Maximin share and minimax envy in fair-division problems

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Abstract

For fair-division or cake-cutting problems with value functions which are normalized positive measures (i.e., the values are probability measures) maximin-share and minimax-envy inequalities are derived for both continuous and discrete measures. The tools used include classical and recent basic convexity results, as well as ad hoc constructions. Examples are given to show that the envy-minimizing criterion is not Pareto optimal, even if the values are mutually absolutely continuous. In the discrete measure case, sufficient conditions are obtained to guarantee the existence of envy-free partitions.

Keywords Fair-division; Cake-cutting; Maximin share; Minimax envy; Envy-free; Optimal partition; Equitable partition

1. Introduction

The subject of this paper is fair-division or cake-cutting inequalities (cf. [5,6,11]), and in particular, the relationship among various notions of optimality such as maximin share, minimax envy, and Dubins–Spanier optimality. A cake Ω is to be divided among n players

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whose relative values $v_1, \ldots, v_n$ of the various parts of the cake may differ. A partition of the cake into n pieces $P_1, \ldots, P_n$ is sought so that the resulting values $v_i(P_j)$ make the minimum perceived share as large as possible, or make the maximum envy as small as possible.

The formal framework is as follows. There are n (countably additive) probability measures $v_1, \ldots, v_n$ on the same measurable space $(\Omega, \mathcal{F})$, where Ω represents the cake and $\mathcal{F}$ is the σ-algebra of subsets of Ω which represents the collection of feasible pieces. For each $P \in \mathcal{F}$ and each i, $v_i(P)$ represents the value of piece P to player i. (Hence, in this setting, the feasible pieces always include the whole cake, and are closed under complements and countable unions; and the value functions are additive.)

Throughout this paper, Π_n will denote the collection of $\mathcal{F}$-measurable n partitions of Ω, that is,

$$\Pi_n = \left\{ (P_1, \ldots, P_n)\colon\ P_i \in \mathcal{F} \text{ for all } i,\ P_i \cap P_j = \emptyset \text{ if } i \neq j, \text{ and } \bigcup_{i=1}^{n} P_i = \Omega \right\},$$

and a typical element $\mathbb{P} \in \Pi_n$ is the partition $\mathbb{P} = (P_1, \ldots, P_n)$ representing allocation of P_i to player i for all $i = 1, \ldots, n$.

This paper is organized as follows: Section 2 contains definitions and examples of the value matrix, maximin optimality and fair partitions, as well as the main compactness and convexity theorem for value matrices due to Dubins and Spanier [6]; Section 3 contains the analogous convexity/compactness result for envy matrices, a proof that even in the mutually absolutely continuous case, a Dubins–Spanier optimal partition need not be envy-free, and several results guaranteeing the existence of *quantifiably* super-fair envy-free and super-envy-free partitions; and Section 4 contains minimax-envy inequalities for general measures (including measures with atoms) whose bounds are functions of the maximum atom size.

2. Fair and Dubins–Spanier-optimal partitions

Denote by $M(n \times n)$ the set of real-valued $n \times n$ matrices.

Definition 2.1. The *value matrix* $M_{\mathrm{V}}(\mathbb{P})$ of a partition $\mathbb{P}$ is the matrix whose entries are the values of the pieces of the partition to the respective players, that is, $M_{\mathrm{V}} : \Pi_n \to M(n \times n)$ is given by

$$M_{\mathrm{V}}(\mathbb{P}) = M_{\mathrm{V}}\big((P_1, \ldots, P_n)\big) = \big(v_i(P_j)\big)_{i,j=1}^{n},$$

and the *set of $\mathcal{F}$-feasible value matrices* $\mathcal{M}_{\mathrm{V}}$ is given by

$$\mathcal{M}_{\mathrm{V}} = \big\{ M_{\mathrm{V}}(\mathbb{P})\colon\ \mathbb{P} \in \Pi_n \big\} \subset M(n \times n).$$

Example 2.2. Let $(\Omega, \mathcal{F}) = ([0,1], \text{Borels})$, $n = 2$, $v_1 =$ uniform distribution on $[0,1]$, and $v_2 =$ probability measure on $[0,1]$ with distribution function $F_2(x) = x^2$, $0 \leqslant x \leqslant 1$. Then for $\mathbb{P}_1 = ([0, 1/2), [1/2, 1])$ and $\mathbb{P}_2 = ([0, (\sqrt{5}-1)/2), [(\sqrt{5}-1)/2, 1])$,

$$M_{\mathrm{V}}(\mathbb{P}_1) = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix} \quad \text{and} \quad M_{\mathrm{V}}(\mathbb{P}_2) = \begin{pmatrix} \frac{\sqrt{5}-1}{2} & \frac{3-\sqrt{5}}{2} \\ \frac{3-\sqrt{5}}{2} & \frac{\sqrt{5}-1}{2} \end{pmatrix},$$

and an easy calculation shows that

$$\mathcal{M}_{\mathrm{V}} = \left\{ \begin{pmatrix} x & 1-x \\ 1-y & y \end{pmatrix} :\ 0 \leqslant x \leqslant 1,\ (1-x)^2 \leqslant y \leqslant 1-x^2 \right\}.$$

Example 2.3. Let $(\Omega, \mathcal{F}) = ([0,1], \text{Borels})$, $n = 2$, $v_1 = v_2 = \delta_{(1/2)}$, the Dirac point mass at $\{1/2\}$, and let $\mathbb{P}_1, \mathbb{P}_2$ be as in Example 2.2. Then

$$M_{\mathrm{V}}(\mathbb{P}_1) = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \qquad M_{\mathrm{V}}(\mathbb{P}_2) = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix},$$

and

$$\mathcal{M}_{\mathrm{V}} = \left\{ \begin{pmatrix} x & 1-x \\ x & 1-x \end{pmatrix} :\ x = 0 \text{ or } x = 1 \right\}.$$

The next result, a consequence of Lyapounov's convexity theorem due to Dubins and Spanier, is one of the main tools in measure-theoretic fair-division problems, and is recorded here for ease of reference. (Recall that a measure v is *atomless* if for every $P \in \mathcal{F}$ with $v(P) > 0$, there exists a set $A \in \mathcal{F}$, $A \subset P$ with $0 < v(A) < v(P)$; for Borel measures on the real line, this is equivalent to $v(\{x\}) = 0$ for every $x \in \mathbb{R}$.)

Proposition 2.4 [6]. *Fix $n \geqslant 1$ and $v_1, \ldots, v_n$ probability measures on $(\Omega, \mathcal{F})$. Then*

(i) $\mathcal{M}_{\mathrm{V}}$ *is compact* (*as a subset of real $n \times n$ matrices*); *and*
(ii) *if each v_i is atomless, then $\mathcal{M}_{\mathrm{V}}$ is convex.*

Remarks. Note that the measures in Example 2.2 are atomless, and hence that the set of feasible value matrices $\mathcal{M}_{\mathrm{V}}$ is convex. In Example 2.3, on the other hand, v_1 and v_2 are purely atomic, and $\mathcal{M}_{\mathrm{V}}$ is far from convex. It is also easy to check that $\mathcal{M}_{\mathrm{V}}$ may be convex even if $\{v_i\}$ are atomic; for example, by taking $n = 2$ and $v_1 = v_2$ defined by $v_1(\{x\}) = x$ for $x = 2^{-n}$, $n = 1, 2, \ldots$, and $v_1(x) = 0$ otherwise, in which case

$$\mathcal{M}_{\mathrm{V}} = \left\{ \begin{pmatrix} x & 1-x \\ x & 1-x \end{pmatrix} :\ 0 \leqslant x \leqslant 1 \right\}.$$

