Clifford Algebra Calculations with Representation Theory & an Introduction to Clifford Algebra

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The goals of this paper are to provide an introduction to vector, exterior and Clifford algebra and to present a method for performing Clifford algebra calculations efficiently using representation theory. Vectors, exterior algebra and other mathematical and historical foundations of Clifford algebra are introduced. Matrix representations of Clifford algebras are elucidated. The computation of rotations using Clifford representations of spin groups is discussed, and a few of the relevant scientific and engineering applications of the Clifford and exterior algebras are noted throughout.

I. Introduction to Vectors and Exterior Algebra

One of the most fundamental ideas used in the study of linear algebra and geometry is the concept of a vector. A vector is defined as an element of a vector space. A definition of a vector space adapted from one given by Kreyzig[1] is as follows: a vector space is a nonempty set *V* of vectors with the same number of components such that any two vectors a and b in *V* and all their linear combinations $a\alpha + b\beta$ (where α and β are scalars, defined as elements of a field *F*) are elements of *V*, and these vectors satisfy the rules of vector addition and scalar multiplication. The definition of a field is too involved for it to be inserted in the middle of the definition of a vector space, but due to the scope of this work the only fields that will be considered hereafter are those of the real numbers and the complex numbers. The rules of vector addition and scalar multiplication are presented in tables 1A and 1B below for any vectors *A*, *B* and *C* of a vector space *V*, and any scalars *c* and *k* of a field *F*.

Table 1A. The Rules of Vector Addition

Commutativity	$A + B = B + A$
Associativity	$A + B + C = (A + B) + C = A + (B + C)$
Existence of Additive Identity, where the zero vector 0 is unique to the vector space	$A + 0 = A$
Existence of Additive Inverse $(-A)$ for all *A*, where $(-A)$ is unique to the vector space	$A + (-A) = 0$

Table 1B. The Rules of Scalar Multiplication

Scalar Distributivity	c(A + B) = cA + cB
Vector Distributivity	(c + k)A = cA + kA
Associativity	c(kA) = (ck)A = ckA
Existence of Multiplicative Identity (where the identity element 1 is unique to the field)	1A = A

One common notation for writing a concrete example of a vector expresses the vector as a summation in which a set of characters serve as units for vector components. An example three-component vector *V* over the real number field can be written as shown in quantity (1a):

$$V \equiv (-5.1i + \pi j + 1.9k) \tag{1a}$$

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where the units i, j and k prevent the summation from simplifying. One benefit of this notation is that the summation of two or more vectors with each other is easily accomplished by grouping like units and simplifying the sum to a single vector. This notation is also satisfying because scalar multiplication immediately follows from the distributive property - each of the vector's components will be multiplied by the scalar when the product is simplified.

Many different physical phenomena can be modeled using vectors, such as positions, velocities and accelerations in classical mechanics. In a three-dimensional Euclidean space, a position vector can be visualized as an arrow drawn from the origin (the zero vector) to a point in the Euclidean space with Cartesian coordinates defined by the vector's components. As an aside, please note that although Cartesian coordinates are often utilized in vector calculations (and any coordinate system mentioned in this paper is assumed to be Cartesian) other coordinate systems such as polar coordinates also exist.

One of the most fundamental concepts in vector algebra is that a vector quantity describes both a scalar magnitude and a spatial direction. When a position vector is considered, for example, it follows that the scalar magnitude of the vector is the distance from the origin to the point in space that the vector is describing. In a Euclidean space, the scalar magnitude of a vector can be calculated using the Euclidean norm, which is the square root of the sum of the squares of the vector components. Using the Euclidean norm, a Euclidean vector can be factored into the product of a scalar magnitude with a vector of unit Euclidean norm. The vector of unit Euclidean norm is sometimes called a direction cosine, because it describes direction, while its scalar coefficient describes a scalar magnitude.

Vectors that do not describe a position (such as an acceleration vector) lack concrete spatial orientations - they describe a scalar magnitude and a direction, but are not intrinsically located anywhere. One of the most common non-position quantities represented by a vector is that of a point-force. In classical mechanics, a summation of all point-force vectors acting on a particle results in a vector that describes the net force on the particle. To ease the visualization of a vector summation in the context of point-forces acting on a particle, one can utilize the origin of the given Cartesian coordinate system as a virtual location at which the particle can be imagined to exist. Consider the following example in which vector summation is employed in a two-dimensional Euclidean space to sum two point-forces *A* and *B* acting on a particle. The summation of two vectors follows the "parallelogram law." A parallelogram can be constructed from the origin using *A* and *B* to define its sides, and the sum of *A* and *B* is a vector from the origin to the opposite vertex of the parallelogram, as illustrated in fig. 1, which was drawn using MATLAB.[2]

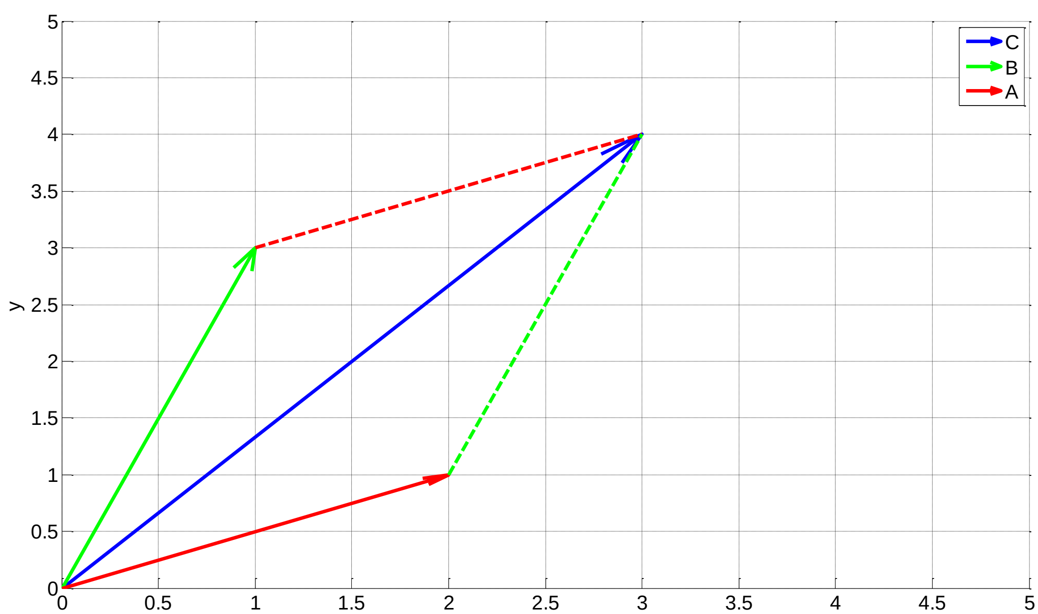

Figure 1. The "parallelogram law" of vector addition, illustrating the vector sum *A* + *B* = *C*.

The fact that two or more vectors (each with the same number of components) can always be summed is quite fundamental to the definition of a vector, as is scalar multiplication. Product operations on two or more vectors are not present in the definition of a vector, but they can be quite useful. One of the oldest and most well-known product operators that can act on two vectors each with n components is called the wedge or exterior product. The exterior product is used in what is today known as exterior algebra. As a historical aside, the exterior product was first seen in the work of Hermann Grassmann. As described by Israel Kleiner in *A History of Abstract Algebra*,[3] many of the fundamental notions in linear algebra, including linear independence and the idea of a vector space, were investigated by Grassmann and published in his work *Die Lineale Ausdehnungslehre* (translated as *Doctrine of Linear Extension*) in 1844, and another work in 1862. According to Kleiner, Grassmann's work influenced Giuseppe Peano in his formulation of the modern definition of a vector space in his *Geometric Calculus* of 1888.

The exterior product may be considered complicated by students first learning it for two reasons: firstly, the exterior product of two vectors does not result in a vector, and secondly, the exterior product is a non-commutative product, so the order in which vectors are multiplied matters. In *Clifford Algebras and Spinors*[4] by Pertti Lounesto, the exterior algebra is introduced by defining a set of rules for the exterior product with respect to the vector units ε_1, ε_2, ... , ε_n where subscripts are used to denote different units instead of individual letters such as i, j and k. The defining rules for the exterior product are provided by relations (1b) and (1c) below, where the wedge operator $\wedge$ denotes the exterior product.

$$\varepsilon_i \wedge \varepsilon_j = -\varepsilon_j \wedge \varepsilon_i \ \ \forall \ \ i \neq j \quad \textbf{(1b)}$$

$$\varepsilon_i \wedge \varepsilon_i = 0 \quad \textbf{(1c)}$$

Relation (1b) shows that the exterior product of two vectors is non-commutative. Relation (1b) also helps to explain why the exterior product of two vectors does not result in a vector; the exterior product of two unlike units does not reduce. The product of two like units reduces to zero, however, as shown in relation (1c).

