Necessary and Sufficient Condition that the Limit of Stieltjes Transforms is a Stieltjes Transform

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Abstract

The pointwise limit \(S\) of a sequence of Stieltjes transforms \((S_n)\) of real Borel probability measures \((P_n)\) is itself the Stieltjes transform of a Borel p.m. \(P\) if and only if \(iyS(iy) \to -1\) as \(y \to \infty\), in which case \(P_n\) converges to \(P\) in distribution. Applications are given to several problems in mathematical physics.

Key words and phrases: real Borel probability measure, convergence in distribution, Stieltjes transform, Lévy continuity theorem, Akhiezer-Krein theorem, weak convergence of probability measures.

Lévy’s classical continuity theorem says that if the pointwise limit of the characteristic functions of a sequence of real Borel probability measures \((P_n)\) exists, then the limit function \(\varphi\) is itself the characteristic function for a probability measure \(P\) if and only if \(\varphi\) is continuous at zero, in which case \(P_n \to P\) in distribution. The purpose of this note is to prove a direct analog of Lévy’s theorem for Stieltjes transforms, complementing those for other representing functions in [HS] and [HK], and to give several examples of applications.

Throughout this note, \(\mathbb{R}\) and \(\mathbb{C}\) denote the real and complex numbers, respectively; p.m. and s.p.m. denote Borel probability measures, and sub-probability (mass \(\leq 1\)) measures, respectively, on \(\mathbb{R}\); and s.p.m.’s \((\mu_n)\) converge vaguely to a s.p.m. \(\mu\) [C, p. 80], if there exists a dense subset \(D\) of \(\mathbb{R}\) such that for all \(a, b \in D, a < b, \mu_n((a, b]) \to \mu((a, b])\). (Thus if \((\mu_n), \mu\) are p.m.’s, vague convergence is equivalent to convergence in distribution.)

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Definition 1. The Stieltjes transform $S_P$ of a p.m. $P$ is the function $S_P : \{\text{Im}(z) > 0\} \to \mathbb{C}$ given by

$$S_P(z) = \int_{-\infty}^{\infty} \frac{1}{w - z} dP(w).$$

A basic property of Stieltjes transforms, which has important applications in the theory of moments (cf. [A], [S1], [ST]) and in mathematical physics ( ), is that they are a representing class for finite measures.

Lemma 1. For s.p.m.’s $P$ and $Q$, $P = Q$ iff $S_P = S_Q$.

Proof. Follows immediately from the Stieltjes transform inversion formula [A, p. 125].

Just as limits of characteristic functions of p.m.’s are in general not characteristic functions, and limits of Hardy-Littlewood functions or expected-extrema functions are not in general Hardy-Littlewood or expected-extrema functions [HK], limits of Stieltjes transforms are not always Stieltjes transforms, as the next easy example shows.

Example 1. For $n = 1, 2, \ldots$, let $P_n = \delta_n$, the Dirac point mass at $n$. Then $S_{P_n}(z) = (n - z)^{-1}$ for all $n$ and all $z$ with $\text{Im}(z) > 0$, so $\lim_{n \to \infty} S_{P_n}(z) \equiv 0$, which is clearly not the Stieltjes transform for any p.m. $P$ (see Lemma 2 below).

On the other hand, just as with Lévy’s theorem, the limit of Stieltjes transforms is itself a Stieltjes transform if and only if it satisfies one single universal limit condition. The next theorem is the main result of this note.

Theorem 1. Suppose that $(P_n)$ are real Borel probability measures with Stieltjes transforms $(S_n)$, respectively. If $\lim_{n \to \infty} S_n(z) = S(z)$ for all $z$ with $\text{Im}(z) > 0$, then there exists a Borel probability measure $P$ with Stieltjes transform $S_P = S$ if and only if

$$\lim_{y \to \infty} iyS(iy) = -1,$$

in which case $P_n \to P$ in distribution.

Corollary 1. If $P, (P_n)$ are real Borel p.m.’s with Stieltjes transforms $S, (S_n)$, respectively, then $P_n \to P$ in distribution if and only if $S_n \to S$ pointwise.
Proof of Corollary. If $S_n \to S$, then $P_n \to P$ in distribution by Theorem 1. Conversely, suppose that $P_n \to P$ in distribution. Since $f_z(w) := (w - z)^{-1}$ is continuous and bounded in $w$ for fixed $z$ in $\{\text{Im}(z) > 0\}$, then $\text{Im}(f_z)$ and $\text{Re}(f_z)$ are also continuous and bounded, so by the basic equivalence of convergence in distribution of p.m.’s and convergence of integrals of bounded continuous functions [C, Theorem 4.4.2], $\int \text{Im}(f_z)dP_n \to \int \text{Im}(f_z)dP$ and $\int \text{Re}(f_z)dP_n \to \int \text{Re}(f_z)dP$, so $S_n(z) \to S(z)$. □

To facilitate the proof of Theorem 1, two additional lemmas are useful, which are stated here for ease of reference.