Definition 2.5. A partition $\mathbb{P} = (P_1, \ldots, P_n)$ is *fair* if $v_i(P_i) \geqslant 1/n$ for all i; is *equitable* if $v_i(P_i) = v_j(P_j)$ for all i, j; is *maximin optimal* if $\min_{1 \leqslant i \leqslant n} v_i(P_i) \geqslant \min_{1 \leqslant i \leqslant n} v_i(\hat{P}_i)$ for all $\hat{\mathbb{P}} = (\hat{P}_1, \ldots, \hat{P}_n) \in \Pi_n$; and is *Dubins–Spanier optimal* (DS optimal) if $(v_{\langle 1 \rangle}(P_{\langle 1 \rangle}), \ldots, v_{\langle n \rangle}(P_{\langle n \rangle})) \succ (v_{\langle 1 \rangle}(\hat{P}_{\langle 1 \rangle}), \ldots, v_{\langle n \rangle}(\hat{P}_{\langle n \rangle}))$ for all $\hat{\mathbb{P}} = (\hat{P}_1, \ldots, \hat{P}_n) \in \Pi_n$, where $v_{\langle i \rangle}(P_{\langle i \rangle})$ are the increasing order statistics of the $\{v_i(P_i)\}$ (i.e., $v_{\langle 1 \rangle}(P_{\langle 1 \rangle}) \leqslant v_{\langle 2 \rangle}(P_{\langle 2 \rangle}) \leqslant \cdots \leqslant$

$v_{\langle n\rangle}(P_{\langle n\rangle}))$, and "$\succ$" is the real lexicographic order. In other words, $\mathbb{P}$ is DS optimal if the smallest share $\min_{1\leqslant i\leqslant n} v_i(P_i)$ is as large as possible among all possible partitions, and among all partitions attaining that maximin, the second smallest share is as large as possible, and so forth.

Remarks. As shown in [6], it follows from Proposition 2.4(i) that maximin-optimal and DS-optimal partitions always exist; and from Proposition 2.4(ii) that if the $\{v_i\}$ are atomless, that fair equitable partitions always exist, and that every DS-optimal partition is fair. Without the assumption of atomless measures, DS-optimal partitions may not be fair, as is easily seen in Example 2.3.

3. Envy-minimizing partitions

A recent alternative to the objective of maximizing one's own share $v_i(P_i)$, is the objective of minimizing one's envy of other's shares $v_i(P_j) - v_i(P_i)$ (cf. [3–5,14]). Clearly the two objectives are related, but as the next example points out, players trying to minimize envy would sometimes reject a given partition in favor of one which gives *every player a much smaller share*. In this example, the players would reject an equitable partition which allocates each player very nearly 50% of his own value of the cake (but with an accompanying miniscule amount of envy) in favor of an envy-free partition which allots each player a piece he feels is worth exactly 1% of the total value. In particular, the example shows that the envy-minimizing objective is not Pareto optimal.

Example 3.1. Let $(\Omega, \mathcal{F}) = ([0, 100], \text{Borels})$, $n = 100$, let v_i be uniform on $[i-1, i+1)$ for $i = 1, \dots, 99$, and let v_{100} be uniform on $[99, 100] \cup [0, 1)$. Let $\mathbb{P} = (P_1, P_2, \dots, P_{100})$ be given by $P_i = [i + 0.0001, i + 1.0001)$, $i = 1, \dots, 98$, $P_{99} = [99.0001, 100] \cup [0, 0.0001)$, and $P_{100} = [0.0001, 1.0001)$; and let $\hat{\mathbb{P}} = (\hat{P}_1, \hat{P}_2, \dots, \hat{P}_{100})$ be $\hat{P}_i = \bigcup_{k=0}^{99} [k + (i-1)/100, k + i/100)$, $i = 1, \dots, 100$. It is easily checked that, for each i, $v_i(P_i) = 0.49995$ and the envy of player i (see Definition 3.2 below) is 0.00005 for each i. On the other hand, with partition $\hat{\mathbb{P}}$, each player receives a piece worth exactly $v_i(\hat{P}_i) = 0.01$, but no player values any other piece more than his own. Thus players seeking to minimize envy would choose $\mathbb{P}_2$ over $\mathbb{P}_1$ and reduce their shares uniformly by nearly a factor of 50.

In the above example, however, it is easy to see that there is a partition (namely $P_i = [i-1, i)$ for all i) which is simultaneously envy-free, DS optimal, equitable and fair, and which assigns each player a share he values exactly 50% of the cake. It is the purpose of this section to record several basic properties of envy, to investigate the interrelationship among these various notions of optimality, and to derive several general inequalities for upper bounds on envy.

Definition 3.2. The *envy of a partition* $\mathbb{P}$ *to player* i, $e_i(\mathbb{P})$, is $e_i(\mathbb{P}) = \max_{1\leqslant j\neq i\leqslant n} v_i(P_j) - v_i(P_i)$; the *maximum envy of* $\mathbb{P}$, $e_{\max}(\mathbb{P})$, is $e_{\max}(\mathbb{P}) = \max_{1\leqslant i\leqslant n} e_i(\mathbb{P})$; the *envy matrix of* $\mathbb{P}$, $M_{\mathrm{E}}(\mathbb{P})$ is the element in $M(n \times n)$ with (i, j)th entry $e_{i,j} = v_i(P_j) - v_i(P_i)$; and the *set of $\mathcal{F}$-feasible envy matrices* $\mathcal{M}_{\mathrm{E}}$ is the subset of $M(n \times n)$ given by $\mathcal{M}_{\mathrm{E}} = \{M_{\mathrm{E}}(\mathbb{P})\colon \mathbb{P} \in \Pi_n\}$.

(Note that the definition of envy here is the negative of that in [14]; here positive envy reflects valuing another's piece more than one's own, and the objective is to minimize envy.)

Example 3.3. (i) For the problem in Example 2.2,

$$M_{\mathrm{E}}(\mathbb{P}_1) = \begin{pmatrix} 0 & 0 \\ -\frac{1}{2} & 0 \end{pmatrix}, \qquad M_{\mathrm{E}}(\mathbb{P}_2) = \begin{pmatrix} 0 & 2-\sqrt{5} \\ 2-\sqrt{5} & 0 \end{pmatrix},$$

and

$$\mathcal{M}_{\mathrm{E}} = \left\{ \begin{pmatrix} 0 & 1-2x \\ 1-2y & 0 \end{pmatrix}\colon\ 0 \leqslant x \leqslant 1,\ (1-x)^2 \leqslant y \leqslant 1-x^2 \right\}.$$

(ii) For the problem in Example 2.3,

$$M_{\mathrm{E}}(\mathbb{P}_1) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad M_{\mathrm{E}}(\mathbb{P}_2) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

and

$$\mathcal{M}_{\mathrm{E}} = \left\{ \begin{pmatrix} 0 & 1-2x \\ 2x-1 & 0 \end{pmatrix}\colon\ x = 0 \text{ or } 1 \right\}.$$

Lemma 3.4. (i) $\dim(\mathcal{M}_{\mathrm{V}}) = \dim(\mathcal{M}_{\mathrm{E}})$; (ii) *the function from* $\mathcal{M}_{\mathrm{V}} \to \mathcal{M}_{\mathrm{E}}$ *defined by* $M_{\mathrm{V}}(\mathbb{P}) \mapsto M_{\mathrm{E}}(\mathbb{P})$ *is one-to-one, onto, and affine.*

Proof. Conclusion (i) is a direct consequence of (ii). To see (ii), note that $\{v_i(P_j)\}_{i,j=1}^n$ clearly determines $\{v_i(P_j) - v_i(P_i)\}_{i,j=1}^n$; conversely, the sum of the envy entries in the ith row,

$$\sum_{j=1}^{n} \bigl(v_i(P_j) - v_i(P_i)\bigr) = \sum_{j=1}^{n} v_i(P_j) - n v_i(P_i) = 1 - n v_i(P_i),$$

so

$$\begin{aligned} v_i(P_j) &= v_i(P_j) - v_i(P_i) + v_i(P_i) \\ &= v_i(P_j) - v_i(P_i) + n^{-1}\left(1 - \sum_{j=1}^{n} \bigl(v_i(P_j) - v_i(P_i)\bigr)\right). \qquad \square \end{aligned}$$

The next theorem is a direct analog of the main compactness–convexity result for value matrices given in Proposition 2.4.