The exterior product has a very interesting Euclidean geometric interpretation. The exterior product of two non-parallel vectors, for example, can be interpreted as resulting in a parallelogram-shaped plane segment called a bivector with a characteristic area and planar attitude. The area magnitude of a general Euclidean bivector is equal to the area of a parallelogram constructed using the two non-parallel vectors as its sides, and can be calculated as the Euclidean norm of the bivector's components. Note that a bivector does not have a complete spatial orientation because it lacks a definite position, much like a vector. In a two-dimensional Euclidean space, an arbitrary bivector *B* is described by quantity (1d):

$$B \equiv a\varepsilon_1 \wedge \varepsilon_2 \quad \textbf{(1d)}$$

and has an area magnitude equal to the absolute value of the scalar *a*. In a three-dimensional Euclidan space, an arbitrary bivector *B* has three components necessary to establish a planar attitude, and can be written as shown in relation (1e):

$$B \equiv a\varepsilon_2 \wedge \varepsilon_3 + b\varepsilon_3 \wedge \varepsilon_1 + c\varepsilon_1 \wedge \varepsilon_2 \quad \textbf{(1e)}$$

where *a*, *b* and *c* are scalars. In this special case of three dimensions, a duality exists between vectors and bivectors:

$$a\varepsilon_1 + b\varepsilon_2 + c\varepsilon_3 \ \leftrightarrow \ a\varepsilon_2\varepsilon_3 + b\varepsilon_3\varepsilon_1 + c\varepsilon_1\varepsilon_2 \quad \textbf{(1f)}$$

and allows for a surface normal vector to be substituted in the place of a bivector without loss of information. As noted by Lounesto, this linear isomorphism between vectors and bivectors in three dimensions explains the existence of the vector cross product. The cross product is widely used in vector analysis, an algebraic system of operations on three-dimensional vectors published by Gibbs in 1901.[4] A vector cross product exists for three-dimensional vectors because it is only in three-dimensional spaces that vectors are dual to bivectors. In any n-dimensional space every bivector has a companion bivector of opposite sign with the same planar attitude and area-magnitude, but in the three-dimensional case there exist a pair of surface normal vectors dual to the described pair of bivectors. These normal vectors can be visualized as being collinear and pointing in opposite directions.

The exterior product has numerous physical applications, one example of which is calculating the moment which results from the application of a point force to a lever. The difference between two bivectors that are equal up to a sign becomes important in this case because a plane of orientation and a magnitude alone are not enough to describe a moment (for example, think of the difference between a "clockwise" and a "counterclockwise" moment of the same magnitude in a given plane of rotation). As an illustration of the exterior product of two vectors, consider fig. 2. in which a vector *A* describing the length of a moment arm and a vector *B* describing a force applied to it are multiplied to produce a bivector *C*. The bivector has signed area with magnitude equal to that of the resulting moment. If *A* were instead the force and *B* the moment arm, the vectors would "wrap around" the other side of the parallelogram area in the opposite sense of rotation, and the sign of the bivector would change but not the area.

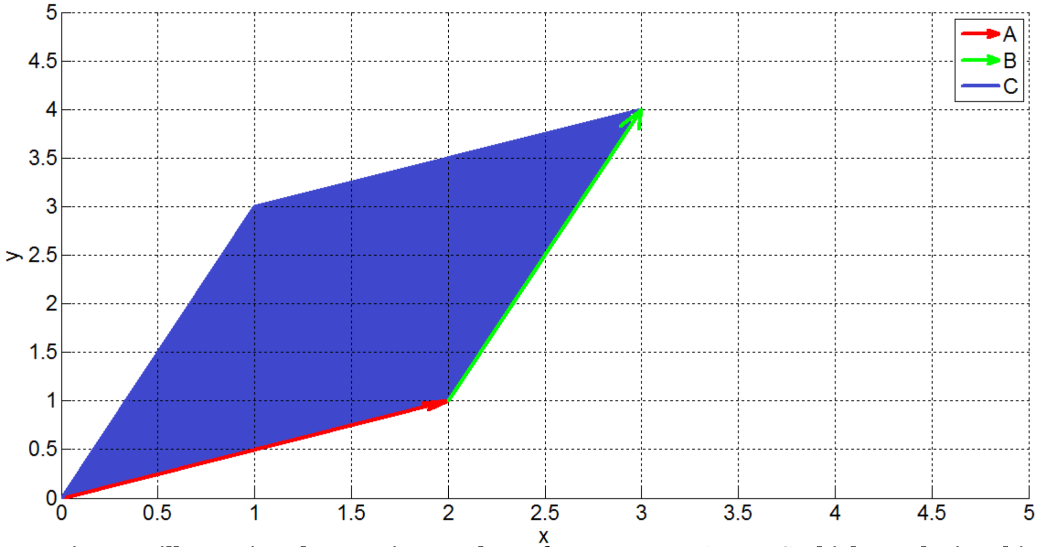

Figure 2. Diagram illustrating the exterior product of two vectors $A \wedge B = C$ which results in a bivector.

The net moment acting on a rigid body can be calculated by summing all the moments acting on the body, where each moment is represented by a bivector. Bivectors are summed in the same way that vectors are summed – by grouping like units and simplifying. Due to the duality between vectors and bivectors in three dimensional spaces, any summation of bivectors can be equivalently represented by a summation of vectors dual to those bivectors, as is the case when a sum of moments is written as a sum of cross products of pairs of vectors in vector analysis.

As a closing remark on bivectors, it should be noted that there exist bivectors in exterior algebras of dimension four and greater that cannot be formed from the exterior product of two vectors.[4] The exterior product of two vectors results in what is known as a simple bivector; a general bivector can be constructed by summing multiple simple bivectors. For example, consider the bivector below that is a sum of two simple bivectors in a four-dimensional space as shown in relation (1g). The two simple bivectors in the sum can be imagined as two planes in a four-dimensional space that intersect only at a single point – the origin of the coordinate system. These two planes in four-dimensional space are thus orthogonal, in almost the same sense as two Euclidean vectors separated by a right angle – they cannot be projected onto one another and hence the magnitude of their exterior product would be the product of their magnitudes.

$$B \equiv a\varepsilon_1 \wedge \varepsilon_2 + b\varepsilon_3 \wedge \varepsilon_4 \tag{1g}$$

The exterior product of three vectors (or equivalently the exterior product of a simple bivector and a vector) results in a simple trivector, which in a Euclidean space has volume-magnitude equal to the volume of a parallelepiped. As was the case with bivectors, an arbitrary trivector in an n-dimensional space can be written as a summation of simple trivectors, and represents a collection of parallelepipeds that intersect only at single point. A key difference between vectors and multivectors (n-vectors with $n > 1$) in some k-dimensional space is that the sum of two orthogonal vectors is always a simple vector, which is not the case with n-vectors because two orthogonal n-vectors sum to a non-simple n-vector.

As will be explained soon in connection with determinants of matrices, an n-vector in an n-dimensional space is of special significance. Just as there is only one bivector component ($e_1 \wedge e_2$) in the two-dimensional exterior algebra, there is only one trivector component ($e_1 \wedge e_2 \wedge e_3$) in the three-dimensional exterior algebra; the highest grade multivector in an exterior algebra always has a single component. Because the components of bivectors, trivectors, and multivectors in general are formed by unique combinations of vector units under the exterior product, the number of components that describe a given grade of multivector in an n-dimensional exterior algebra is given by binomial coefficients - and hence can be nicely presented in the form of Pascal's triangle. As seen in fig. 3, in a four-dimensional exterior algebra there are six components in a bivector, four components in a trivector, and one component in a quadvector.

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
1 & & 4 & & 6 & & 4 & & 1
\end{array}
$$

Figure 3. An illustrative example of the first five rows of Pascal's triangle. The number of linear components that are used to describe an arbitrary multivector are given by binomial coefficients.

All of the concepts that were mentioned thus far in exterior algebra are also applicable to Clifford algebras, which are the focus of this paper. To obtain a definition for Clifford algebras in Euclidean geometry, a tiny adjustment is needed in the rules of the exterior product to obtain the Clifford product. There is, however, one more aspect of the exterior algebra that should at least be briefly mentioned first due to its great significance in linear algebra.