Lemma 2. Let $S : \{\text{Im}(z) > 0\} \to \mathbb{C}$ be analytic. Then there exists a p.m. $P$ with $S_P(z) = S(z)$ for all $z$ with $\text{Im}(z) > 0$ if and only if (1) holds and

$$\text{Im}(S(z)) > 0 \quad \text{for all } z \text{ with } \text{Im}(z) > 0. \quad (2)$$

Proof. By the classical Akhiezer-Krein theorem [A, p. 93], $S = S_P$ for some finite Borel measure $P$ if and only if: $S$ is analytic in $\{\text{Im}(z) > 0\}$; $S$ satisfies (2); and

$$\sup_{y \geq 1} |yS(iy)| < \infty. \quad (3)$$

Suppose $P$ is a p.m. with $S = S_P$. The Akhiezer-Krein theorem implies that (2) holds, and (1) follows immediately from the definition of $S_P$. Conversely, suppose that $S$ is analytic and satisfies (1) and (2). Since $yS(iy)$ is continuous in $y$, (1) easily implies (3), so by the Akhiezer-Krein theorem again, there is a finite Borel measure $P$ with $S_P = S$. Since clearly $\lim_{y \to \infty} [-iyS_P(iy)] = \text{mass}(P)$, (1) implies that $P$ is a p.m. □

Lemma 3. Let $\mathcal{F}$ be a family of functions analytic in a connected open domain $D$. If for each compact $K \subset D$ there exists a constant $M(K) < \infty$ such that

$$|f(z)| \leq M(K) \quad \text{for all } z \in K \text{ and all } f \in \mathcal{F}, \quad (4)$$

then all pointwise limits of functions in $\mathcal{F}$ are also analytic in $D$. 
**Proof.** ([H, Theorem 15.2.3]). □

**Proof of Theorem 1.** If \( \lim S_n = S = S_P \) for some p.m. \( P \), then (1) follows by Lemma 2.

Conversely, suppose that \( S = \lim S_n \) satisfies (1). Let \( F = \{ \bigcup S_n \} \), and for \( K \subset D := \{ \Im(z) > 0 \} \), let \( d(K) = \inf \{ \|y - z\| : y \in \mathbb{R}, z \in K \} \) be the smallest distance from \( K \) to the real line. Clearly \( 0 < d(K) < \infty \) for all compact \( K \subset D \), and \( M(K) = 1/d(K) \) satisfies (4), so Lemma 3 implies that \( S = \lim S_n \) is analytic in \( D \). By Lemma 2, \( \Im(S_n(z)) > 0 \) for all \( z \in D \), so \( \Im(S(z)) \geq 0 \) for all \( z \in D \). Suppose, by way of contradiction to (2), that \( \Im(S(z_0)) = 0 \) for some \( z_0 \in D \). Since \( S \) is analytic, \( \Im(S) \) and \( \Re(S) \) are harmonic on \( D \) [K, p. 590]. By the maximum principle [K, p. 760], a non-constant function which is harmonic in a simply connected bounded open set \( G \) has neither a maximum nor a minimum in \( G \), so since \( \Im(S(z)) \geq 0 \) on \( G \) for every simply connected open bounded set \( G \) with \( z_0 \in G \subset D \), it follows (taking \( G_t = \{ z \in D : \|z\| < t \} \), and letting \( t \to \infty \)) that \( \Im(S(z)) \equiv 0 \) for all \( z \in D \), which contradicts (1). Thus (2) holds, and since \( S \) is analytic and (1) holds by assumption, Lemma 2 implies there exists a real Borel p.m. \( P \) with \( S_P = S \).

For the convergence in distribution conclusion, suppose that \( S_n = S_{P_n} \to S_P \) pointwise in \( D \) for p.m.’s \((P_n)\), \( P \). By the Helly selection theorem [C, Theorem 4.3.3], there exists a s.p.m. \( Q \) and a subsequence \((P_{n_k})\) of \((P_n)\) such that \( P_{n_k} \to Q \) vaguely. Fix \( z \) in \( D \), and let \( f_z : \mathbb{R} \to \mathbb{C} \) be given by \( f_z(w) = (w - z)^{-1} \). Since \( f_z \) is continuous in \( w \) and vanishes at infinity, \( \Re(f_z) \) and \( \Im(f_z) \) are continuous and vanish at infinity, so it follows by the equivalence of vague convergence of s.p.m.’s and convergence of integrals of continuous functions which vanish at infinity [C, Theorem 4.4.1] that \( S_{P_{n_k}}(z) \to S_Q(z) \) as \( k \to \infty \) for all \( z \in D \). By hypothesis, \( S_{P_n} \to S_P \), so \( S_P = S_Q \), which by Lemma 1 implies that \( P = Q \). Since every vaguely convergent subsequence of \((P_n)\) thus converges to \( P \), this implies [C, Theorem 4.3.4] that \( P_n \) converges vaguely to \( P \), that is, since \((P_n)\) and \( P \) are p.m.’s, \( P_n \) converges to \( P \) in distribution. □

**Sketch of Alternative Proof.** (B. Simon [S2]). The functions \( \{S_{P_n}\} \) are Herglotz functions, so the limit \( S \) is Herglotz, and pointwise convergence \( S_n \to S \) implies weak convergence for the measures \((1 + x^2)^{-1}dP \) to a measure \( P \) on \([-\infty, \infty]\), where \((1 + x^2)^{-1}dP \) is finite. Given that \( S(iy) \to 0 \) as \( y \to \infty \), \( P\{\infty, \infty\} = 0 \), so it follows using the fact...
that $S_{P_n} \to S_P$, the Herglotz representation theorem, and the monotone convergence theorem, that $P$ is a p.m. Then weak convergence of $P_n$ to $P$ can be shown given the weak convergence of the measures when multiplied by $(1 + x^2)^{-1}$. (For related ideas, see pp. 129–130 in [S1]).
References