Theorem 3.5. *Fix $n \geqslant 1$, and $v_1, v_2, \ldots, v_n$ probability measures on $(\Omega, \mathcal{F})$. Then*

(i) $\mathcal{M}_{\mathrm{E}}$ *is compact*; *and*
(ii) *if each v_i is atomless, then $\mathcal{M}_{\mathrm{E}}$ is convex.*

Proof. Conclusion (i) follows from Lemma 3.4 and Proposition 2.4, since $\mathcal{M}_{\mathrm{E}}$ is a continuous image of the compact set $\mathcal{M}_{\mathrm{V}}$, and (ii) follows similarly since in the atomless measure case, $\mathcal{M}_{\mathrm{E}}$ is the image of the convex set $\mathcal{M}_{\mathrm{V}}$ under an affine transformation. □

Note that in Example 3.3, $\mathcal{M}_{\mathrm{E}}$ is convex in case (i), and not convex in (ii); in both cases it is compact.

Definition 3.6. A partition $\mathbb{P}^* \in \Pi_n$ is *envy-free* if $e_{\max}(\mathbb{P}^*) \leqslant 0$; is *minimax envy optimal* if $e_{\max}(\mathbb{P}^*) = \min\{e_{\max}(\mathbb{P})\colon\ \mathbb{P} \in \Pi_n\}$; and is *DS minimax envy optimal* if it attains the minimum, lexicographically, of the set of feasible ordered envy vectors $\{(e_{\langle 1\rangle}(\mathbb{P}), \dots, e_{\langle n\rangle}(\mathbb{P}))\colon\ \mathbb{P} \in \Pi_n\}$ (cf. Definition 3.2).

Example 3.7. The partition $\mathbb{P}_2$ in Example 2.2 is the unique (up to sets of measure zero) DS-minimax-envy-optimal partition and is also envy-free (see Example 3.3(i)); *every* partition in Example 2.3 is DS minimax envy optimal with maximum possible envy $+1$ for one of the players, and no partition is envy-free.

Theorem 3.8. *Fix $n \geqslant 1$, and $v_1, \dots, v_n$ probability measures on $(\Omega, \mathcal{F})$. Then*

(i) *Minimax-envy-optimal and DS-minimax-envy-optimal partitions always exist*;
(ii) *If a partition is envy-free, then it is fair*;
(iii) *If $\{v_i\}_1^n$ are atomless, then envy-free partitions always exist*;
(iv) *If $\{v_i\}_1^n$ are atomless and linearly independent, then super-envy-free partitions ($e_{\max} < 0$) always exist.*

Proof. Conclusion (i) follows easily from Theorem 3.5(i) since the mapping $\mathcal{M}_{\mathrm{E}} \to [-1, 1]$ given by $M_{\mathrm{E}}(\mathbb{P}) \mapsto e_{\max}(\mathbb{P})$ ($\mapsto (e_{\langle 1\rangle}(\mathbb{P}), \dots, e_{\langle n\rangle}(\mathbb{P}))$, respectively) is continuous, so its minimum is attained; (ii) is trivial since $v_i(P_j) - v_i(P_i) \leqslant 0$ for all i, j implies that $v_i(P_i) \geqslant 1/n$ for all i; (iii) follows by Theorem 3.5(ii) by considering the n partitions $\mathbb{P}^1 = (\Omega, \emptyset, \dots, \emptyset)$, $\mathbb{P}^2 = (\emptyset, \Omega, \emptyset, \dots, \emptyset)$, $\dots$, $\mathbb{P}^n = (\emptyset, \dots, \emptyset, \Omega)$, and noting that $\sum_{j=1}^n M_{\mathrm{E}}(\mathbb{P}^j)$ is the zero matrix; and (iv) is the main result in [3]. □

Contrary to a claim in [15], the next example shows that even for three mutually absolutely continuous measures v_1, v_2, v_3, a DS-optimal partition need not be envy-free. (Recall that in Example 3.1, $\mathbb{P}$ was strictly better value-wise for each player than the envy-free partition $\hat{\mathbb{P}}$, but $\mathbb{P}$ was not DS optimal.)

Example 3.9. Let $(\Omega, \mathcal{F}) = ([0, 3], \text{Borels})$, $n = 3$, and (letting $I(a, b)$ denote the indicator function $I(a, b)(x) = 1$ if $a < x < b$, and $= 0$ otherwise) let v_1, v_2, v_3 be the continuous distributions with density functions f_1, f_2, f_3, respectively, given by

$$f_1 = 0.4I(0, 1) + 0.1I(1, 2) + 0.5I(2, 3),$$
$$f_2 = 0.3I(0, 1) + 0.4I(1, 2) + 0.3I(2, 3),$$
$$f_3 = 0.3I(0, 1) + 0.3I(1, 2) + 0.4I(2, 3).$$

Then, as will be proved in the next theorem, the partition $\mathbb{P} = (P_1, P_2, P_3) = ([0, 1), [1, 2), [2, 3])$ yields the uniquely maximin-optimal vector $(v_1(P_1), v_2(P_2), v_3(P_3)) = (0.4, 0.4, 0.4)$, but $\mathbb{P}$ is not envy-free. Thus every envy-free partition is strictly suboptimal in the maximin criterion, and hence also strictly suboptimal in the DS criterion.

Theorem 3.10. (i) *If $n = 2$ and v_1, v_2 are atomless, then every maximin-optimal partition is envy-free*; *and* (ii) *for each $n \geqslant 3$, there exist mutually absolutely continuous atomless measures $v_1, \ldots, v_n$ such that no maximin-optimal partition is envy-free.*

Proof. To see (i), note that $\left(\begin{smallmatrix} 1 & 0 \\ 1 & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ are in $\mathcal{M}_{\mathrm{V}}$ (taking $\mathbb{P}_1 = (\Omega, \emptyset)$, $\mathbb{P}_2 = (\emptyset, \Omega)$), so by Proposition 2.4, $\left(\begin{smallmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{smallmatrix}\right) \in \mathcal{M}_{\mathrm{V}}$, and thus every maximin-optimal partition $\mathbb{P}$ satisfies $v_1(\mathbb{P}) \geqslant 1/2$, $v_2(\mathbb{P}) \geqslant 1/2$. By additivity, this implies that $v_1(P_1) \geqslant v_1(P_2)$ and $v_2(P_2) \geqslant v_2(P_1)$, and hence that $\mathbb{P}$ is envy-free.

