The \((q, r)\)-Simon Newcomb Problem

DON RAWLINGS

A new statistic, the \(r\)-major index, is defined for sequences. A linear recurrence is then derived that enumerates sequences by \(r\)-descent number and \(r\)-major index.

1. INTRODUCTION

The classic Simon Newcomb problem may be described as follows. Let \(Js = (j_1, j_2, \ldots, j_s)\) be a sequence of non-negative integers with \(n = j_1 + j_2 + \cdots + j_s\). Denote by \(R(\text{Js})\) the set of sequences \(f = (f(1), f(2), \ldots, f(m))\) in which the integer \(m\) occurs \(j_m\) times. For an integer \(r \geq 1\), the set of \(r\)-descents of \(f\) is defined as

\[
\{i : f(i) \geq f(i+1) + r, 1 \leq i \leq n-1\}. \tag{1.1}
\]

The cardinality of set (1.1) is denoted by \(r\)-des \(f\). Then, the problem is to determine the number of sequences \(f \in R(\text{Js})\) with \(1\)-des \(f = k\).

The present paper has three objectives. First, the \(r\)-major index as introduced in [12] for permutations is extended to sequences. Denoted by \(r\)-maj \(f\), the \(r\)-major index of \(f\) is defined to be the sum of the elements in set (1.1) plus the cardinality of the set

\[
\{(i,j) : 1 \leq i < j \leq n, f(i) > f(j) > f(i) - r\}. \tag{1.2}
\]

In the two special cases \(r = 1\) and \(r \geq s\), the \(r\)-major index respectively reduces to what are commonly known as the major index and the inversion number. It will be shown that if the \(q\)-analog, \(q\)-factorial, and \(q\)-multinomial
coefficient of $n$ are respectively defined as

(a)  $[n] = 1 + q + \cdots + q^{n-1}$

(b)  $[n]! = [1][2] \cdots [n]$

(c)  $\left[ \begin{array}{c} n \\ j_1 j_2 \cdots j_s \end{array} \right] = \frac{[n]!}{[j_1]![j_2]! \cdots [j_s]!}$

(1.3)

then the generating function for the $r$-major index is

$$\sum_{f} q^{-r\text{-maj } f} = \left[ \begin{array}{c} n \\ j_1 j_2 \cdots j_s \end{array} \right]$$

(1.4)

summed over $R(J_s)$, which generalizes the observation of MacMahon [11] that the major index and inversion number have the same generating function.

Second, as identity (1.4) suggests, for integers $b, c \geq 1$ there is a bijection $\Phi_{b,c}: R(J_s) \to R(J_s)$ with the property that

$$b\text{-maj } f = c\text{-maj } \Phi_{b,c}(f)$$

(1.5)

for all $f \in R(J_s)$. The construction of $\Phi_{b,c}$ in section 6 will even provide a proof of (1.4). Foata [6] has given another such bijection in the case $b = 1$ and $c \geq s$.

The final and main objective is to solve the $(q, r)$-Simon Newcomb problem of enumerating sequences by $r$-descents and $r$-major index. It will be shown that for $J_s = (j_1, j_2, \ldots, j_s)$ with

(a)  $j_s \geq 1$

(b)  $j(m, r) = \begin{cases} j_m + j_{m-1} + \cdots + j_{r+m-r+1} & \text{if } m \geq r \\ j_m + j_{m-1} + \cdots + j_1 & \text{otherwise} \end{cases}$

(c)  $J_s - 1 = (j_1, j_2, \ldots, j_{s-1}, j_s - 1)$

(1.6)

the polynomial

$$M[J_s, k, r] = \sum_{f} q^{-r\text{-maj } f}$$

(1.7)

summed over sequences $f \in R(J_s)$ with $r\text{-des } f = k$ satisfies the recurrence

$$[j_s] M[J_s, k, r] = [k + j(s, r)] M[J_s - 1, k, r]$$

$$+ q^{k + j(s, r) - 1} [n + 1 - k - j(s, r)] M[J_s - 1, k - 1, r]$$

(1.8)

where $M[J_s, 0, r] = \left[ \begin{array}{c} n \\ j_1 j_2 \cdots j_s \end{array} \right]$. The more explicit formula

$$M[J_s, k, r] = \sum_{l=0}^{k} (-1)^l \left[ \begin{array}{c} n + 1 \\ l \end{array} \right] q^{(l)} \prod_{m=1}^{s} \left[ k - l + j(m, r) \right]$$

