The (q, r)-Simon Newcomb Problem

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A new statistic, the r-major index, is defined for sequences. A linear recurrence is then derived that enumerates sequences by r-descent number and r-major index.

1. INTRODUCTION

The classic Simon Newcomb problem may be described as follows. Let $Js = (j_1, j_2, \ldots, j_s)$ be a sequence of non-negative integers with $n = j_1 + j_2 \cdots + j_s$. Denote by $R(Js)$ the set of sequences $f = (f(1), f(2), \ldots, f(n))$ in which the integer m occurs j_m times. For an integer $r \geqslant 1$, the set of r-descents of f is defined as

$$\{i : f(i) \geqslant f(i+1) + r,\ 1 \leqslant i \leqslant n-1\}. \tag{1.1}$$

The cardinality of set (1.1) is denoted by r-des f. Then, the problem is to determine the number of sequences $f \in R(Js)$ with 1-des $f = k$.

The present paper has three objectives. First, the r-major index as introduced in [12] for permutations is extended to sequences. Denoted by r-maj f, the r-major index of f is defined to be the sum of the elements in set (1.1) plus the cardinality of the set

$$\{(i, j) : 1 \leqslant i < j \leqslant n,\ f(i) > f(j) > f(i) - r\}. \tag{1.2}$$

In the two special cases $r = 1$ and $r \geqslant s$, the r-major index respectively reduces to what are commonly known as the major index and the inversion number. It will be shown that if the q-analog, q-factorial, and q-multinomial coefficient of n are respectively defined as

$$\text{(a)}\ [n] = 1+q+\cdots+q^{n-1}$$

$$\text{(b)}\ [n]! = [1][2]\ldots[n]$$

$$\text{(c)}\ \begin{bmatrix} n \\ j_1 j_2 \ldots j_s \end{bmatrix} = \frac{[n]!}{[j_1]![j_2]!\ldots[j_s]!} \tag{1.3}$$

then the generating function for the r-major index is

$$\sum_f q^{r\text{-maj}\, f} = \begin{bmatrix} n \\ j_1 j_2 \ldots j_s \end{bmatrix} \tag{1.4}$$

summed over $R(Js)$, which generalizes the observation of MacMahon [11] that the major index and inversion number have the same generating function.

Second, as identity (1.4) suggests, for integers b, $c \geqslant 1$ there is a bijection $\Phi_{b,c}: R(Js) \to R(Js)$ with the property that

$$b\text{-maj}\ f = c\text{-maj}\ \Phi_{b,c}(f) \tag{1.5}$$

for all $f \in R(Js)$. The construction of $\Phi_{b,c}$ in section 6 will even provide a proof of (1.4). Foata [6] has given another such bijection in the case $b = 1$ and $c \geqslant s$.

The final and main objective is to solve the (q, r)-Simon Newcomb problem of enumerating sequences by r-descents and r-major index. It will be shown that for $Js = (j_1, j_2, \ldots, j_s)$ with

$$\text{(a)}\ j_s \geqslant 1$$

$$\text{(b)}\ j(m, r) = \begin{cases} j_m + j_{m-1} + \cdots + j_{m-r+1} & \text{if } m \geqslant r \\ j_m + j_{m-1} + \cdots + j_1 & \text{otherwise} \end{cases}$$

$$\text{(c)}\ Js - 1 = (j_1, j_2, \ldots, j_{s-1}, j_s - 1) \tag{1.6}$$

the polynomial

$$M[Js, k, r] = \sum_f q^{r\text{-maj}\, f} \tag{1.7}$$

summed over sequences $f \in R(Js)$ with r-des $f = k$ satisfies the recurrence

$$\begin{aligned}[] [j_s]M[Js, k, r] &= [k + j(s, r)]M[Js - 1, k, r] \\ &\quad + q^{k + j(s,r) - 1}[n + 1 - k - j(s, r)]M[Js - 1, k - 1, r] \end{aligned} \tag{1.8}$$

where $M[Jr, 0, r] = \begin{bmatrix} n \\ j_1 j_2 \ldots j_r \end{bmatrix}$. The more explicit formula

$$M[Js, k, r] = \sum_{l=0}^{k} (-1)^l \begin{bmatrix} n+1 \\ l \end{bmatrix} q^{\binom{l}{2}} \prod_{m=1}^{s} \begin{bmatrix} k - l + j(m, r) \\ j_m \end{bmatrix} \tag{1.9}$$

will then be derived from (1.8).