To see (ii), note that [6, last remark on p. 17] for any n, when the measures are mutually absolutely continuous the DS-optimal solution is equitable. Therefore, all maximin-optimal solutions are DS optimal and equitable.

For $n = 3$, consider the measures v_1, v_2, and v_3 of Example 3.9. The set of all possible partitions of $[0, 3]$ can be described as follows: the interval $[0, 1)$ is divided into three parts with player 1 (respectively, player 2) receiving a piece of length p_1 (respectively, p_2) and player 3 getting the rest, i.e., $1 - p_1 - p_2$. Similarly, $[1, 2)$ is split into three parts of length q_1, q_2, and $1 - q_1 - q_2$, respectively, and $[2, 3]$ is partitioned as r_1, r_2, and $1 - r_1 - r_2$.

Every equitable partition is obtained as a solution of the following system of linear equations and inequalities:

$$\begin{cases} 0.4p_1 + 0.1q_1 + 0.5r_1 = \alpha, \\ 0.3p_2 + 0.4q_2 + 0.3r_2 = \alpha, \\ 0.3(1 - p_1 - p_2) + 0.3(1 - q_1 - q_2) + 0.4(1 - r_1 - r_2) = \alpha, \\ p_1, p_2, q_1, q_2, r_1, r_2 \geqslant 0, \\ p_1 + p_2 \leqslant 1, \\ q_1 + q_2 \leqslant 1, \\ r_1 + r_2 \leqslant 1, \end{cases}$$

and the largest value of α is sought that keeps this system admissible. The corresponding solutions for the p_i's, q_i's, and r_i's describe all possible maximin-optimal solutions.

Solving the first equation for p_1 in terms of α, q_1, and r_1, and the second for q_2 in terms of α, p_2, and r_2, and substituting these expressions in the third equation yields

$$0.3p_2 + 0.9q_1 + 0.1r_1 + 0.7r_2 = 4 - 10\alpha. \tag{3.1}$$

If $\alpha > 0.4$, Eq. (3.1) has no solution with nonnegative variables.

If, instead, $\alpha = 0.4$, (3.1) admits only the solution

$$p_2 = q_1 = r_1 = r_2 = 0. \tag{3.2}$$

(Thus, the segment $[2, 3]$ is given in its entirety to player 3, who reaches his “quota” of 0.4 and has no interest in the other parts of the cake.)

Hence, since $\alpha = 0.4$, $p_1 + p_2 = 1$ and $q_1 + q_2 = 1$. This fact and (3.2) imply that $p_1 = q_2 = 1$, which shows that $\mathbb{P} = ([0, 1), [1, 2), [2, 3])$ is, up to sets of Lebesgue measure zero, the only minimax-optimal (and DS-optimal) solution. But

$$\mathcal{M}_{\mathrm{E}}(\mathbb{P}) = \begin{pmatrix} 0 & -0.3 & 0.1 \\ -0.1 & 0 & -0.1 \\ -0.1 & -0.1 & 0 \end{pmatrix},$$

which shows that this partition is not envy-free. This completes the case $n = 3$ (and establishes the claim in Example 3.9).

For $n > 3$, let $(\Omega, \mathcal{F}) = ([0, n], \text{Borels})$ and fix ε with $0 < \varepsilon < 1$. Consider the continuous distributions $v_1, \dots, v_n$ with density functions $f_1, \dots, f_n$, respectively, that have constant values in each interval $[i - 1, i)$, $i = 1, \dots, n$, with values shown in the following table:

	$[0, 1)$	$[1, 2)$	$[2, 3)$	$[3, 4)$	$[4, 5)$	$\dots$	$[n-1, n]$
f_1	$0.4(1-\varepsilon)$	$0.1(1-\varepsilon)$	$0.5(1-\varepsilon)$	$\varepsilon/(n-3)$	$\varepsilon/(n-3)$	$\dots$	$\varepsilon/(n-3)$
f_2	$0.3(1-\varepsilon)$	$0.4(1-\varepsilon)$	$0.3(1-\varepsilon)$	$\varepsilon/(n-3)$	$\varepsilon/(n-3)$	$\dots$	$\varepsilon/(n-3)$
f_3	$0.3(1-\varepsilon)$	$0.3(1-\varepsilon)$	$0.4(1-\varepsilon)$	$\varepsilon/(n-3)$	$\varepsilon/(n-3)$	$\dots$	$\varepsilon/(n-3)$
f_4	$\varepsilon/(n-1)$	$\varepsilon/(n-1)$	$\varepsilon/(n-1)$	$1-\varepsilon$	$\varepsilon/(n-1)$	$\dots$	$\varepsilon/(n-1)$
f_5	$\varepsilon/(n-1)$	$\varepsilon/(n-1)$	$\varepsilon/(n-1)$	$\varepsilon/(n-1)$	$1-\varepsilon$	$\dots$	$\varepsilon/(n-1)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
f_n	$\varepsilon/(n-1)$	$\varepsilon/(n-1)$	$\varepsilon/(n-1)$	$\varepsilon/(n-1)$	$\varepsilon/(n-1)$	$\dots$	$1-\varepsilon$

As in the $n = 3$ case, the distributions are mutually absolutely continuous and all minimax-optimal solutions are DS optimal and equitable.

Denote by $p_{i,j}$ $(i, j = 1, \dots, n)$ the length of the part of $[i - 1, i)$ assigned to player j with the usual constraints $p_{i,j} \geqslant 0$ and $\sum_{j=1}^{n} p_{i,j} = 1$ for all i. The partition, defined by $p_{i,i} = 1$ for all i, has $0.4(1 - \varepsilon)$ as its lowest value, so the minimax-optimal value cannot be smaller than this value.

Consider now the first three players only. Since the minimax value is at least $0.4(1 - \varepsilon)$ and since $\int_3^n f_i \, dv_i = \varepsilon$, $i = 1, 2, 3$, each of the players $1, 2, 3$ must receive something worth at least $0.4 - 1.4\varepsilon$ from the interval $[0, 3)$, so the following system of inequalities must be satisfied by any minimax-optimal solution:

$$0.4p_{1,1} + 0.1p_{2,1} + 0.5p_{3,1} \geqslant \beta + O(\varepsilon), \tag{3.3a}$$

$$0.3p_{1,2} + 0.4p_{2,2} + 0.3p_{3,2} \geqslant \beta + O(\varepsilon), \tag{3.3b}$$

$$0.3p_{1,3} + 0.3p_{2,3} + 0.4p_{3,3} \geqslant \beta + O(\varepsilon), \tag{3.3c}$$

$$\beta \geqslant 0.4 + O(\varepsilon). \tag{3.3d}$$

A simple consequence of the normalizing constraints for the $p_{i,j}$ is that

$$p_{i,3} \leqslant 1 - p_{i,1} - p_{i,2}, \quad i = 1, 2, 3. \tag{3.4}$$

This, with (3.3c), implies that

$$0.3(1 - p_{1,1} - p_{1,2}) + 0.3(1 - p_{2,1} - p_{2,2}) + 0.4(1 - p_{3,1} - p_{3,2}) \geqslant \beta + O(\varepsilon).$$