(1.9)

will then be derived from (1.8).
Numerous special cases of (1.8) and (1.9) may be found in the list references. When \( r = 1 \) identity (1.9) gives the solution to the \( q \)-Simon Newcomb problem as obtained by MacMahon [10, v. 2 p. 211]. Also see Gessel [9, p. 98]. The case \( r = 1 = q \) is discussed in [1, 2, 5, 9, 10]. However, even in these cases, recurrence (1.8) seems to be new.

In the permutation case, \( j_m = 1 \) for \( 1 \leq m \leq s \), (1.7) becomes

\[
A[n, k, r] = \sum_{\sigma} q^{r-\text{maj}} \sigma
\]

summed over permutations \( \sigma \) of \( \{1, 2, \ldots, n\} \) with \( k \) \( r \)-descents and (1.8) reduces to

\[
A[n, k, r] = [k + r]A[n - 1, k, r] + q^{k+r-1}[n+1-k-r]A[n-1, k-1, r]
\]  

(1.11)

where \( A[r, 0, r] = [r]! \). Recurrence (1.11) defines the \((q, r)\)-Eulerian numbers as developed in [12]. The \((1, r)\)-Eulerian numbers have been studied in [7, 8, 13] and Carlitz [4] obtained recurrence (1.11) for the \((q, 1)\)-Eulerian numbers.

2. THE INSERTION LEMMA

The construction of \( \Phi_{k, \varepsilon} \) and the proof of (1.8) are based on insertion. It will be convenient to regard a sequence \( f \in R(Js) \) as the word \( f(1) f(2) \ldots f(n) \) obtained by juxtaposing the letters \( f(1), f(2), \ldots, f(n) \).

A colored word \( F \) is obtained by inserting an underlined \( s \) into an \( f \in R(Js - 1) \). The color of \( F \), denoted by \( C(F) \), is defined to be the number of times that \( s \) appears to the right of the underlined \( s \) in \( F \). For example, with \( s = 5 \) and \( f = 2152431552 \in R(2, 3, 1, 1, 2) \) one may obtain

\[
F = 2152431552
\]

(2.1)

where \( C(F) = 1 \). The color is just a way of indicating which \( s \) was inserted.

To observe the effect that the insertion has on the \( r \)-major index and the \( r \)-descent number, the \( n \) possible insertion positions between the letters of \( f = f(1) f(2) \ldots f(n-1) \in R(Js - 1) \) are labeled as follows. Using the labels \{0, 1, \ldots, n-1\} in order, first read from right to left and label the positions that will not result in the creation of a new \( r \)-descent. Then, reading back from left to right the remaining positions that will create an \( r \)-descent are labeled. For instance, if \( r = 3 \), \( s = 5 \), and \( f = 215243152 \in R(2, 3, 1, 1, 2) \), then the labels in the top and bottom rows of

\[
\begin{array}{cccccccc}
\bullet & \bullet & 6 & 5 & 4 & 3 & \bullet & 2 & 1 & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
2 & 1 & 5 & 2 & 4 & 3 & 1 & 5 & 2 \\
\uparrow & \uparrow \\
7 & 8 & \bullet & \bullet & \bullet & \bullet & 9 & \bullet & \bullet \\
\end{array}
\]

(2.2)
respectively indicate the positions that will not and that will result in a new 3-descent.

