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A Sequential Search Distribution: Proofreading, Russian Roulette, and the Incomplete q-Eulerian Polynomials

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The distribution for the number of searches needed to find k of n lost objects is expressed in terms of a refinement of the q-Eulerian polynomials, for which formulae are developed involving homogeneous symmetric polynomials. In the case when $k = n$ and the find probability remains constant, relatively simple and efficient formulas are obtained. From our main theorem, we further (1) deduce the inverse absorption distribution and (2) determine the expected number of times the survivor pulls the trigger in an n-player game of Russian roulette.

Keywords: sequential search distribution, inverse absorption distribution, proofreading, Russian roulette, q-Eulerian polynomials

1 Main Results: An Overview

There are many contexts in which a sequence of searches is conducted for lost objects. The proofreading of a manuscript is a case in point: Each complete reading is a search for errors. Ideally, the process is halted when all objects have been located. However, practical considerations may require termination of the process when some specified number of the lost objects have been found.

Each search is assumed to be conducted in the same order (*e. g.*, each reading of a manuscript typically proceeds from the first to the last page). If the lost objects are labeled 1 through n in accordance with the search order, then the first search may be viewed as a left-to-right scan of the list $1\,2\,\ldots n$. For convenience, each object found is subscripted by the number of the search on which it is found. For instance, the subscripted list

$$\omega = 1\ \ 2_1\ \ 3_4\ \ 4_1\ \ 5_5\ \ 6 \tag{1}$$

is the outcome of a sequential search for 4 of 6 objects where objects 2 and 4 are found on the first search, no finds are made on the second and third searches, object 3 is found on the fourth search, and object 5 is found on the fifth search. The process is terminated with the fourth find.

Another sequential search strategy based on Blomqvists' [2] absorption process in which at most one find is allowed per search (so each find, except the k^{th} one, signals the start of a new search) has been considered by Dunkl [4] for proofreading and by Kemp [5] in preserving endangered species. Although clearly different than the sequential search strategy considered herein, we will show in section 7 that Dunkl's proofreading distribution (known as the inverse absorption distribution) follows from our results.

The outcome of a sequential search for k of n objects may also be expressed as a sequence of Bernoulli trials: Upon encountering a lost object, a coin is tossed. The object is found if heads occurs. So the outcome in (1) may be represented as below. Each block constitutes a search.

$$\omega = THTHTT\ \ TTTT\ \ TTTT\ \ THTT\ \ TH \tag{2}$$

As resources may be depleted or as knowledge may be gained in making finds, it is natural to assume that each find probability depends on the prior number of finds: Specifically, the probability of an object being found after the occurence of j^{th} find is assumed to be $1-q_{k-j}$ where $0 \le q_{k-j} < 1$. In the Bernoulli trials context, this amounts to changing coins each time heads occurs where the probability of the j^{th} coin landing tails up is q_{k-j+1}. So ω in (2) occurs with probability $q_4(1-q_4)q_3(1-q_3)q_2^{11}(1-q_2)q_1^3(1-q_1)$. Varying the find probabilities will be particularly useful in section 3 where we compute the expected number of times the survivor pulls the trigger in an n-player game of Russian roulette.

Let $p_{n,k}(m)$ denote the probability that m searches are needed to find k of n lost objects. In section 4, we prove

Theorem 1.1 *The probability generating function for* $\{p_{n,k}(m)\}_{m\ge 1}$ *is*

$$P_{n,k}(z) \doteq \sum_{m\ge 1} p_{n,k}(m)z^m = \frac{z(1-q_1)(1-q_2)\cdots(1-q_k)E_{n,k}(z)}{(1-zq_1^{n-k+1})(1-zq_2^{n-k+2})\cdots(1-zq_k^n)} \tag{3}$$

where $E_{n,k}(z)$ *denotes the* $(n,k)^{th}$ *incomplete* **q**-*Eulerian polynomial. Moreover, the expected number of searches needed to find* k *of* n *objects is*

$$P'_{n,k}(1) = 1 + \frac{E'_{n,k}(1)}{[n-k+1]_1[n-k+2]_2\cdots[n]_k} + \sum_{j=1}^{k} \frac{q_j^{n-k+j}}{1-q_j^{n-k+j}}$$

where $[j]_i = (1-q_i^j)/(1-q_i)$ *denotes the* q_i*-analog of* j.