Rearranging (3.3a)–(3.3c) yields

$$p_{1,1} \geqslant \frac{5}{2}\beta - \frac{1}{4}p_{2,1} - \frac{5}{4}p_{3,1} + O(\varepsilon), \tag{3.5a}$$

$$p_{2,2} \geqslant \frac{5}{2}\beta - \frac{3}{4}p_{1,2} - \frac{3}{4}p_{3,2} + O(\varepsilon), \tag{3.5b}$$

$$0.3p_{1,1} + 0.3p_{1,2} + 0.3p_{2,1} + 0.3p_{2,2} + 0.4p_{3,1} + 0.4p_{3,2} \leqslant 1 - \beta + O(\varepsilon), \tag{3.5c}$$

$$\beta \geqslant 0.4 + O(\varepsilon). \tag{3.5d}$$

Substituting (3.5a) and (3.5b) into (3.5c) yields

$$0.3p_{1,2} + 0.9p_{2,1} + 0.1p_{3,1} + 0.7p_{3,2} \leqslant 4 - 10\beta + O(\varepsilon), \tag{3.6a}$$

$$\beta \geqslant 0.4 + O(\varepsilon). \tag{3.6b}$$

These imply that

$$0.3p_{1,2} + 0.9p_{2,1} + 0.1p_{3,1} + 0.7p_{3,2} \leqslant 4 - 10\beta + O(\varepsilon) = O(\varepsilon). \tag{3.7}$$

Since all variables in (3.7) are nonnegative, they all satisfy

$$p_{1,2} = O(\varepsilon), \qquad p_{2,1} = O(\varepsilon), \qquad p_{3,1} = O(\varepsilon), \qquad p_{3,2} = O(\varepsilon). \tag{3.8}$$

From (3.5a) and (3.5d), and the fact that $p_{2,1}$ and $p_{3,1}$ are $O(\varepsilon)$, it follows that $p_{1,1} = 1 + O(\varepsilon)$, and hence that

$$p_{1,3} = O(\varepsilon). \tag{3.9}$$

Similarly, (3.5b), (3.5d), and (3.8) imply that $p_{2,2} = 1 + O(\varepsilon)$, so

$$p_{2,3} = O(\varepsilon). \tag{3.10}$$

Finally, (3.5c), (3.5d), (3.9), and (3.10) imply that

$$p_{3,3} = 1 + O(\varepsilon). \tag{3.11}$$

Thus, player 1's evaluation of his own share in any minimax-optimal solution is

$$v_1(P_1) = 0.4p_{1,1} + 0.1p_{2,1} + 0.5p_{3,1} + O(\varepsilon) = 0.4 + O(\varepsilon).$$

Player 1's evaluation of player 3's share, on the other hand, is

$$v_1(P_3) = 0.4p_{1,3} + 0.1p_{2,3} + 0.5p_{3,3} + O(\varepsilon) = 0.5 + O(\varepsilon),$$

where the last equality follows from (3.11). Therefore,

$$e_{1,3} = v_1(P_3) - v_1(P_1) \to 0.1 \text{ as } \varepsilon \searrow 0,$$

so asymptotically, in every maximin-optimal partition, player 1 envies player 3's share by an amount arbitrarily close to 0.1. □

The final theorem in this section gives sharp bounds for fairness of envy-free partitions and minimax envy, in the case where the measures are atomless and have known upper and lower bounds, respectively.

For measures $v_1, \ldots, v_n$, the function $\bigvee_{i=1}^n v_i : \mathcal{F} \to [0,1]$, called the maximum of $\{v_1, \ldots, v_n\}$, is the smallest set function which dominates each of the $\{v_i\}$; $\bigwedge_{i=1}^n v_i$ is the analogous minimum. It is easy to check that both $\bigvee^n v_i$ and $\bigwedge^n v_i$ are also countably additive measures on $(\Omega, \mathcal{F})$, and letting v^*, v_* denote the total masses of $\bigvee^n v_i$, $\bigwedge^n v_i$, respectively, that $n \geqslant v^* \geqslant 1 \geqslant v_*$, with equality if and only if $v_1 = v_2 = \cdots = v_n$. (When $\{v_i\}$ are absolutely continuous with densities $\{f_i\}$, v^* is simply the total area under the outer envelope $\max_{1 \leqslant i \leqslant n} f_i$ of $\{f_i\}$, and v_* is the area under $\min_{1 \leqslant i \leqslant n} f_i$.) In fair-division problems, v^* represents the cooperative value of Ω, that is the total value to the coalition of all players if each piece is given to the player who values it most, and these values are added together. Similarly, v_* represents the "worst-case" allocation if the values are added (cf. [8,10]).

Example 3.11. For the measures in Example 2.2, $v^* = 5/4$ and $v_* = 3/4$; in Example 2.3, $v^* = 1 = v_*$.

Theorem 3.12. *Fix $n \geqslant 1$ and $v_1, \ldots, v_n$ atomless probability measures on $(\Omega, \mathcal{F})$. Then there exist partitions $\mathbb{P}^{(1)}, \mathbb{P}^{(2)}, \mathbb{P}^{(3)}, \mathbb{P}^{(4)}$ in Π_n such that*

(i) $\mathbb{P}^{(1)}$ *is envy-free and* $v_i(P_i^{(1)}) = (n - v^* + 1)^{-1}$ *for all* i;
(ii) $\mathbb{P}^{(2)}$ *is envy-free and* $v_i(P_i^{(2)}) = (n + v_* - 1)^{-1}$ *for all* i;
(iii) $e_{\max}(\mathbb{P}^{(3)}) \leqslant \min\left\{0, \frac{n - v^* - 1}{n - v^* + 1}\right\}$; *and*
(iv) $e_{\max}(\mathbb{P}^{(4)}) \leqslant \min\left\{0, \frac{n + v_* - 3}{n + v_* - 1}\right\}$,

and these bounds are best possible.

Recall that $v^* = 1$ if and only if $v_* = 1$ if and only if $v_1 = \cdots = v_n$, so the bounds $(n - v^* + 1)^{-1}$ and $(n + v_* - 1)^{-1}$ in (i) and (ii) are strictly bigger than $1/n$ whenever the $\{v_i\}$ are not identical. Thus, in that case, (i) and (ii) guarantee the existence of envy-free super-fair partitions, with super-fairness quantifiably greater than $1/n$. Similarly, for v^* sufficiently large, or v_* sufficiently small ($v^* > n - 1$, $v_* < 3 - n$), (iii) and (iv) guarantee the existence of super-envy-free partitions with envy quantifiably strictly negative (cf. [6] and [3] for nonquantifiable super-fair and for super-envy-free partitions, respectively).

Proof of Theorem 3.12. Let $\mu = \sum_{i=1}^n v_i$. Every v_i is absolutely continuous with respect to μ, so, by the Radon–Nikodym theorem, there exists a function f_i, called the density function of v_i, such that $v_i(A) = \int_A f_i \, d\mu$ for all $A \in \mathcal{F}$.