Let \((f; \ell)\) denote the colored word obtained by inserting the underlined \(s\) into position \(\ell\). For example (2.2), the colored word \((f; 2)\) is given in (2.1). Note that \(3\text{-des}(f; 2) = 3\text{-des} f\) and that \(3\text{-maj}(f; 2) + C(f; 2) = 2 + 3\text{-maj} f\). This demonstrates the

**Insertion Lemma.** For \(f \in R(Js - 1)\) and \(0 \leq \ell \leq n - 1\)

(a) \(r\text{-des}(f; \ell) =
\begin{cases}
  r\text{-des} f & \text{if } 0 \leq \ell \leq r\text{-des} f + j(s, r) - 1 \\
  1 + r\text{-des} f & \text{otherwise}
\end{cases}
\)

(b) \(r\text{-maj}(f; \ell) + C(f; \ell) = l + r\text{-maj} f\).

**Proof.** For (a), note that there are \(r\text{-des} f + j(s, r)\) insertion positions that will not result in a new \(r\)-descent: preceding any of the \(j(s, r) - 1\) integers greater than \(s-r\), in any of the \(r\)-descents, or at the extreme right end of \(f\). As these positions are labeled first, (a) is immediate.

For (b), let \(m\) be the number of \(r\)-descents and integers in \(\{s-r+1, s-r+2, \ldots, s-1\}\) that are to the right of position \(\ell\). Let \(c\) be the number of times \(s\) appears to the right of position \(\ell\). In the case \(0 \leq \ell \leq r\text{-des} f + j(s, r) - 1\), it follows from (1.1) and (1.2) that inserting \(s\) into position \(\ell\) will increase the \(r\)-major index by \(m\). The color of the resulting word is \(c\). By the labeling, \(l = m+c\) and (b) follows in this case. For \(r\text{-des} f + j(s, r) \leq \ell \leq n - 1\), note that there are \(l-m-c-1\) integers in \(f\) to the left of position \(\ell\). As a new \(r\)-descent is created in this case, the color of the resulting word plus the increase in the \(r\)-major index is \(c + (l-m-c) + m = l\).

**3. Proof of Recurrence (1.8)**

Let \(R(Js, k, r) = \{ f \in R(Js) : r\text{-des} f = k \}\). Part (a) of the insertion lemma implies that the set of ordered pairs

\[
\{(F, C(F)) : F \in R(Js, k, r), C(F) \in \{0, 1, \ldots, Js - 1\}\}
\]

regarded as colored words is the disjoint union of the sets

(a) \(\{(f; \ell) : f \in R(Js - 1, k, r), 0 \leq \ell \leq k + j(s, r) - 1\}\)

(b) \(\{(g; m) : g \in R(Js - 1, k - 1, r), k + j(s, r) - 1 \leq m \leq n - 1\}\).
The following calculation based on (b) of the insertion lemma

\[ [j_s] M[Js, k, r] = \sum_{(F, C(F))} q^{r - \text{maj}(F) + C(F)} \]

\[ = \sum_{(F; L)} q^{r - \text{maj}(f; L) + C(f; L)} + \sum_{(g; m)} q^{r - \text{maj}(g; m) + C(g; m)} \]

\[ = \sum_{f} q^{f} \sum_{f} q^{r - \text{maj}(f)} + \sum_{m} q^{m} \sum_{g} q^{r - \text{maj}(g)} \]

summed respectively over the sets of (3.1) and (3.2) establishes (1.8).

To show that \( M[Js, o, r] \) is equal to the \( q \)-multinomial coefficient of \( n \), first note that \( M[Js, o, r] = 1 \) and for all \( s \leq r \) identity (1.8) reduces to

\[ [j_s] M[Js, o, r] = [n] M[Js - 1, o, r]. \tag{3.3} \]

Iteration of (3.3) yields

\[ M[Js, o, r] = \frac{[n]}{[j_s]} M[Js - 1, o, r] = \begin{bmatrix} n \\ j_1 j_2 \ldots j_r \end{bmatrix}. \]

4. THE EXPLICIT SOLUTION

Let \( (t; q)_{n+1} = (1-t)(1-tq) \ldots (1-tq^n) \) and \( (q)_n = (q; q)_n = (1-q)(1-q^2) \ldots (1-q^n) \). For a reference of the identities

(a) \( (t; q)_{n+1} = \sum_{k \geq 0} (-1)^k \binom{n+1}{k} q^{\frac{t}{2} k^2} \)

(b) \( (t; q)^{-1}_{n+1} = \sum_{k \geq 0} \binom{n+k}{k} t^k \) \tag{4.1}

see [2, p. 36]. Let \( B[Js, k, r] \) denote the polynomial on the right-hand side of

\( (1.9) \). To prove (1.9) it suffices to show that

\[ B[Js, k, r] = \begin{cases} n \\ j_1 j_2 \ldots j_r \end{cases} \begin{cases} \text{if} & k = 0 \\ \text{otherwise} & \end{cases} \tag{4.2} \]

and that \( B[Js, k, r] \) satisfies recurrence (1.8).