Numerous special cases of (1.8) and (1.9) may be found in the list references. When $r = 1$ identity (1.9) gives the solution to the q-Simon Newcomb problem as obtained by MacMahon [10, v. 2 p. 211]. Also see Gessel [9, p. 98]. The case $r = 1 = q$ is discussed in [1, 2, 5, 9, 10]. However, even in these cases, recurrence (1.8) seems to be new.

In the permutation case, $j_m = 1$ for $1 \leqslant m \leqslant s$, (1.7) becomes

$$A[n, k, r] = \sum_{\sigma} q^{r\text{-maj}\,\sigma} \tag{1.10}$$

summed over permutations σ of $\{1, 2, \ldots, n\}$ with k r-descents and (1.8) reduces to

$$A[n, k, r] = [k+r]A[n-1, k, r] + q^{k+r-1}[n+1-k-r]A[n-1, k-1, r] \tag{1.11}$$

where $A[r, 0, r] = [r]!$. Recurrence (1.11) defines the (q, r)-Eulerian numbers as developed in [12]. The $(1, r)$-Eulerian numbers have been studied in [7, 8, 13] and Carlitz [4] obtained recurrence (1.11) for the $(q, 1)$-Eulerian numbers.

2. THE INSERTION LEMMA

The construction of $\Phi_{b,c}$ and the proof of (1.8) are based on insertion. It will be convenient to regard a sequence $f \in R(Js)$ as the word $f(1)f(2)\ldots f(n)$ obtained by juxtaposing the letters $f(1), f(2), \ldots, f(n)$.

A colored word F is obtained by inserting an underlined s into an $f \in R(Js-1)$. The color of F, denoted by $C(F)$, is defined to be the number of times that s appears to the right of the underlined s in F. For example, with $s = 5$ and $f = 215243152 \in R(2, 3, 1, 1, 2)$ one may obtain

$$F = 2\,1\,5\,2\,4\,3\,1\,\underline{5}\,5\,2 \tag{2.1}$$

where $C(F) = 1$. The color is just a way of indicating which s was inserted.

To observe the effect that the insertion has on the r-major index and the r-descent number, the n possible insertion positions between the letters of $f = f(1)f(2)\ldots f(n-1) \in R(Js-1)$ are labeled as follows. Using the labels $\{0, 1, \ldots, n-1\}$ in order, first read from right to left and label the positions that will *not* result in the creation of a new r-descent. Then, reading back from left to right the remaining positions that *will* create an r-descent are labeled. For instance, if $r = 3$, $s = 5$, and $f = 215243152 \in R(2, 3, 1, 1, 2)$, then the labels in the top and bottom rows of

$$\begin{array}{cccccccccccccccccccc}
 & \bullet & & \bullet & & 6 & & 5 & & 4 & & 3 & & \bullet & & 2 & & 1 & & 0 \\
 & & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow \\
f = & & 2 & & 1 & & 5 & & 2 & & 4 & & 3 & & 1 & & 5 & & 2 & \\
 & \uparrow & & \uparrow & & & & & & & & & & \uparrow & & & & & & \\
 & 7 & & 8 & & \bullet & & \bullet & & \bullet & & \bullet & & 9 & & \bullet & & \bullet & & \bullet
\end{array} \tag{2.2}$$

respectively indicate the positions that will not and that will result in a new 3-descent.