To prove (i), let $\mathbb{P}^* = (P_1^*, \ldots, P_n^*)$ be the partition of Ω which assigns each element of Ω to the player whose density is highest in that point. In case of ties, the point is allocated to the player identified by the lowest number. More formally, let

$$P_1^* = \left\{ x \in \Omega \colon \; f_1(x) = \max_m f_m(x) \right\}$$

and

$$P_k^* = \Big\{x \in \Omega\colon\ f_k(x) = \max_m f_m(x)\Big\} \Big\backslash \bigcup_{i=1}^{k-1} P_i^*, \quad k = 2, \ldots, n.$$

Let $M_{\mathrm{V}}(\mathbb{P}^*) = (v_i(P_j^*))_{i,j=1}^n$ be the value matrix associated with the partition $\mathbb{P}^*$. Then

$$\sum_{i=1}^n v_i\big(P_i^*\big) = \sum_{i=1}^n \int_{P_i^*} f_i\, d\mu = \sum_{i=1}^n \int_{P_i^*} \max_m f_m\, d\mu = \int_{\Omega} \max_m f_m\, d\mu = v^* \tag{3.12}$$

and

$$v_i\big(P_i^*\big) = \int_{P_i^*} f_i\, d\mu = \int_{P_i^*} \max_m f_m\, d\mu \geqslant \int_{P_i^*} f_j\, d\mu = v_j\big(P_i^*\big)$$
$$\text{for all } i, j = 1, \ldots, n. \tag{3.13}$$

Now, for each $k = 1, \ldots, n$, consider the partition $\mathbb{P}^k = (P_1^k, \ldots, P_n^k)$ which assigns the whole set Ω to player k, i.e.,

$$P_j^k = \begin{cases} \Omega & \text{if } j = k, \\ \emptyset & \text{otherwise.} \end{cases} \tag{3.14}$$

Clearly, the value matrix $M_{\mathrm{V}}(\mathbb{P}^k)$ satisfies

$$v_i\big(P_j^k\big) = \begin{cases} 1 & \text{if } j = k, \\ 0 & \text{otherwise,} \end{cases} \quad \text{for all } i = 1, \ldots, n.$$

Since the v_i are atomless, Proposition 2.4(ii) implies that $\mathcal{M}_{\mathrm{V}}$ is convex. Therefore, for any choice of $\beta_1, \ldots, \beta_n, \beta_{n+1}$ with $\beta_i \geqslant 0$ for all $i = 1, \ldots, n+1$ and $\sum_{i=1}^{n+1} \beta_i = 1$, there exists a partition $\mathbb{P}^{(1)} = (P_1^{(1)}, \ldots, P_n^{(1)})$ such that

$$M_{\mathrm{V}}(\mathbb{P}^{(1)}) = \sum_{k=1}^n \beta_k M_{\mathrm{V}}(\mathbb{P}^k) + \beta_{n+1} M_{\mathrm{V}}(\mathbb{P}^*).$$

Define the coefficients $\{\beta_i\}$ as follows (cf. [12,13]):

$$\beta_k = \frac{1 - v_k(P_k^*)}{n - v^* + 1}, \quad k = 1, \ldots, n, \quad \text{and} \quad \beta_{n+1} = \frac{1}{n - v^* + 1}.$$

The $\{\beta_i\}$ are all nonnegative since $v^* \leqslant n$, and satisfy $\sum_i \beta_i = 1$ by (3.12). The elements of $M_{\mathrm{V}}(\mathbb{P}^{(1)})$ satisfy

$$\begin{aligned} v_i\big(P_j^{(1)}\big) &= \sum_{k=1}^n \beta_k v_i\big(P_j^k\big) + \beta_{n+1} v_i\big(P_j^*\big) = \beta_j + \beta_{n+1} v_i\big(P_j^*\big) \\ &= \frac{1 - v_j(P_j^*) + v_i(P_j^*)}{n - v^* + 1} \leqslant \frac{1}{n - v^* + 1} = v_i\big(P_i^{(1)}\big). \end{aligned} \tag{3.15}$$

The inequality in (3.15) follows by (3.13), with the roles of i and j reversed. Therefore, $\mathbb{P}^{(1)}$ is envy-free and allots the value $(n - v^* + 1)^{-1}$ to each player.

The proof of (ii) also requires the following inversion principle (cf. [10, Proposition 2.3]):

$$M_{\mathrm{V}} \in \mathcal{M}_{\mathrm{V}} \quad \Rightarrow \quad (\mathbf{1} - M_{\mathrm{V}})/(n-1) \in \mathcal{M}_{\mathrm{V}}, \tag{3.16}$$

where $\mathbf{1}$ is the $n \times n$ matrix whose elements are all 1's.

This time, the partition $\mathbb{P}_*$, which assigns each point to the player with the lowest density, is

$$P_{*1} = \Big\{x \in \Omega\colon\ f_1(x) = \min_m f_m(x)\Big\}$$

and

$$P_{*k} = \Big\{x \in \Omega\colon\ f_k(x) = \min_m f_m(x)\Big\} \Big\backslash \bigcup_{i=1}^{k-1} P_{*i}, \quad k = 2, \dots, n.$$

It is easy to see that

$$\sum_{i=1}^{n} v_i(P_{*i}) = v_* \tag{3.17a}$$

and

$$v_i(P_{*i}) \leqslant v_j(P_{*i}) \quad \text{for all } i, j = 1, \dots, n. \tag{3.17b}$$

By (3.16), $(\mathbf{1} - M_{\mathrm{V}}(\mathbb{P}_*))/(n-1) \in \mathcal{M}_{\mathrm{V}}$ and, therefore, by Proposition 2.4(ii), there exists a partition $\mathbb{P}^{(2)} = (P_1^{(2)}, \dots, P_n^{(2)})$ whose value matrix satisfies

$$M_{\mathrm{V}}(\mathbb{P}^{(2)}) = \sum_{k=1}^{n} \hat{\beta}_k M_{\mathrm{V}}(\mathbb{P}^k) + \hat{\beta}_{n+1} \frac{\mathbf{1} - M_{\mathrm{V}}(\mathbb{P}_*)}{n-1},$$

where coefficients $\hat{\beta}_i$ are given by

$$\hat{\beta}_k = \frac{v_k(P_{*k})}{n + v_* - 1}, \quad k = 1, \dots, n, \quad \text{and} \quad \hat{\beta}_{n+1} = \frac{n-1}{n + v_* - 1}.$$

It is easy to check that $\hat{\beta}_i \geqslant 0$, and, by (3.17a), $\sum_{i=1}^{n+1} \hat{\beta}_i = 1$.

From (3.17b) it follows that

$$\begin{aligned} v_i\big(P_j^{(2)}\big) &= \sum_{k=1}^{n} \hat{\beta}_k v_i\big(P_j^k\big) + \hat{\beta}_{n+1} \frac{1 - v_i(P_{*j})}{n-1} = \hat{\beta}_j + \hat{\beta}_{n+1} \frac{1 - v_i(P_{*j})}{n-1} \\ &= \frac{v_j(P_{*j}) + 1 - v_i(P_{*j})}{n + v_* - 1} \leqslant \frac{1}{n + v_* - 1} = v_i\big(P_i^{(2)}\big), \end{aligned} \tag{3.18}$$

so $\mathbb{P}^{(2)}$ is envy-free and allots the value $(n + v_* - 1)^{-1}$ to each player.

Statements (iii) and (iv) are a direct consequence of (i) and (ii), respectively. In particular, to prove (iii), again consider the partition $\mathbb{P}^{(1)}$. It was shown in (i) that this partition is envy-free, so $e_{\max}(\mathbb{P}^{(1)}) \leqslant 0$. Also, by (3.15), $v_i(P_i^{(1)}) = (n - v_* + 1)^{-1}$ and

$$v_i\big(P_j^{(1)}\big) \leqslant 1 - v_i\big(P_i^{(1)}\big) = \frac{n - v^*}{n - v^* + 1} \quad \text{for all } j \neq i.$$

Therefore

$$v_i\big(P_j^{(1)}\big) - v_i\big(P_i^{(1)}\big) \leqslant \frac{n - v^* - 1}{n - v^* + 1} \quad \text{for all } j \neq i,$$

which completes the proof of (iii).