First, note that (a) of (4.1) implies

\[ \sum_{k \geq 0} B[Js, k, r] t^k = (t; q)_{n+1} \sum_{k \geq 0} t^k \prod_{m=1}^{s} \binom{k+j(m, r)}{j_m}. \tag{4.3} \]

In the case \( r = s \), (4.3) and (b) of (4.1) show that
\[ \sum_{k \geq 0} B[J_r, k, r] t^k = \left[ \begin{array}{c} n \\ j_1 j_2 \ldots j_r \end{array} \right] (t; q)_{n+1} \sum_{k \geq 0} \left[ \begin{array}{c} n+k \\ k \end{array} \right] t^k = \left[ \begin{array}{c} n \\ j_1 j_2 \ldots j_r \end{array} \right] \]

which checks (4.2). Then using the identities

(a) \[ \left[ \begin{array}{c} k-l+j(s, r) \\ j_s \end{array} \right] = \left[ \begin{array}{c} k-l+j(s, r) \\ j_s-1 \end{array} \right] - \left[ \begin{array}{c} k-l+j(s, r) \\ j_s \end{array} \right] \]

(b) \[ [k-l+j(s, r)] = [k+j(s, r)] - q^{k-l+j(s, r)} [l] \]

(c) \[ \left[ \begin{array}{c} n+1 \\ l \end{array} \right] = \left[ \begin{array}{c} n \\ l \end{array} \right] + q^{n+1-l} \left[ \begin{array}{c} n \\ l-1 \end{array} \right] \]

it may be tediously verified that \( B[J_s, k, r] \) does indeed satisfy recurrence (1.8).

It is known that \( r! \) divides the \((1, r)\)-Eulerian numbers. Identity (1.9) may be rewritten as

\[ M[J_s, k, r] = \left[ \begin{array}{c} j(r, r) \\ j_1 j_2 \ldots j_r \end{array} \right] \sum_{s=0}^{k} (-1)^i \left[ \begin{array}{c} n+1 \\ l \end{array} \right] \left[ \begin{array}{c} k-l+j(r, r) \\ k-l \end{array} \right] q^{i(l)} \]

\[ \times \prod_{m=r+1}^{s} \left[ \begin{array}{c} k-l+j(m, r) \\ j_m \end{array} \right] \]

which shows that the \( q \)-multinomial coefficient of \( j(r, r) \) divides the polynomial \( M[J_s, k, r] \).

5. THE GENERATING FUNCTION FOR THE \( R \)-MAJOR INDEX

Note that

\[ \sum_{k \geq 0} M[J_s, k, r] = \sum_f q^{|r\text{-maj}_f|} \quad (5.1) \]

summed over \( R(J_s) \). Identity (4.3) with \( t = 1 \) and \( |q| < 1 \), and the fact that \([n]! = (q)_n(1-q)^{-n}\) imply

\[ \sum_{k \geq 0} M[J_s, k, r] = (q)_n \prod_{m=1}^{s} \lim_{k \to \infty} \left[ \begin{array}{c} k+j(m, r) \\ j_m \end{array} \right] \]

\[ = (q)_n \prod_{m=1}^{s} (q)_j^{-1} = \left[ \begin{array}{c} n \\ j_1 j_2 \ldots j_s \end{array} \right] \quad (5.2) \]

which verifies (1.4).