II. Exterior Algebra and Linear Independence

One of the pioneering ideas of linear algebra that was recognized early on by Hermann Grassmann was the notion of linear independence. A set of vectors is linearly independent if none of the vectors in the set can be formed by linear combinations of any of the other vectors in the set.[4] A linearly independent set of vectors can serve as a basis for a vector space, a set of vectors that is closed with respect to linear combinations of its constituent vectors. The concept of linear independence is fundamental to linear algebra, but can be somewhat difficult to understand due to the abstract nature of the idea. The following non-technical explanation may help.

Consider a set containing lists of numbers (vectors) where each of these lists can be summed with another list to produce a resultant list containing information from only the two lists that were summed to create it. Due to the geometric interpretation of vectors, the "information" that they contain can be interpreted as regarding a certain linear attitude in space, hence two parallel vectors are not linearly independent because they both equally well describe the attitude of a certain line. Although it is inaccurate to make this analogy due to modern knowledge of genetics and genetic mutations, it may help to think of the resultant vector as being an "offspring vector" that inherits only the information of its parents. A set of vectors that is linearly independent contains no vectors that could be offspring or clones; each of the vectors in the set contains some unique information. Eliminating any one of the vectors in a linearly independent set of vectors results in a net loss of information; the vector space generated by the remaining vectors would have a smaller range.

Grassmann's exterior algebra can be a useful tool for determining whether a set of vectors is linearly independent. It can be shown that a set of vectors is linearly independent if and only if the exterior product of all of the vectors in the set produces a non-zero result. In linear algebra, it is well known that the determinant of a square matrix can serve as a test for whether the column vectors (or equivalently the row vectors) of a square matrix form a linearly independent set. The geometric meaning of determinant of a square matrix can be clearly explained in terms of the exterior product. For example, consider the matrix determinant below.

$$
\begin{vmatrix} 2 & 7 & 5 \\ 1 & 4 & 6 \\ 9 & 3 & 8 \end{vmatrix} = 185 \tag{2a}
$$

The determinant of a square matrix such as the example above can be computed as the exterior product of its column vectors (or equivalently its row vectors).

$$
\begin{aligned}
&(2\varepsilon_1 + 1\varepsilon_2 + 9\varepsilon_3)\wedge(7\varepsilon_1 + 4\varepsilon_2 + 3\varepsilon_3)\wedge(5\varepsilon_1 + 6\varepsilon_2 + 8\varepsilon_3) \\
&= (-33\varepsilon_2\wedge\varepsilon_3 + 57\varepsilon_3\wedge\varepsilon_1 + 1\varepsilon_1\wedge\varepsilon_2)\wedge(5\varepsilon_1 + 6\varepsilon_2 + 8\varepsilon_3) \\
&= 185\varepsilon_1\wedge\varepsilon_2\wedge\varepsilon_3
\end{aligned} \tag{2b}
$$

As such, the geometric interpretation of a matrix determinant can be verified using the geometric interpretation of the exterior product. In linear algebra, the determinant of an n by n square matrix over the real number field is the signed n-volume of a n-dimensional parallelepiped formed by its column vectors; this is exactly what one would expect from the exterior product of n linearly independent n-dimensional vectors. In a singular matrix, at least one of the column vectors (or equivalently, one of its row vectors) is not linearly independent of the others, so the parallelepiped is degenerate and the determinant is zero.

Although exterior algebra can be quite useful in linear algebra and geometry, the exterior product has some undesirable algebraic characteristics. Because the exterior product serves as a test for linear independence, multivectors in exterior algebra cannot be squared, nor can they have inverses with respect to the exterior product. This is not the case with Clifford algebras, however.

III. Euclidean Clifford Algebras

Clifford algebras are very close relatives of Grassmann's exterior algebra. Their namesake, William Clifford, was one of the few early proponents of exterior algebra. According to Lounesto,[4] Clifford first defined n-dimensional Euclidean Clifford algebras in 1882 by altering Grassmann's rules for the exterior product. The definition of the Clifford product for a Euclidean Clifford algebra $C\ell_n$ is provided below.

$$\varepsilon_i \varepsilon_j = -\varepsilon_j \varepsilon_i \;\; \forall \;\; i \neq j \tag{3a}$$

$$\varepsilon_i \varepsilon_i = 1 \tag{3b}$$

Relation (3a) is identical to relation (1d) (note that the wedge operator is no longer used to avoid confusion with the Clifford product), but relation (3b) shows the key change that Clifford made in the rules defining the exterior product. The Clifford product of two like units in $C\ell_n$ simplifies to 1 instead of zero. This small change has enormous consequences for the algebra. For example, the Clifford product of two vectors generally does not result in a bivector, but instead is the sum of a scalar with a bivector, a quantity called a parabivector. The concept of a scalar summed with a bivector may seem unusual, but parabivectors can be very useful, especially in the Clifford algebras of two and three dimensional Euclidean spaces $C\ell_2$ and $C\ell_3$.

A parabivector in $C\ell_n$ encodes more information about the vectors that were multiplied to produce it due to its additional scalar part. Specifically, a parabivector has a characteristic area, planar attitude, and characteristic angle. The geometric interpretation of the bivector part of a parabivector remains unchanged from the exterior product, and represents the area of an oriented parallelogram-shaped plane segment. Unlike a bivector however, a parabivector has scalar part that contains the information necessary to describe a characteristic angle. This scalar results from the inner or "grade-lowering" part of the Clifford product that is absent from the exterior product. The inner product part of the Clifford product is valuable because it preserves information that would be discarded by the exterior product, allowing Clifford algebras to be compatible with the notion of multiplicative inverses, unlike the exterior algebra. For example, the square of a Clifford vector is a scalar, so the multiplicative inverse of a vector is simply the vector divided by its square.

When a parabivector is the result of the product of two non-parallel Euclidean vectors, the characteristic angle is equal to the offset angle between the two vectors that were multiplied. The Euclidean Clifford product of two vectors *A* and *B* can be written in the following form:

$$AB = \|A\|\|B\|\left(\cos(\theta) + \sin(\theta)\frac{A \wedge B}{\|A \wedge B\|}\right) \tag{3c}$$

where θ is the characteristic angle, $\|A\|\|B\|$ is the product of the Euclidean norms of A and B, $A \wedge B$ is the bivector, and $\|A \wedge B\|$ is the Euclidean norm of the bivector. A parabivector thus can be visualized as having a primary vertex at which two of the sides of the plane segment meet to form the characteristic angle, as shown in fig. 4.

Parabivectors are significant because they can be used to represent a single plane of rotation. A plane of rotation is best described as a directed angle paired with a planar attitude; together they describe a unique simple rotation of less than a full revolution. In two and three-dimensional Euclidean spaces, only simple rotations can be performed as orthogonal planes are not found in these low-dimensional cases. As observed earlier in connection with the exterior product, orthogonal planes first appear in four-dimensional Euclidean spaces. In a four-dimensional Euclidean space, a bivector can be written as a sum of at most two simple bivectors; an arbitrary rotation in a four-dimensional Euclidean space can have at most two planes of rotation.[4]

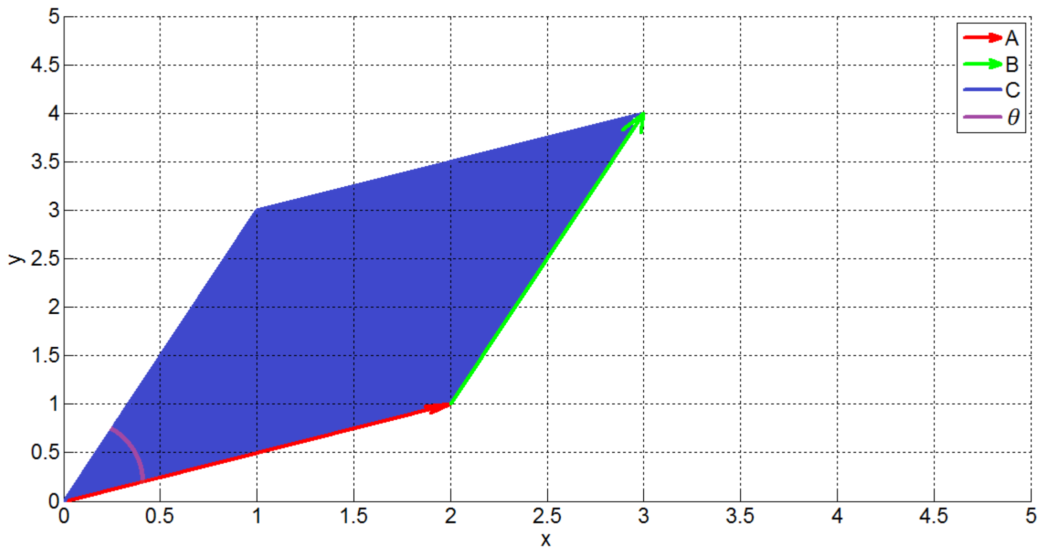

Figure 4. Diagram illustrating the Clifford product of two vectors *AB* = *C* which results in a parabivector.