Similarly, to obtain (iv), note that $e_{\max}(\mathbb{P}^{(2)}) \leqslant 0$ and, by (3.18),

$$v_i\big(P_j^{(2)}\big) \leqslant 1 - v_i\big(P_i^{(2)}\big) = \frac{n + v_* - 2}{n + v^* - 1} \quad \text{for all } j \neq i,$$

so

$$v_i\big(P_j^{(2)}\big) - v_i\big(P_i^{(2)}\big) \leqslant \frac{n + v_* - 3}{n + v_* - 1} \quad \text{for all } j \neq i. \qquad \square$$

Example 3.13. For the measures in Example 2.2, Theorem 3.12(i) guarantees the existence of an envy-free partition with equitable share $(2 - v^* + 1)^{-1} = 4/7 \cong 0.57$ for each player, whereas for these particular measures even more is possible (namely $(\sqrt{5} - 1)/2 \cong 0.61$, see Example 2.2). Similarly, Theorem 3.12(iii) guarantees the existence of a super-envy-free partition with maximum envy $\leqslant (2 - v^* - 1)/(2 - v^* + 1) = -1/7$, whereas even smaller maximum envy $2 - \sqrt{5}$ is possible (cf. Example 3.3).

4. Minimax-envy inequalities for measures with atoms

For atomless measures, fair and envy-free partitions always exist (cf. Theorem 3.12), as a consequence of the convexity of the value and envy matrix ranges, respectively (Proposition 2.4, Theorem 3.5). For measures with atoms, however, in general the sets of $\mathcal{F}$-feasible value matrices and envy matrices are not convex, and neither fair nor envy-free partitions exist (cf. Examples 2.3 and 3.3(ii)). It is the purpose of this section to establish bounds on the nonconvexity, and upper bounds on envy based on the mass of the largest atom, analogous to the bounds found in [7] for value matrices. The underlying intuition is simply that if the atoms are all very small, then the envy-matrix range must be nearly convex, and hence nearly envy-free partitions must exist.

For $\alpha \in (0, 1)$, let $\mathcal{P}(\alpha)$ denote the set of value functions with no atom mass greater than α. That is,

$$\mathcal{P}(\alpha) = \big\{v\colon\ v \text{ is a probability measure on } (\Omega, \mathcal{F}) \\ \text{with } v(A) \leqslant \alpha \text{ for all } v\text{-atoms } A \in \mathcal{F}\big\}.$$

The next theorem gives an upper bound on how far from convex the set of feasible envy matrices can be as a function of the maximum atom size and the number of measures. Here $\operatorname{co}(S)$ denotes the convex hull of the set S.

Theorem 4.1. *Fix $n \geqslant 1$ and $\alpha \in (0, 1)$, and let $v_i \in \mathcal{P}(\alpha)$, $i = 1, \ldots, n$. Then for every $C = (c_j)_{i,j=1}^n \in \operatorname{co}(\mathcal{M}_{\mathrm{E}})$ there exists $\mathbb{P} \in \Pi_n$ with*

$$\big|e_{i,j}(\mathbb{P}) - c_{i,j}\big| \leqslant \alpha(2n)^{3/2} \quad \textit{for all } i, j = 1, \ldots, n.$$

Proof. Fix $C = (c_{i,j})_{i,j=1}^n \in \mathrm{co}(\mathcal{M}_\mathrm{E})$. By Lemma 3.4, there exists $D = (d_{i,j})_{i,j=1}^n \in \mathrm{co}(\mathcal{M}_\mathrm{V})$ such that

$$c_{i,j} = d_{i,j} - d_{i,i} \quad \text{for all } i, j = 1, \ldots, n. \tag{4.1}$$

Since $v_1, \ldots, v_n \in \mathcal{P}(\alpha)$, by a theorem of Allaart [2, Theorem 2.11(i)], the Hausdorff euclidean distance between $\mathcal{M}_\mathrm{V}$ and its convex hull is no more than $\sqrt{2}\alpha n^{3/2}$, so there exists $M = (m_{i,j})_{i,j=1}^n \in \mathcal{M}_\mathrm{V}$ with

$$\left[\sum_{i,j=1}^{n} (m_{i,j} - d_{i,j})^2\right]^{1/2} \leqslant \sqrt{2}\alpha n^{3/2}. \tag{4.2}$$

Since $M \in \mathcal{M}_\mathrm{V}$, there exists a partition $\mathbb{P} = (P_1, \ldots, P_n) \in \Pi_n$ with

$$v_i(P_j) = m_{i,j} \quad \text{for all } i, j = 1, \ldots, n. \tag{4.3}$$

Since $\max\{|a_1|, \ldots, |a_m|\} \leqslant (\sum_{k=1}^m a_k^2)^{1/2}$, (4.2) and (4.3) imply that

$$\left|v_i(P_j) - d_{i,j}\right| \leqslant \sqrt{2}\alpha n^{3/2} \quad \text{for } i, j = 1, \ldots, n. \tag{4.4}$$

By definition of envy, $e_{i,j}(\mathbb{P}) = v_i(P_j) - v_i(P_i)$, so (4.1) implies that

$$\begin{aligned}\left|e_{i,j}(\mathbb{P}) - c_{i,j}\right| &= \left|v_i(P_j) - v_i(P_i) - (d_{i,j} - d_{i,i})\right| \\ &\leqslant \left|v_i(P_j) - d_{i,j}\right| + \left|v_i(P_i) - d_{i,i}\right| \leqslant 2\sqrt{2}\alpha n^{3/2} = \alpha(2n)^{3/2},\end{aligned}$$

where the last inequality follows by (4.4). □

Allaart has also found the sharp bound for the Hausdorff distance between the *partition range* and its convex hull [1, Theorem 2.5] in terms of α, which has direct application to maximin-share but not to minimax-envy inequalities. The next result is an example of an application of Theorem 4.1 to establish the existence of envy-free partitions in some fair-division problems with atoms. Recall that v^* and v_* are the total masses of the smallest measure dominating, and the largest measure dominated by, respectively, all the measures $v_1, \ldots, v_n$ (cf. Example 3.11).

Theorem 4.2. *Fix $n \geqslant 1$ and $\alpha \in (0, 1)$, and let $v_i \in \mathcal{P}(\alpha)$ for all $i = 1, \ldots, n$. Then if either*

(i) $\alpha < \left(\frac{-n+v^*+1}{n-v^*+1}\right)(2n)^{-3/2}$ *or*
(ii) $\alpha < \left(\frac{-n-v_*+3}{n+v_*-1}\right)(2n)^{-3/2}$,

then there exists a super-envy-free partition $\mathbb{P} \in \Pi_n$.

Proof. To see (i), assume without loss of generality, that $(-n + v^* + 1) > 0$, for otherwise the conclusion is trivial. Enlarging Ω if necessary (e.g., replacing A by $A \times [0, 1]$ for every v-atom A in $\mathcal{F}$), it may be assumed without loss of generality that there exists a σ-algebra $\hat{\mathcal{F}} \supset \mathcal{F}$, and atomless measures $u_1, \ldots, u_n$ on $(\Omega, \hat{\mathcal{F}})$ such that

$$u^* = v^* \quad \text{and} \quad u_i(P) = v_i(P) \quad \text{for all } P \in \mathcal{F}. \tag{4.5}$$

Letting

$$\hat{\mathcal{M}}_{\mathrm{E}} = \left\{ \left(u_i(\hat{P}_j)\right)_{i,j=1}^n : \hat{P}_1, \ldots, \hat{P}_n \in \hat{\mathcal{F}},\ \bigcup \hat{P}_i = \Omega,\ \hat{P}_i \cap \hat{P}_j = \emptyset \text{ if } i \neq j \right\},$$

it follows by the definition of $\{u_i\}$ and $\hat{\mathcal{F}}$ that

$$\hat{\mathcal{M}}_{\mathrm{E}} = \mathrm{co}(\mathcal{M}_{\mathrm{E}}). \tag{4.6}$$