Let $(f; l)$ denote the colored word obtained by inserting the underlined s into position l. For example (2.2), the colored word $(f; 2)$ is given in (2.1). Note that 3-des$(f; 2)$ = 3-des f and that 3-maj $(f; 2)+C(f; 2) = 2+$3-maj f. This demonstrates the

INSERTION LEMMA. For $f \in R(Js-1)$ and $0 \leqslant l \leqslant n-1$

(a) $r\text{-des}(f; l) = \begin{cases} r\text{-des}\, f & \text{if } 0 \leqslant l \leqslant r\text{-des}\, f + j(s,r)-1 \\ 1+r\text{-des}\, f & \text{otherwise} \end{cases}$

(b) $r\text{-maj}(f; l)+C(f; l) = l + r\text{-maj}\, f$.

Proof For (a), note that there are r-des $f+j(s, r)$ insertion positions that will not result in a new r-descent: preceding any of the $j(s, r)-1$ integers greater than $s-r$, in any of the r-descents, or at the extreme right end of f. As these positions are labeled first, (a) is immediate.

For (b), let m be the number of r-descents and integers in $\{s-r+1, s-r+2, \ldots, s-1\}$ that are to the right of position l. Let c be the number of times s appears to the right of position l. In the case $0 \leqslant l \leqslant r$-des $f+j(s,r)-1$, it follows from (1.1) and (1.2) that inserting s into position l will increase the r-major index by m. The color of the resulting word is c. By the labeling, $l = m+c$ and (b) follows in this case. For r-des $f+j(s,r) \leqslant l \leqslant n-1$, note that there are $l-m-c-1$ integers in f to the left of position l. As a new r-descent is created in this case, the color of the resulting word plus the increase in the r-major index is $c+(l-m-c)+m = l$.

3. PROOF OF RECURRENCE (1.8)

Let $R(Js, k, r) = \{f \in R(Js): r\text{-des}\, f = k\}$. Part (a) of the insertion lemma implies that the set of ordered pairs

$$\{(F, C(F)): F \in R(Js, k, r), C(F) \in \{0, 1, \ldots, j_s-1\}\} \tag{3.1}$$

regarded as colored words is the disjoint union of the sets

$$\text{(a)}\quad \{(f; l): f \in R(Js-1, k, r), 0 \leqslant l \leqslant k+j(s,r)-1\}$$

$$\text{(b)}\quad \{(g; m): g \in R(Js-1, k-1, r), k+j(s,r)-1 \leqslant m \leqslant n-1\}. \tag{3.2}$$

The following calculation based on (b) of the insertion lemma

$$
\begin{aligned}
[j_s]M[Js, k, r] &= \sum_{(F, C(F))} q^{r\text{-maj } F + C(F)} \\
&= \sum_{(f;\,l)} q^{r\text{-maj}(f;\,l) + C(f;\,l)} + \sum_{(g;\,m)} q^{r\text{-maj}(g;\,m) + C(g;\,m)} \\
&= \sum_{l} q^{l} \sum_{f} q^{r\text{-maj } f} + \sum_{m} q^{m} \sum_{g} q^{r\text{-maj } g}
\end{aligned}
$$

summed respectively over the sets of (3.1) and (3.2) establishes (1.8).

To show that $M[Jr, o, r]$ is equal to the q-multinomial coefficient of n, first note that $M[J1, o, r] = 1$ and for all $s \leqslant r$ identity (1.8) reduces to

$$[j_s]M[Js, o, r] = [n]M[Js-1, o, r]. \tag{3.3}$$

Iteration of (3.3) yields

$$M[Jr, o, r] = \frac{[n]}{[j_r]} M[Jr-1, o, r] = \begin{bmatrix} n \\ j_1 j_2 \ldots j_r \end{bmatrix}.$$

