196, Springer, New York.