6. THE CONSTRUCTION OF \( \Phi_{c,b} \)

Denote by \( U(j, l) \) the set of words \( u = u(1)u(2) \ldots u(j) \) with letters \( u(m) \in \{0, 1, \ldots, l-1\} \) for \( 1 \leq m \leq j \) and \( u(1) \leq u(2) \leq \cdots \leq u(j) \). Let \( W(J_s) \) be the set of
words \( w = u_1 u_2 \ldots u_s \) obtained by juxtaposing words \( u_m \in U(j_m, 1 + j(m-1, m-1)) \) for \( 1 \leq m \leq s \). Define \( \Sigma(z) \) to be the sum of the letters in a word \( z \). First, a bijection \( \Gamma_s : R(Js) \to W(Js) \) is constructed such that

\[
\text{if } \Gamma_s(f) = w \text{ then } r \text{-maj } f = \Sigma(w). \tag{6.1}
\]

It is then clear that the composition \( \Phi_{b,c} = \Gamma_c^{-1} \Gamma_b \) satisfies (1.5).

The construction of \( \Gamma_s \) is inductive. When \( s = 1 \), \( R(J1) \) and \( W(J1) \) consist respectively of the words \( f = 11 \ldots 1 \) and \( w = 00 \ldots 0 \) of length \( j_1 \). Clearly, \( r \text{-maj } f = \Sigma(w) = 0 \).

In general, from \( f \in R(Js) \) remove the \( s \) that has the minimum effect on the \( r \text{-major index} \). If there is more than one such \( s \), take the one furthest to the right. Let \( u_s(1) \) denote the resulting decrease in the \( r \text{-major index} \). Remove the next \( s \) with minimum effect and set \( u_s(2) \) equal to the decrease in the \( r \text{-major index} \). Repeat until all \( j_s \) letters equal to \( s \) have been removed from \( f \). Let \( g \) denote the resulting word and let \( u_s = u_s(1)u_s(2) \ldots u_s(j_s) \). It follows from the order of removal and the definition of \( u_s \) that

\[
\begin{align*}
(\text{a}) & \quad r \text{-maj } f = r \text{-maj } g + \Sigma(u_s) & \tag{6.2} \\
(\text{b}) & \quad u_s(1) \leq u_s(2) \leq \ldots \leq u_s(j_s). 
\end{align*}
\]

Part (b) of the insertion lemma implies that \( u_s(j_s) = j(s-1, s-1) \) showing that \( u_s \in U(j_s, 1 + j(s-1, s-1)) \). Induction then completes the construction.

As an example of the process, for \( r = 3 \) the subscripts of the letters \( s = 5 \) in the word

\[
f = 215_224315_25_12 \in R(2, 3, 1, 1, 3) \tag{6.3}
\]

indicate the order of removal of the first \( j_5 = 3 \) letters. Removal gives \( g = 2124312 \) and \( u_5 = 167 \). Iteration then leads to \( \Gamma_5(f) = u_1u_2 \ldots u_5 = 00012222167 \in W(2, 3, 1, 1, 3) \) with \( 3 \text{-maj } f = 21 = \Sigma(u_1u_2 \ldots u_5) \).

The correspondence \( \Gamma_r \) provides another proof of (1.4). Using \( \Gamma_r \) and the identity

\[
\sum q^{\Sigma(u_m)} = \left[ \begin{array}{c} j(m, m) \\ j_m \end{array} \right] \tag{6.4}
\]

summed over \( u_m \in U(j_m, 1 + j(m-1, m-1)) \), which may be found in Carlitz [3], the calculation

\[
\sum_f q^{r \text{-maj } f} = \prod_{m=1}^s \sum_{u_m} q^{\Sigma(u_m)} = \prod_{m=1}^s \left[ \begin{array}{c} j(m, m) \\ j_m \end{array} \right] = \left[ \begin{array}{c} n \\ j_1j_2 \ldots j_s \end{array} \right] \tag{6.5}
\]

where the first sum is over \( f \in R(Js) \), proves again that the \( q \)-multinomial coefficient of \( n \) is the generating function for the \( r \text{-major index} \). Carlitz [3] used calculation (6.5) to prove (1.4) in the case \( r \geq s \).
References


[12] D. Rawlings, The \( r \)-major index (submitted to *J. Combinatorial Theory* (A)).