4. THE EXPLICIT SOLUTION

Let $(t;q)_{n+1} = (1-t)(1-tq)\ldots(1-tq^n)$ and $(q)_n = (q;q)_n = (1-q)(1-q^2)\ldots(1-q^n)$. For a reference of the identities

$$
\begin{aligned}
&\text{(a)}\ (t;q)_{n+1} = \sum_{k \geqslant 0} (-1)^k \begin{bmatrix} n+1 \\ k \end{bmatrix} q^{\binom{k}{2}} t^k \\
&\text{(b)}\ (t;q)_{n+1}^{-1} = \sum_{k \geqslant 0} \begin{bmatrix} n+k \\ k \end{bmatrix} t^k
\end{aligned}
\tag{4.1}
$$

see [2, p. 36]. Let $B[Js, k, r]$ denote the polynomial on the right-hand side of (1.9). To prove (1.9) it suffices to show that

$$B[Jr, k, r] = \begin{cases} \begin{bmatrix} n \\ j_1 j_2 \ldots j_r \end{bmatrix} & \text{if } \ k = 0 \\ 0 & \text{otherwise} \end{cases} \tag{4.2}$$

and that $B[Js, k, r]$ satisfies recurrence (1.8).

First, note that (a) of (4.1) implies

$$\sum_{k \geqslant 0} B[Js, k, r] t^k = (t;q)_{n+1} \sum_{k \geqslant 0} t^k \prod_{m=1}^{s} \begin{bmatrix} k + j(m, r) \\ j_m \end{bmatrix}. \tag{4.3}$$

In the case $r = s$, (4.3) and (b) of (4.1) show that

$$\sum_{k\geq 0} B[Jr,k,r]t^k = \begin{bmatrix} n \\ j_1 j_2 \ldots j_r \end{bmatrix} (t;q)_{n+1} \sum_{k\geq 0} \begin{bmatrix} n+k \\ k \end{bmatrix} t^k = \begin{bmatrix} n \\ j_1 j_2 \ldots j_r \end{bmatrix}$$

which checks (4.2). Then using the identities

$$\text{(a)}\quad [j_s]\begin{bmatrix} k-l+j(s,r) \\ j_s \end{bmatrix} = [k-l+j(s,r)]\begin{bmatrix} k-l+j(s,r)-1 \\ j_s-1 \end{bmatrix} \tag{4.4}$$

$$\text{(b)}\quad [k-l+j(s,r)] = [k+j(s,r)] - q^{k-l+j(s,r)}[l]$$

$$\text{(c)}\quad \begin{bmatrix} n+1 \\ l \end{bmatrix} = \begin{bmatrix} n \\ l \end{bmatrix} + q^{n+1-l}\begin{bmatrix} n \\ l-1 \end{bmatrix}$$

it may be tediously verified that $B[Js,k,r]$ does indeed satisfy recurrence (1.8).

It is known that $r!$ divides the $(1,r)$-Eulerian numbers. Identity (1.9) may be rewritten as

$$M[Js,k,r] = \begin{bmatrix} j(r,r) \\ j_1 j_2 \ldots j_r \end{bmatrix} \sum_{l=0}^{k} (-1)^l \begin{bmatrix} n+1 \\ l \end{bmatrix} \begin{bmatrix} k-l+j(r,r) \\ k-l \end{bmatrix} q^{\binom{l}{2}}$$
$$\times \prod_{m=r+1}^{s} \begin{bmatrix} k-l+j(m,r) \\ j_m \end{bmatrix}$$

which shows that the q-multinomial coefficient of $j(r,r)$ divides the polynomial $M[Js,k,r]$.

5. THE GENERATING FUNCTION FOR THE R-MAJOR INDEX

Note that

$$\sum_{k\geq 0} M[Js,k,r] = \sum_f q^{r\text{-maj}\, f} \tag{5.1}$$

summed over $R(Js)$. Identity (4.3) with $t = 1$ and $|q| < 1$, and the fact that $[n]! = (q)_n(1-q)^{-n}$ imply

$$\sum_{k\geq 0} M[Js,k,r] = (q)_n \prod_{m=1}^{s} \lim_{k\to\infty} \begin{bmatrix} k+j(m,r) \\ j_m \end{bmatrix}$$
$$= (q)_n \prod_{m=1}^{s} (q)_{jm}^{-1} = \begin{bmatrix} n \\ j_1 j_2 \ldots j_s \end{bmatrix} \tag{5.2}$$

which verifies (1.4).