In the Euclidean Clifford algebras $C\ell_2$ and $C\ell_3$ parabivectors make up the even part of the algebra. Every multivector in a Clifford algebra can be identified as being either an odd and even grade multivector; the distinction between odd and even grades is based on whether the multivector results from the Clifford product of an odd or even number of vectors.[5] Scalars result from the product of two vectors and thus can be considered multivectors of zero grade (0-vectors). The Clifford algebra $C\ell_3$ has two odd grade multivectors (vectors and trivectors) and two even grade multivectors (bivectors and scalars). The even grade multivectors of a Euclidean Clifford algebra form a subalgebra useful for describing rotational phenomena. In $C\ell_3$ for example, the even subalgebra is isomorphic as an algebra to (has an algebraic structure identical to) the algebra of quaternions discovered by Sir William Hamilton in 1843. Quaternions are widely used for modeling three-dimensional rotations in applications such as vehicle navigation. The modeling of rotations using spin groups and Clifford algebra will be discussed further in the application section of this work.

Clifford algebra can also provide an ideal algebraic structure for vector calculus. As described by Lounesto,[4] the divergence and curl operations on three-dimensional Euclidean vector fields are easily generalized to operations on n-dimensional vector fields using the Dirac operator (named after the physicist Paul Dirac) and Clifford algebra. The Dirac operator is a first-order differential operator that acts on an n-dimensional vector field, and is defined as:

$$\nabla \equiv \varepsilon_1 \frac{\partial}{\partial x_1} + \varepsilon_2 \frac{\partial}{\partial x_2} + \varepsilon_3 \frac{\partial}{\partial x_3} + \cdots + \varepsilon_n \frac{\partial}{\partial x_n} \tag{3d}$$

where the Clifford product of this operator with an n-dimensional vector field results in the sum of the divergence (a scalar) with the curl (a bivector). Assuming that the vector field is defined in an n-dimensional Euclidean space, the Euclidean Clifford product can be utilized in this operation. Clifford algebras of non-Euclidean signature also exist, however, and some of them find applications in modern physics.

IV. Signatures and Spin Groups

Unlike the exterior algebra, a Clifford algebra always has a metric signature associated with it, which is used to define the "grade-lowering" part of the Clifford product. One of the interesting features of Clifford algebra is the ease with which it can describe non-Euclidean geometries. A Euclidean Clifford algebra $C\ell_n$ is more accurately classified as a Clifford algebra with signature $C\ell_{n,0}$ because its vector basis is formed from n "space-like" units that square to 1, and zero "time-like" units that square to -1. One example a non-Euclidean metric signature exists in the algebra $C\ell_{3,1}$ which has a Minkowski space-time metric which can be useful for calculations involving relativistic phenomena in modern physics. A Clifford algebra of arbitrary signature p,q can be defined using the rules below for the $C\ell_{p,q}$ Clifford product.

$$\varepsilon_i \varepsilon_j = -\varepsilon_j \varepsilon_i \ \forall \ i \neq j \qquad \textbf{(4a)}$$

$$\varepsilon_i \varepsilon_j = +1 \ \forall \ i = j \leq p \qquad \textbf{(4b)}$$

$$\varepsilon_i \varepsilon_j = -1 \ \forall \ i = j > p \qquad \textbf{(4c)}$$

Apart from the Minkowski space-time and time-space signature algebras ($C\ell_{3,1}$ and $C\ell_{1,3}$ respectively) the Euclidean and anti-Euclidean families of Clifford algebras ($C\ell_{n,0}$ and $C\ell_{0,n}$ respectively) may be more commonly encountered in literature due to their homogeneous signatures; these were also the first two families of Clifford algebras investigated by Clifford himself.

Some interesting patterns exist in these two families of algebras. For example, the even part of a Euclidean Clifford algebra $C\ell_{n,0}$ is always isomorphic as an algebra to the anti-Euclidean algebra $C\ell_{0,n-1}$. The n-dimensional anti-Euclidean signature Clifford algebras always contain representations of certain abstract groups (sets of elements closed with respect to a single binary operation) called Euclidean spin groups, Spin(n), which are useful for describing rotations in n-dimensional Euclidean (and anti-Euclidean) spaces. A spin group with arbitrary signature p,q is denoted Spin(p,q), and a representation for it exists within the even subalgebra of the Clifford algebra $C\ell_{p,q}$.

The Euclidean spin groups Spin(n) can be represented by sets of Clifford numbers in which every number can be written as a product of exponentials of bivectors in the algebra $C\ell_{n,0}$ or $C\ell_{0,n}$. For example, the group Spin(2) is isomorphic to the circle group, which has a convenient representation as the set of all complex numbers with unit modulus, with ordinary multiplication being the group operation; the even subalgebras of $C\ell_{2,0}$ and $C\ell_{0,2}$ are both isomorphic to the complex number algebra.[4] As will be further covered in the application section of this work, the Euclidean spin groups Spin(n) provide a way to calculate rotations in n-dimensional Euclidean spaces using Clifford algebra. Rotations in Spin(n) are composed using the Clifford product; a single rotation equivalent to a sequence of rotations is given by the Clifford product of the rotations in sequence in the order that they are to occur. For example, consider a sequence of three rotations A, B, and C in which A occurs before B, and B occurs before C. A single rotation R that is equivalent to this rotation sequence is computed as the product given in relation (4d).

$$\mathrm{R} = \mathrm{ABC} \qquad \textbf{(4d)}$$

In order to investigate the Euclidean spin groups further, more fundamentals of Clifford algebra should be introduced such as conjugation operators.

V. Conjugation Operators, Euclidean Spin Groups and Normed Division Algebras

Clifford algebras have conjugation operators; for example, the algebra $C\ell_{0,1}$ is isomorphic to the complex number algebra, and has a conjugation operator identical to complex conjugation. In fact, there are two fundamental conjugation operators that act on Clifford numbers, but the Clifford algebras of one-dimensional Euclidean and anti-Euclidean signatures possess only a single conjugation operator (and $C\ell_{0,0}$ does not have any). The conjugation operator in the complex number algebra is mathematically classified as an automorphism of the algebra – a map from the algebra to itself that preserves the real linear structure of the algebra, the multiplication of numbers in the algebra, and the unity (identity) element of the algebra.[4] In Clifford algebra, this conjugation operation is called the grade automorphism, and it acts as a spatial inversion.[5] As shown in relation (5a) the grade automorphism satisfies the property that the grade automorphism of a Clifford number equals the product of the grade automorphisms of its factors because the multiplication of numbers in the algebra is preserved.

$$\overline{(ABC)} = \bar{A}\bar{B}\bar{C} \qquad \textbf{(5a)}$$

The grade automorphism maps the vector basis units that generate the Clifford algebra to their negatives. The one-dimensional Clifford algebras $C\ell_{0,1}$ and $C\ell_{1,0}$ are generated by a single vector basis unit, so the grade automorphism changes only the sign of that unit. In all other Clifford algebras, the exterior part of the Clifford product is nonzero, and multivectors such as bivectors and trivectors can exist. A spatial inversion mapping the generating units of the Clifford algebra to their negatives will change the sign of only the odd-grade multivectors, so the even elements of the algebra are unaffected[5] – the grade involution is thus aptly named.