By (4.5) and Theorem 3.12(iii), there exists a partition $\hat{\mathbb{P}} = (\hat{P}_1, \ldots, \hat{P}_n)$ with $\hat{P}_i \in \hat{\mathcal{F}}$ for all i, and satisfying

$$u_i(\hat{P}_j) - u_i(\hat{P}_i) \leqslant \frac{n - u^* - 1}{n - u^* + 1} \quad \text{for all } i = 1, \ldots, n,\ i \neq j. \tag{4.7}$$

By (4.6), $(u_i(\hat{P}_j))_{i,j=1}^n \in \mathrm{co}(\mathcal{M}_{\mathrm{E}})$, so by Theorem 4.1 there exists a partition $\mathbb{P} \in \Pi_n$ with

$$\begin{aligned} \left| e_{i,j}(\mathbb{P}) - \left(u_i(\hat{P}_j) - u_i(\hat{P}_i)\right) \right| &\leqslant \alpha(2n)^{3/2} < \left(\frac{-n + v^* + 1}{n - v^* + 1} \right) (2n)^{-3/2} (2n)^{3/2} \\ &= \frac{-n + v^* + 1}{n - v^* + 1}, \quad i \neq j, \end{aligned}$$

so by (4.7) and the fact that $u^* = v^*$,

$$e_{i,j}(\mathbb{P}) < \frac{-n + v^* + 1}{n - v^* + 1} + u_i(\hat{P}_j) - u_i(\hat{P}_i) \leqslant 0,$$

so $e_{\max}(\mathbb{P}) < 0$ and $\mathbb{P}$ is super-envy-free, which proves (i). The argument for (ii) is similar, using Theorem 3.12(iv). $\square$

Example 4.3. Suppose that v_1 and v_2 are probability measures with $v^* = 5/4$. If no atom in v_1 or v_2 has mass greater than $\left(\frac{-2+v^*+1}{2-v^*+1}\right)(4)^{-3/2} = 1/56$, then there is a super-envy-free partition. (Compare with Example 3.13, where v_1 and v_2 are *atomless* with the same outer measure $v^* = 5/4$.)

The next proposition, which is recorded here for ease of reference, gives the sharp guaranteed maximin share as a function of maximum atom size and number of measures; it will be used here to establish upper bounds on maximum envy also as a function of atom size and number of measures.

Definition 4.4. $V_n : [0, 1] \to [0, 1]$ is the unique nonincreasing function satisfying $V_n(x) = 1 - k(n-1)x$ for all $x \in [(k+1)k^{-1}((k+1)n - 1)^{-1}, (kn-1)^{-1}]$, $k = 1, 2, \ldots$.

Proposition 4.5 [9]. *Fix $n \geqslant 1$ and let $v_1, \ldots, v_n \in \mathcal{P}(\alpha)$. Then there exists a partition $\mathbb{P} = (P_1, \ldots, P_n) \in \Pi_n$ satisfying*

$$v_i(P_i) \geqslant V_n(\alpha) \quad \textit{for all } i = 1, \ldots, n,$$

and this bound is attained.

Theorem 4.6. *Fix $n \geqslant 1$ and $\alpha \in (0,1)$ and let $v_1, \ldots, v_n \in \mathcal{P}(\alpha)$. Then there exist partitions $\mathbb{P}^{(1)}, \mathbb{P}^{(2)}$ in Π_n satisfying*

(i) $e_{\max}(\mathbb{P}^{(1)}) \leqslant \alpha(2n)^{3/2}$; *and*
(ii) $e_{\max}(\mathbb{P}^{(2)}) \leqslant 1 - 2V_n(\alpha)$.

Proof. Let $u_1, \ldots, u_n$, $\hat{\mathcal{F}}$, and $\hat{\mathcal{M}}_{\mathrm{E}}$ be as in the proof of Theorem 4.2. Theorem 3.8(iii) implies the existence of an envy-free partition $\hat{\mathbb{P}}$ for $u_1, \ldots, u_n$, and via correspondence (4.6), this implies that there is an element $C = (c_{i,j})_{i,j=1}^n \in \mathrm{co}(\mathcal{M}_{\mathrm{E}})$ with $c_{i,j} \leqslant 0$ for all $i, j = 1, \ldots, n$. Conclusion (i) then follows immediately from Theorem 4.1.

To see (ii), let $\mathbb{P} = (P_1, \ldots, P_n) \in \Pi_n$ be as in Proposition 4.5. By additivity of the measures $\{v_i\}$, $v_i(P_j) \leqslant 1 - v_i(P_i)$ for all $j \neq i$, so $v_i(P_j) - v_i(P_i) \leqslant 1 - 2v_i(P_i) \leqslant 1 - 2V_n(\alpha)$. $\square$

Example 4.7. Let $\alpha = 0.01$, that is, no participant values any crumb more than one hundredth of the total value of the cake. If there are two players, the bound in Theorem 4.6(i) is 0.08 and checking that $V_2(0.01) = 50/101$, the bound in (ii) is $1/101$, which is sharper. If there are three players, then the bound in (i) is $(0.01)6^{3/2} \cong 0.1470$, and that in (ii) (checking that $V_3(0.01) = 33/101$) is $35/101$, which in this case is substantially weaker than the bound given by (i).

References

[1] P. Allaart, A sharp nonconvexity bound for partition ranges of vector measures with atoms, J. Math. Anal. Appl. 235 (1999) 326–348.
[2] P. Allaart, Bounds on the non-convexity of ranges of vector measures with atoms, Contemp. Math. 234 (1999) 1–11.
[3] J. Barbanel, Super envy-free cake division and independence of measures, J. Math. Anal. Appl. 197 (1996) 54–60.
[4] S. Brams, A. Taylor, An envy-free cake division protocol, Amer. Math. Monthly 102 (1996) 9–18.
[5] S. Brams, A. Taylor, Fair Division: From Cake-Cutting to Dispute Resolution, Cambridge Univ. Press, Cambridge, UK, 1996.
[6] L. Dubins, E. Spanier, How to cut a cake fairly, Amer. Math. Monthly 68 (1961) 1–17.
[7] J. Elton, T. Hill, A generalization of Lyapounov's convexity theorem to measures with atoms, Proc. Amer. Math. Soc. 296 (1987) 297–304.
[8] J. Elton, T. Hill, R. Kertz, Optimal partitioning equalities for nonatomic probability measures, Trans. Amer. Math. Soc. 296 (1986) 703–725.
[9] T. Hill, Partitioning general probability measures, Ann. Probab. 15 (1987) 804–813.
[10] T. Hill, A sharp partitioning-inequality for non-atomic probability measures based on the mass of the infimum of the measures, Prob. Theory Related Fields 75 (1987) 143–147.
[11] T. Hill, Partitioning inequalities in probability and statistics, IMS Lecture Notes 22 (1993) 116–132.
[12] J. Legut, Inequalities for α-optimal partitioning of a measurable space, Proc. Amer. Math. Soc. 104 (1988) 1249–1251.
[13] J. Legut, M. Wilczynski, Optimal partitioning of a measurable space, Proc. Amer. Math. Soc. 104 (1988) 262–264.
[14] J. Robertson, W. Webb, Cake-Cutting Algorithms, A.K. Peters, Natick, MA, 1998.
[15] D. Weller, Fair division of a measurable space, J. Math. Econom. 14 (1985) 5–17.