6. THE CONSTRUCTION OF $\Phi_{c,b}$

Denote by $U(j,l)$ the set of words $u = u(1)u(2)\ldots u(j)$ with letters $u(m) \in \{0, 1, \ldots, l-1\}$ for $1 \leq m \leq j$ and $u(1) \leq u(2) \leq \cdots \leq u(j)$. Let $\mathrm{W}(Js)$ be the set of words $w = u_1 u_2 \ldots u_s$ obtained by juxtaposing words $u_m \in U(j_m, 1+j(m-1, m-1))$ for $1 \leqslant m \leqslant s$. Define $\Sigma(z)$ to be the sum of the letters in a word z. First, a bijection $\Gamma_r: R(Js) \to W(Js)$ is constructed such that

$$\text{if} \quad \Gamma_r(f) = w \quad \text{then} \quad r\text{-maj } f = \Sigma(w). \tag{6.1}$$

It is then clear that the composition $\Phi_{b,c} = \Gamma_c^{-1}\Gamma_b$ satisfies (1.5).

The construction of Γ_r is inductive. When $s = 1$, $R(J1)$ and $W(J1)$ consist respectively of the words $f = 1\,1 \ldots 1$ and $w = 0\,0 \ldots 0$ of length j_1. Clearly, $r\text{-maj } f = \Sigma(w) = 0$.

In general, from $f \in R(Js)$ remove the s that has the minimum effect on the r-major index. If there is more than one such s, take the one furthest to the right. Let $u_s(1)$ denote the resulting decrease in the r-major index. Remove the next s with minimum effect and set $u_s(2)$ equal to the decrease in the r-major index. Repeat until all j_s letters equal to s have been removed from f. Let g denote the resulting word and let $u_s = u_s(1)u_s(2) \ldots u_s(j_s)$. It follows from the order of removal and the definition of u_s that

$$\begin{aligned} &\text{(a)}\ \ r\text{-maj } f = r\text{-maj } g + \Sigma(u_s) \\ &\text{(b)}\ \ u_s(1) \leqslant u_s(2) \leqslant \cdots \leqslant u_s(j_s). \end{aligned} \tag{6.2}$$

Part (b) of the insertion lemma implies that $u_s(j_s) \leqslant j(s-1, s-1)$ showing that $u_s \in U(j_s, 1+j(s-1, s-1))$. Induction then completes the construction.

As an example of the process, for $r = 3$ the subscripts of the letters $s = 5$ in the word

$$f = 2\,1\,5_2\,2\,4\,3\,1\,5_3\,5_1\,2 \in R(2, 3, 1, 1, 3) \tag{6.3}$$

indicate the order of removal of the first $j_5 = 3$ letters. Removal gives $g = 2\,1\,2\,4\,3\,1\,2$ and $u_5 = 1\,6\,7$. Iteration then leads to $\Gamma_3(f) = u_1 u_2 \ldots u_5 = 0\,0\,0\,1\,2\,2\,2\,1\,6\,7 \in W(2, 3, 1, 1, 3)$ with $3\text{-maj } f = 21 = \Sigma(u_1 u_2 \ldots u_5)$.

The correspondence Γ_r provides another proof of (1.4). Using Γ_r and the identity

$$\sum q^{\Sigma(u_m)} = \begin{bmatrix} j(m, m) \\ j_m \end{bmatrix} \tag{6.4}$$

summed over $u_m \in U(j_m, 1+j(m-1, m-1))$, which may be found in Carlitz [3], the calculation

$$\sum_f q^{r\text{-maj } f} = \prod_{m=1}^{s} \sum_{u_m} q^{\Sigma(u_m)} = \prod_{m=1}^{s} \begin{bmatrix} j(m, m) \\ j_m \end{bmatrix} = \begin{bmatrix} n \\ j_1 j_2 \cdots j_s \end{bmatrix} \tag{6.5}$$

where the first sum is over $f \in R(Js)$, proves again that the q-multinomial coefficient of n is the generating function for the r-major index. Carlitz [3] used calculation (6.5) to prove (1.4) in the case $r \geqslant s$.

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