The second conjugation operation fundamental to Clifford algebras is Hermitian conjugation. In the case of a Euclidean Clifford algebra, Hermitian conjugation can also be called reversion conjugation.[5] The Hermitian conjugate of a Clifford number changes the sign of all Clifford multivector units that square to negative unity. When a Clifford number is multiplied by its Hermitian conjugate, the resulting Clifford number is equal to its own Hermitian conjugate. Hermitian conjugation is not an automorphism, but instead an antiautomorphism of a Clifford algebra because it reverses the order of multiplication.[4]

$$(ABC)^* = C^* B^* A^* \tag{5b}$$

Note that in the algebra $C\ell_{0,1}$ multiplication is commutative so the Hermitian conjugation and grade automorphism operators of the algebra coincide. In the two Clifford algebras apart from $C\ell_{0,0}$ that are normed division algebras (the algebras $C\ell_{0,1}$ and $C\ell_{0,2}$ which are isomorphic to the complex number, and quaternion algebras respectively) the Hermitian conjugate can be used to define the norm; the norm of a number in any of these algebras is the square root of the positive scalar obtained by multiplying the number by its Hermitian conjugate. The multiplicative inverse of any number in these algebras is equal to the Hermitian conjugate of the number divided by its norm as shown in relation (5d).

$$\|X\| = \sqrt{XX^*} \tag{5c}$$

$$X^{-1} = \frac{X^*}{\|X\|} \tag{5d}$$

Representations of Euclidean spin groups exists in $C\ell_{0,1}$ and $C\ell_{0,2}$ where the elements of the spin group are represented by the set of all Clifford numbers in the algebra with norm equal to unity. Hence the inverse of an element of a spin group is simply its Hermitian conjugate. In fact, this is true of all Euclidean spin groups. A Clifford algebra representation of an element of a Euclidean spin group is obtained by computing the exponential of a bivector in a Euclidean Clifford algebra,[4] and all bivector elements of a Euclidean Clifford algebra square to negative unity.[5] Thus the Hermitian conjugate of a Euclidean bivector B is its negative so the exponentials of a bivector and its Hermitian conjugate are multiplicative inverses of each other. The Hermitian conjugate of an element of a Euclidean spin group is its multiplicative inverse then, as can be shown by writing the exponential function as a Taylor series in relation (5f).

$$\exp(B^*) = \exp\,(-B) = \exp(B)^{-1} \tag{5e}$$

$$\exp(B)^{-1} = \exp(B^*) = \sum_{k=0}^{n} \frac{(B^*)^k}{\mathrm{k!}} = \left(1 + B^* + \frac{B^* B^*}{2} + \cdots\right) = \left(1 + B + \frac{BB}{2} + \cdots\right)^* = \exp(B)^* \tag{5f}$$

Taking the Hermitian conjugate of an element of a spin group can be thought of as reversing the sense of rotation of each plane of rotation that the element describes, resulting in a rotation that would cancel that of the original spin group element (multiplicative unity obviously represents no rotation). The normed division algebras $C\ell_{0,1}$ and $C\ell_{0,2}$ are distinct from other Clifford algebras of anti-Euclidean signature because they contain representations of Euclidean spin groups (Spin(2) and Spin(3) respectively) whose elements describe at most one plane of rotation. The $C\ell_{0,3}$ Clifford algebra contains a representation of Spin(4), where orthogonal planes of rotation can exist. The algebra $C\ell_{0,3}$ is isomorphic to the direct sum algebra $C\ell_{0,2}\oplus C\ell_{0,2}$ and thus a representation of Spin(4) is isomorphic to Spin(3)⊕Spin(3). Lounesto[4] gives an explicit example of this isomorphism and describes how any rotation in a four-dimensional Euclidean space can be written as a direct sum of two isoclinic rotations – non-simple rotations in which each plane of rotation has the same rotation angle. In the context of a four-dimensional Euclidean space, a quaternion of unit-norm (which can represent an element of Spin(3)) can instead represent an isoclinic rotation.[4] If each of the unit-norm quaternions in the direct sum representation of Spin(4) are instead replaced by numbers in the quaternion algebra (which again is isomorphic to $C\ell_{0,2}$) then the norm of an arbitrary Clifford number in $C\ell_{0,3}$ (isomorphic to $C\ell_{0,2}\oplus C\ell_{0,2}$) can clearly be seen not to exist because this would require that the norms of each of the quaternions in the direct-sum pair be equal. The existence of orthogonal bivectors and non-simple rotations in the Euclidean spaces of dimension four and greater can thus serve as one rationalization for why Clifford algebras $C\ell_{0,n}$ are not normed division algebras for n > 2.

VI. Clifford Algebras as Matrices

As noted by Lounesto,[4] the Clifford algebra $C\ell_{n,n}$ can always be represented by the algebraic ring of all 2^n by 2^n matrices over the real numbers. As previously noted, one rationale for finding matrix representations of Clifford algebras is that efficient algorithms for computing matrix exponentials, logarithms, and multiplicative inverses are commonly employed in software such as MATLAB.[2] To the author's knowledge, it remains to be seen whether efficient algorithms exist for computing the multiplicative inverse, exponential and logarithm of arbitrary Clifford numbers. Of course exceptions exist for specific Clifford numbers such as vectors. As noted earlier, the square of a vector is a scalar so the inverse of any vector with nonzero square can be trivially computed by dividing the vector by its square. Computations in low-dimensional Clifford algebras can more efficient in their standard form – only insight in representation theory can be gained by representing in matrix form a Clifford number that is isomorphic to a complex number or a quaternion. The few examples involving 2 by 2 matrices below are thus of didactic rather than practical interest.

In the matrix representation of the Clifford algebra $C\ell_{1,1}$ the odd part of the algebra is a vector, and can be identified with the anti-diagonal part of a 2 by 2 real matrix, while the even part of the algebra can be identified with the matrix diagonal. The isomorphism between 2 by 2 real matrices and the algebra $C\ell_{1,1}$ can be defined as shown in (6a).

$$1 \equiv \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} \qquad \varepsilon_1 \equiv \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix} \qquad \varepsilon_2 \equiv \begin{bmatrix}0 & -1\\1 & 0\end{bmatrix} \qquad \varepsilon_1\varepsilon_2 \equiv \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix} \tag{6a}$$

The Clifford algebras $C\ell_{2,0}$ and $C\ell_{0,2}$ which have Euclidean and anti-Euclidean signatures respectively can be constructed as algebraic rings within the space of 2 by 2 matrices over the complex numbers by multiplying one of the vector basis units of $C\ell_{1,1}$ by the imaginary unit. For example, the algebra $C\ell_{2,0}$ can be defined by the basis matrices below.

$$1 \equiv \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} \qquad \varepsilon_1 \equiv \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix} \qquad \varepsilon_2 \equiv \begin{bmatrix}0 & -i\\i & 0\end{bmatrix} \qquad \varepsilon_1\varepsilon_2 \equiv \begin{bmatrix}i & 0\\0 & -i\end{bmatrix} \tag{6b}$$

The algebra $C\ell_{0,2}$ is isomorphic to the quaternion algebra, and can be defined using the basis given in (6c).

$$1 \equiv \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} \qquad \varepsilon_1 \equiv \begin{bmatrix}0 & i\\i & 0\end{bmatrix} \qquad \varepsilon_2 \equiv \begin{bmatrix}0 & -1\\1 & 0\end{bmatrix} \qquad \varepsilon_1\varepsilon_2 \equiv \begin{bmatrix}i & 0\\0 & -i\end{bmatrix} \tag{6c}$$

Any Clifford algebra $C\ell_{p,q}$ where p + q = 2n can be obtained from the $C\ell_{n,n}$ real matrix representation by selectively scalar multiplying $C\ell_{n,n}$ vector basis matrices by the imaginary unit to obtain a set of vector basis matrices with the desired signature.

The algebraic ring of all 2 by 2 matrices over the complex numbers can be mapped to the Clifford algebras $C\ell_{3,0}$ and $C\ell_{1,2}$. The Euclidean Clifford algebra $C\ell_{3,0}$ was popularized in a 2 by 2 matrix form discovered by the Physicist Wolfgang Pauli and used to describe the behavior of the nonrelativistic electron (and is thus sometimes called the Pauli algebra). It is defined below by taking the matrix representation of the Euclidean algebra $C\ell_{2,0}$ above and dividing its bivector matrix by the imaginary unit to define a third space-like vector basis unit, as seen in (6d).

$$\begin{gathered}1 \equiv \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} \qquad \varepsilon_1 \equiv \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix} \qquad \varepsilon_2 \equiv \begin{bmatrix}0 & -i\\i & 0\end{bmatrix} \qquad \varepsilon_3 \equiv \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}\\ \varepsilon_1\varepsilon_2\varepsilon_3 \equiv \begin{bmatrix}i & 0\\0 & i\end{bmatrix} \qquad \varepsilon_2\varepsilon_3 \equiv \begin{bmatrix}0 & i\\i & 0\end{bmatrix} \qquad \varepsilon_3\varepsilon_1 \equiv \begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix} \qquad \varepsilon_1\varepsilon_2 \equiv \begin{bmatrix}i & 0\\0 & -i\end{bmatrix}\end{gathered} \tag{6d}$$

As described earlier, the Clifford algebra $C\ell_{n,n}$ can always be represented by the algebraic ring of all 2^n by 2^n block matrices over the real numbers. One efficient way to construct vector basis units within 2^n by 2^n real matrices makes use of the Kronecker product (also called the matrix tensor product). This follows from the mixed-product property of tensor products. As described in *Matrix Mathematics: Theory, Facts, and Formulas* by Bernstein,[6] for matrices *A*, *B*, *C* and *D*, a matrix obtained from the Kronecker product of A with B can be matrix-multiplied with a second matrix obtained from the Kronecker product of C with D using the following identity (where ⊗ denotes the Kronecker product):

$$(A\otimes B)(C\otimes D) = AC\otimes BD \tag{6e}$$

if and only if A and B are m by n and p by q respectively and C and D are n by m and q by p respectively. In the example below, the four vector basis units of the algebra $C\ell_{2,2}$ (σ_1, σ_2, σ_3, σ_4) are defined by taking Kronecker products of the matrices used in the representation of $C\ell_{1,1}$ with units 1, ε_1, ε_2, $\varepsilon_1\varepsilon_2$. The vector basis units of all algebras $C\ell_{n,n}$ can also be defined by taking Kronecker products of the matrices used in the 2 by 2 matrix representation of of $C\ell_{1,1}$ as demonstrated in (6f). One observation worth noting at this point is that 2^n by 2^n matrices can be interpreted as 2 by 2 block matrices with 2^{n-1} by 2^{n-1} matrices as entries – the Kronecker product can be thought of as a binary operation that results in a block matrix. This suggests that the even and odd parts of any Clifford algebra $C\ell_{n,n}$ can be identified with the diagonal and anti-diagonal parts of a 2 by 2 block matrix with 2^{n-1} by 2^{n-1} real matrix entries, as exemplified in earlier in (6a) and in the vector basis unit representations shown in (6f) below.

$$\sigma_1 \equiv \varepsilon_1 \otimes \varepsilon_1\varepsilon_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$$

$$\sigma_2 \equiv \varepsilon_1 \otimes \varepsilon_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\sigma_3 \equiv \varepsilon_1 \otimes \varepsilon_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \qquad \textbf{(6f)}$$

$$\sigma_4 \equiv \varepsilon_2 \otimes 1 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

The highest grade multivector (which is a quadvector) in the $C\ell_{2,2}$ algebra is represented by:

$$\sigma_1\sigma_2\sigma_3\sigma_4 \equiv (\varepsilon_1 \otimes \varepsilon_1\varepsilon_2)(\varepsilon_1 \otimes \varepsilon_1)(\varepsilon_1 \otimes \varepsilon_2)(\varepsilon_2 \otimes 1) = (\varepsilon_1\varepsilon_2 \otimes 1) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \qquad \textbf{(6g)}$$

and as was the case with the 2 by 2 matrix representations, the algebraic ring of all 4 by 4 complex matrices can be identified with a few specific Clifford algebras when the $C\ell_{2,2}$ quadvector matrix is employed in the definition of an additional vector basis unit. This process can be extended to all Clifford algebras $C\ell_{n,n}$; when the highest grade multivector of the algebra $C\ell_{n,n}$ is used to define an additional vector basis unit, the new multivector of highest grade in the resulting Clifford algebra $C\ell_{p,q}$ where p + q = n + 1 is identified with the imaginary unit. In this way, the algebraic ring of all 4 by 4 complex matrices is seen to be isomorphic with the Clifford algebras $C\ell_{4,1}$, $C\ell_{2,3}$, and $C\ell_{0,5}$. It may be worth noting that 2^n by 2^n complex matrices always are isomorphic to a Clifford algebra with either Euclidean or anti-Euclidean signature, with the Euclidean cases occurring when n is odd, and the anti-Euclidean cases occurring when n is even.

When working with matrix representations of Clifford algebras, one problem that the author encountered was that of “reading” a matrix representation of a Clifford algebra. As emphasized earlier, the algebra $C\ell_{n,n}$ is always isomorphic to the algebraic ring of all 2^n by 2^n real matrices. Mapping an arbitrary 2^n by 2^n real matrix to a Clifford number in the algebra $C\ell_{n,n}$ is not a trivial matter however. Fortunately, the author believes he has found a reliable transformation method to accomplish this task.

VII. Mapping 2^n by 2^n Matrices to the Algebra $C\ell_{n,n}$

The following method for mapping an arbitrary 2^n by 2^n real matrix to a Clifford number in the algebra $C\ell_{n,n}$ is influenced by the fact that Kronecker products of the 2 by 2 matrix algebra representation of $C\ell_{1,1}$ provide the simplest means to find basis matrices for representing the algebra $C\ell_{n,n}$. Let the notation given in (7a) be utilized for the Clifford units in the 2 by 2 matrix representation of $C\ell_{1,1}$.

$$\varepsilon_0 \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \varepsilon_1 \equiv \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \varepsilon_2 \equiv \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad \varepsilon_{12} \equiv \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{7a}$$

A Clifford number in $C\ell_{1,1}$ can be rewritten as a 2 by 2 real matrix as shown in (7b).

$$w\varepsilon_0 + x\varepsilon_1 + y\varepsilon_2 + z\varepsilon_{12} = \begin{bmatrix} w & 0 \\ 0 & w \end{bmatrix} + \begin{bmatrix} 0 & x \\ x & 0 \end{bmatrix} + \begin{bmatrix} 0 & -y \\ y & 0 \end{bmatrix} + \begin{bmatrix} z & 0 \\ 0 & -z \end{bmatrix} = \begin{bmatrix} w+z & x-y \\ x+y & w-z \end{bmatrix} \tag{7b}$$

The relationship between the components of the Clifford number and the entries in the matrix is a linear transformation that can be written in matrix form from a sum of Kronecker products of the $C\ell_{1,1}$ basis matrices.

$$\begin{bmatrix} w+z \\ x+y \\ x-y \\ w-z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} \tag{7c}$$

This linear transformation is almost involutory; up to a scalar multiple, the transformation matrix is its own inverse.

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} w+z \\ x+y \\ x-y \\ w-z \end{bmatrix} \tag{7d}$$

The Kronecker product is commonly used in conjunction with the vec operator, which constructs a column vector from the columns of a matrix.[6] Consider the action of the vec operator on a 2 by 2 matrix *X* composed from a sum of Clifford basis matrices, as demonstrated in relation (7e).

$$vec(X) = vec\left(\begin{bmatrix} w+z & x-y \\ x+y & w-z \end{bmatrix} \right) = \begin{bmatrix} w+z \\ x+y \\ x-y \\ w-z \end{bmatrix} \tag{7e}$$

The following useful identity (7f) exists:[6]

$$vec(AXB) = B^T \otimes A vec(X) \tag{7f}$$

which can be used to show that for any 2 by 2 matrix *X* that relation (7g) holds.

$$vec(\varepsilon_1 X \varepsilon_1) + vec(\varepsilon_0 X \varepsilon_{12}) = (\varepsilon_1 \otimes \varepsilon_1 + \varepsilon_{12} \otimes \varepsilon_0) vec(X) \tag{7g}$$

Thus any 2 by 2 matrix can be "read" as a Clifford algebra using relation (7h).

$$\frac{1}{2}\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} w+z & x-y \\ x+y & w-z \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} w+z & x-y \\ x+y & w-z \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right) = \begin{bmatrix} w & x \\ y & z \end{bmatrix} \tag{7h}$$

A deep relationship may exist between the $C\ell_{1,1}$ basis matrices used in the transformation and a set of matrices called Walsh matrices. A Walsh matrix is a binary (has +1 and -1 as its only entries) 2^n by 2^n matrix that is symmetric and has orthogonal column (and row) vectors so that the inner product of any two columns (or rows) of the matrix is zero. The smallest nontrivial Walsh matrix can be expressed as a sum of $C\ell_{1,1}$ basis matrices, as shown in relation (7i).

$$\varepsilon_1 + \varepsilon_{12} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \tag{7i}$$

It can be shown that matrices resulting from Kronecker products of the 2 by 2 Walsh matrix with itself are also Walsh matrices. It thus follows that a Walsh matrix can be expressed as a sum of 2^n unique basis matrices from the 2^n by 2^n real matrix representation of the algebra $C\ell_{n,n}$, which are also formed by Kronecker products of $C\ell_{1,1}$ basis matrices. In relation (7j) below the 4 by 4 Walsh matrix is expressed in this manner.

$$(\varepsilon_1 + \varepsilon_{12}) \otimes (\varepsilon_1 + \varepsilon_{12}) = (\varepsilon_1 \otimes \varepsilon_1) + (\varepsilon_1 \otimes \varepsilon_{12}) + (\varepsilon_{12} \otimes \varepsilon_1) + (\varepsilon_{12} \otimes \varepsilon_{12}) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \tag{7j}$$

The $C\ell_{n,n}$ basis matrices that are summed to create the 2^n by 2^n Walsh matrix may be related to the linear transformation needed to map a 2^n by 2^n real matrix to the Clifford algebra $C\ell_{n,n}$. For example, consider a 4 by 4 real matrix composed as a sum of the 16 Clifford units in the algebra $C\ell_{2,2}$ as shown in (7k).

$$\begin{bmatrix} a+f+u+z & b-e+v-y & c+h-s-x & d-g-t+w \\ b+e+v+y & a-f+u-z & d+g-t-w & c-h-s+x \\ c+h+s+x & d-g+t-w & a+f-u-z & b-e-v+y \\ d+g+t+w & c-h+s-x & b+e-v-y & a-f-u+z \end{bmatrix} \tag{7k}$$

The linear transformation that maps the Clifford algebra $C\ell_{2,2}$ to a 4 by 4 real matrix (and vice versa) is given below.

$$\begin{bmatrix} a+f+u+z \\ b+e+v+y \\ c+h+s+x \\ d+g+t+w \\ b-e+v-y \\ a-f+u-z \\ d-g+t-w \\ c-h+s-x \\ c+h-s-x \\ d+g-t-w \\ a+f-u-z \\ b+e-v-y \\ d-g-t+w \\ c-h-s+x \\ b-e-v+y \\ a-f-u+z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \\ g \\ h \\ s \\ t \\ u \\ v \\ w \\ x \\ y \\ z \end{bmatrix} \tag{7l}$$

The transformation matrix is equal to the following sum of Kronecker products of the $C\ell_{1,1}$ basis matrices:

$$\varepsilon_1 \otimes \varepsilon_1 \otimes \varepsilon_1 \otimes \varepsilon_1 + \varepsilon_1 \otimes \varepsilon_{12} \otimes \varepsilon_1 \otimes \varepsilon_0 + \varepsilon_{12} \otimes \varepsilon_1 \otimes \varepsilon_0 \otimes \varepsilon_1 + \varepsilon_{12} \otimes \varepsilon_{12} \otimes \varepsilon_0 \otimes \varepsilon_0 \tag{7m}$$

and using the Kronecker product identity (7f) it can be shown that:

$$(\varepsilon_1 \otimes \varepsilon_1 \otimes \varepsilon_1 \otimes \varepsilon_1 + \varepsilon_1 \otimes \varepsilon_{12} \otimes \varepsilon_1 \otimes \varepsilon_0 + \varepsilon_{12} \otimes \varepsilon_1 \otimes \varepsilon_0 \otimes \varepsilon_1 + \varepsilon_{12} \otimes \varepsilon_{12} \otimes \varepsilon_0 \otimes \varepsilon_0) vec(X) = \tag{7n}$$

$$vec\big((\varepsilon_1 \otimes \varepsilon_1) X (\varepsilon_1 \otimes \varepsilon_1)\big) + vec\big((\varepsilon_1 \otimes \varepsilon_0) X (\varepsilon_1 \otimes \varepsilon_{12})\big) + vec\big((\varepsilon_0 \otimes \varepsilon_1) X (\varepsilon_{12} \otimes \varepsilon_1)\big) + vec\big((\varepsilon_0 \otimes \varepsilon_0) X (\varepsilon_{12} \otimes \varepsilon_{12})\big)$$

where the matrices $\varepsilon_1 \otimes \varepsilon_1$, $\varepsilon_1 \otimes \varepsilon_{12}$, $\varepsilon_{12} \otimes \varepsilon_1$ and $\varepsilon_{12} \otimes \varepsilon_{12}$ are the four $C\ell_{2,2}$ basis matrices which when summed result in a 4 by 4 Walsh matrix. These matrices are always right-multiplied in the transformation, while the left-multiplied matrices substitute ε_0 for ε_{12} in the Kronecker product of $C\ell_{1,1}$ basis matrices. The author has verified the relationship between the Walsh matrices and the linear map from a 2^n by 2^n real matrix to the Clifford algebra $C\ell_{n,n}$ for n < 5 and conjectures that this relationship holds for all n.

As a closing remark, the ties between Walsh matrices and Clifford algebras are very deep. Lounesto devotes part of one chapter in his book[4] to binary definitions of Clifford algebras using the Walsh functions (which are the binary sequences given by the columns of a Walsh matrix). Lounesto does not remark upon the relationship between the Walsh matrices and matrix representations of Clifford algebras in his book, however. As an illustration of this relationship, consider how any vector with 2^n components can be decomposed into summation of the 2^n column vectors of a 2^n by 2^n Walsh matrix, each with a scalar coefficient, as in the decomposition relation (7o).

$$\begin{bmatrix} 3.14 \\ -2.57 \\ 8.63 \\ 1.95 \end{bmatrix} = 2.7875 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + 3.0975 \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} - 2.5025 \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} - 0.2425 \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} \tag{7o}$$

Regions of a 2^n by 2^n matrix such as the matrix diagonal and matrix antidiagonal can be regarded as vector-like modules with respect to matrix summation, and indeed can be decomposed into a summation of 2^n Clifford basis matrices, each of which carries the binary sequence of a different Walsh function. Any 2^n by 2^n matrix thus can be decomposed into a summation of 2^n vector-like regions (such as the matrix diagonal), where each of these regions can be decomposed into a summation of 2^n Clifford basis matrices. In this sense, the Walsh matrices allow for matrix representations of Clifford algebras to exist.

VIII. Application: Euclidean Spin Groups and Rotations

One of the most prominent motivations for working with Clifford algebra stems from the fact that Clifford algebras are the natural parents of the spin groups. The success of spin groups and spinor theory in modeling physical phenomena such as the angular momentum of an electron is evidenced by the pioneering works of Wolfgang Pauli, Paul Dirac and others in the early twentieth century, and such theories remain important to active physicists today. As mentioned earlier, spin groups are also useful for calculating rotation transformations. For example, a Euclidean spin group Spin(n) is related to the special orthogonal group SO(n) which can be represented by the set of all n by n orthogonal matrices with unit determinant (sometimes called rotation matrices) and both can be used to model rotations in an n-dimensional Euclidean space. In the language of mathematical topology, Spin(n) is both a universal covering group and a two-fold covering group of the SO(n) group.[4]

To elaborate on the relationship between Spin(n) and SO(n), consider that an n by n rotation matrix serves as a unique rotation transformation in an n-dimensional Euclidean space; in contrast, there exist two different Clifford numbers in the Clifford algebra representation of Spin(n), either of which can be used in a computation of the same rotation transformation. As an example of this phenomenon, consider the group Spin(2) and the everyday experience of navigating the surface of Earth. If one is facing north and desires to face east instead, one can rotate either clockwise or counterclockwise to do so; the topology of Spin(2) and Spin(3) allows one to distinguish between equivalent "clockwise" and "counterclockwise" rotations of less than a full revolution (quotations emphasizing that the terms clockwise and counterclockwise are ambiguous and used for their analogy only). The ability to distinguish between two equivalent rotations of less than a full revolution may be desirable in some applications such as robotics and computer animation.

As described earlier, the exponential function has special significance in representations of both the SO(n) and Spin(n) groups. A matrix representation of an element in the group SO(n) can be generated by taking the matrix exponential of a real-valued n by n matrix which changes only its sign when operated on by the matrix transpose (or equivalently the Hermitian conjugate); such matrices are called skew-symmetric. Likewise, an Clifford algebra representation of an element in the group Spin(n) can be generated by the exponential of a bivector in the algebra $C\ell_n$. Euclidean bivectors also change only their sign when operated on by the Hermitian conjugate. In the special case of a two-dimensional Euclidean space, representations of elements in the groups Spin(2) and SO(2) can both be generated by the matrix exponential of a 2 by 2 skew-symmetric matrix because the Clifford algebra $C\ell_2$ can be represented using 2 by 2 real matrices. The difference between rotations calculated using representations of Spin(2) instead of SO(2) in this case results only from the difference between the actions of these group representations on vectors. Rotation matrices, for example, act on matrix column vectors, while the Clifford spin groups act on Clifford vectors. When Clifford algebras are represented by matrices, Clifford vectors take the form of square matrices instead of column vectors, and the rotation is computed as a matrix similarity transformation as opposed to the action of a column vector on a matrix.

Consider the following 2 by 2 rotation matrix *R* calculated to the nearest four decimal places:

$$R = \exp\left(\begin{bmatrix} 0 & -5 \\ 5 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0.2837 & 0.9589 \\ -0.9589 & 0.2837 \end{bmatrix} \quad \textbf{(8a)}$$

and its action on the column vector defined below:

$$V = \begin{bmatrix} 7 \\ 4 \end{bmatrix} \quad \textbf{(8b)}$$

where the *V* is multiplied by *R* to compute the rotated vector as shown.

$$\begin{bmatrix} 5.8213 \\ -5.5778 \end{bmatrix} = \begin{bmatrix} 0.2837 & 0.9589 \\ -0.9589 & 0.2837 \end{bmatrix} \begin{bmatrix} 7 \\ 4 \end{bmatrix} \quad \textbf{(8c)}$$

Because a rotation in two spatial dimensions has a single plane of rotation (unlike rotations in n-dimensional Euclidean spaces where $n > 3$) one can identify the rotation composed in eq. (8a) as a "counterclockwise" rotation of 5 radians, which is approximately 286.5 degrees. This rotation is indistinguishable from a rotation of about –1.2832 radians, where the negative sign indicates "clockwise" rotation; a single rotation matrix represents both of these rotations, as demonstrated in the example below.

$$R = \exp\left(\begin{bmatrix} 0 & -5 \\ 5 & 0 \end{bmatrix}\right) = \exp\left(\begin{bmatrix} 0 & 1.2832 \\ -1.2832 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0.2837 & 0.9589 \\ -0.9589 & 0.2837 \end{bmatrix} \quad \textbf{(8d)}$$

In contrast with the matrix representations of special orthogonal groups (which act on matrix column vectors) the Clifford algebra representations of spin groups act on Clifford vectors using a coordinate transformation called similarity transformation. If *S* is an element in Spin(n) then it can rotate a Clifford vector *U* in the algebra $C\ell_n$ using the following similarity relation:

$$(S^{-1})US = V \quad \textbf{(8e)}$$

where *V* is the vector that results from the rotation transformation acting on *U*. The sign of the spin element *S* in the rotation similarity transformation has no impact on the result of the transformation because negative signs cancel. This effect is responsible for the phenomenon by which two different Clifford numbers can be used in the computation of the same rotation transformation. Equivalent "clockwise" and "counterclockwise" rotations of less than a full revolution in the algebras $C\ell_2$ and $C\ell_3$ can be distinguished by the sign of the *S* term in eq. (8e). The previous example in which a rotation was computed using SO(2) will now be reworked using Spin(2) to demonstrate rotation calculations in Clifford algebra. First, consider the matrix representation of the algebra $C\ell_2$ defined below.

$$1 \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \varepsilon_1 \equiv \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \varepsilon_2 \equiv \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad \varepsilon_1\varepsilon_2 \equiv \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad \textbf{(8f)}$$

An element *S* of the group Spin(n) can be written as the exponential of some bivector *B* in the $C\ell_n$ algebra, but when composing rotations, it is customary to scalar multiply the bivector by one-half before the exponential is computed; doing so provides clarity because the bivector then can be interpreted as having units of radians. Thus a rotation in the $C\ell_n$ algebra can be composed using relation (8g):

$$\exp\left(\frac{-1}{2}B\right) U \exp\left(\frac{1}{2}B\right) = V \quad \textbf{(8g)}$$

where *B* is a bivector, *U* is a Clifford vector to be rotated and *V* is the resultant vector. Due to the factor of one-half in the exponential this similarity transformation is sometimes called a half-angle rotation transformation, particularly in the case where the bivector represents a single plane of rotation. In the special case of Spin(2), one can make use of the isomorphism between elements of Spin(2) and elements of SO(2) and realize that eq. (8g) can be equivalently written in this case as:

$$\frac{1}{\pm\sqrt{R}} U\left(\pm\sqrt{R}\right) = V \quad \textbf{(8h)}$$

where R is a 2 by 2 rotation matrix. One can then reuse the results of the rotation calculation in the SO(2) example and calculate the square root of the rotation matrix:

$$\pm\sqrt{R} = \pm\begin{bmatrix} 0.2837 & 0.9589 \\ -0.9589 & 0.2837 \end{bmatrix}^{\frac{1}{2}} = \pm\begin{bmatrix} 0.8011 & 0.5985 \\ -0.5985 & 0.8011 \end{bmatrix} \quad \textbf{(8i)}$$

and then verify the difference in sign that exists between the two equivalent rotations computed using $C\ell_2$:

$$\exp\left(\frac{1}{2}\begin{bmatrix} 0 & 1.2832 \\ -1.2832 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0.8011 & 0.5985 \\ -0.5985 & 0.8011 \end{bmatrix} \quad \textbf{(8j)}$$

$$\exp\left(\frac{1}{2}\begin{bmatrix} 0 & -5 \\ 5 & 0 \end{bmatrix}\right) = \begin{bmatrix} -0.8011 & -0.5985 \\ 0.5985 & -0.8011 \end{bmatrix} \quad \textbf{(8k)}$$

giving an example of how the sign of the S term in eq. (8e) allows one to distinguish between "clockwise" and "counterclockwise" versions of the same rotation in two-dimensional space. The vector used in the SO(2) example, when written as a Clifford vector in the 2 by 2 real matrix representation of $C\ell_2$ has the form:

$$7\varepsilon_1 + 4\varepsilon_2 = 7\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + 4\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 4 & 7 \\ 7 & -4 \end{bmatrix} \quad \textbf{(8l)}$$

and the equivalence of the Clifford algebra and rotation matrix methods can be shown as demonstrated below.

$$\begin{bmatrix} 0.8011 & -0.5985 \\ 0.5985 & 0.8011 \end{bmatrix}\begin{bmatrix} 4 & 7 \\ 7 & -4 \end{bmatrix}\begin{bmatrix} 0.8011 & 0.5985 \\ -0.5985 & 0.8011 \end{bmatrix} = \begin{bmatrix} -5.5778 & 5.8213 \\ 5.8213 & 5.5778 \end{bmatrix} \quad \textbf{(8m)}$$

In the previous examples, it was interesting to see how the elements of Spin(2) and SO(2) can both be represented by 2 by 2 orthogonal matrices, which are algebraically isomorphic to complex numbers with unit modulus. When these 2 by 2 orthogonal matrices were used to compute rotations of Euclidean vectors however, they acted on completely different matrices (matrix column vectors versus matrix representations of Clifford vectors). Clifford vector rotations composed using representations of Spin(2) were seen to use the half-angle rotation formula, while SO(2) rotations could be composed from the exponential of a 2 by 2 skew-symmetric matrix without the factor of one-half; this fundamental difference in computation is observed between all Clifford vector rotations composed using Spin(n) and all matrix column vector rotations using SO(n), as noted by Lounesto.[4]

Before concluding, a brief note on the relationship between quaternions and Clifford algebra with respect to the computation of three-dimensional rotations seems appropriate. As previously described, the even subalgebra of $C\ell_3$ is the subalgebra of all parabivectors in $C\ell_3$ and is isomorphic to the quaternion algebra. By exploiting the duality between vectors and bivectors that exists in a three-dimensional Euclidean, a bivector can be substituted for a vector (and vice versa) as the object that the parabivector acts on in the rotation transformation. Thus rotations in a three-dimensional Euclidean space can be computed using only the even part of $C\ell_3$, or equivalently the algebra of quaternions.

IX. Conclusion

Although Clifford algebras are primarily known for their use in constructing representations of spin groups, they also provide a useful algebraic structure for calculations involving vectors. For example, Clifford algebra not only generalizes vector calculus operations such as divergence and curl to n-dimensional vector spaces, but also allows for calculations involving non-Euclidean vectors to be easily handled. The applications of Clifford algebra are numerous, and the scope of this work greatly limited the number of applications of Clifford algebra that could be discussed; in particular, the ability of Clifford algebra to describe conformal transformations and the use of Clifford algebra in projective geometry were both completely neglected. Its usefulness in calculations involving relativistic phenomena were mentioned only in passing. Any reader interested in learning relativistic electrodynamics should consider reading Baylis' book *Electrodynamics: A Modern Geometric Approach*, which provides a good introduction to Clifford algebra before applying it to physics usually addressed using cumbersome space-time tensors. In closing, the author highly recommends *Clifford Algebras and Spinors* by Lounesto as an excellent reference which greatly assisted in the production of this work.

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