A combinatorial definition, recurrence relationship, and determinant formula for $E_{n,k}(z)$ are presented in section 2. When $k = n$ and $q_n = \cdots = q_1 = q$, our combinatorial interpretation is equivalent to one due to Carlitz [3] for the ordinary q-Eulerian polynomials and, as verified in section 5, Theorem 1 leads to a closed formula for $p_{n,n}(m)$ and to a relatively efficient means for computing the expected number of searches $P'_{n,n}(1)$:

Corollary 1.2 *If* $q_n = \cdots = q_1 = q$, *then* $p_{n,n}(m) = (1-q^m)^n - (1-q^{m-1})^n$ *and*

$$P'_{n,n}(1) = 1 + \sum_{j=1}^{n} (-1)^{j-1} \binom{n}{j} \frac{q^j}{1-q^j}.$$

2 The Incomplete **q**-Eulerian Polynomials

Let $\mathcal{I}_{n,k}$ be the set of injections from $\{1,2,\ldots,k\}$ to $\{1,2,\ldots,n\}$. An injection $f \in \mathcal{I}_{n,k}$ will be expressed as the list $f(1)f(2)\ldots f(k)$ of its range elements. The *Descent set*, *descent number*, and *comajor index* of $f \in \mathcal{I}_{n,k}$ are respectively

$$\operatorname{Des} f = \{j : 1 \le j < k, f(j) > f(j+1)\}, \quad \operatorname{des} f = |\operatorname{Des} f|, \quad \text{and}$$
$$\operatorname{cmj} f = |\{1,2,\ldots,f(k)\} \setminus \{f(1),f(2),\ldots,f(k)\}| + \sum_{j \in \operatorname{Des} f} (n-j).$$

For $k = n$, $\mathcal{I}_{n,n}$ is just the set of permutations on $\{1,2,\ldots,n\}$ and the comajor index is a close relative of a statistic that has become known as the major index first considered by MacMahon [6]. For $1 \le i \le k$, the i^{th} *cyclic interval* of f is defined as

$$[f,i]_c = \begin{cases} \{f(i-1), f(i-1)+1, \ldots, f(i)\} & \text{if } f(i-1) < f(i) \\ \{f(i-1), f(i-1)+1, \ldots, n, 1, \ldots, f(i)\} & \text{if } f(i-1) > f(i) \end{cases}$$

with the convention that $f(0) = 0$. The i^{th} *cyclic run* in f is

$$\operatorname{run}(f,i) = |[f,i]_c \setminus \{0, f(1), f(2), \ldots, f(i)\}|.$$

For $f = 2435 \in \mathcal{I}_{6,4}$, note that $\operatorname{Des} f = \{2\}$, $\operatorname{des} f = 1$, $\operatorname{cmj} f = |\{1\}| + (6-2) = 5$, $\operatorname{run}(f,1) = 1 = \operatorname{run}(f,2)$, $\operatorname{run}(f,3) = 3$, and $\operatorname{run}(f,4) = 0$. Although not of importance here, the fact that the sum of the runs equals the comajor index noted in [7] for $k = n$ also holds for $1 \le k < n$.

We define the $(n,k)^{th}$ *incomplete* **q**-*Eulerian polynomial* as

$$E_{n,k}(z) = \sum_{f \in \mathcal{I}_{n,k}} q_k^{\operatorname{run}(f,1)} q_{k-1}^{\operatorname{run}(f,2)} \cdots q_1^{\operatorname{run}(f,k)} z^{\operatorname{des} f}. \tag{4}$$

As each injection is uniquely characterized by its sequence of runs, $E_{n,k}(z)$ contains $|\mathcal{I}_{n,k}| = n(n-1)\cdots(n-k+1)$ distinct monomials. The permutation case $k = n$ with $q_n = q_{n-1} = \ldots = q_1 = q$ is the n^{th} q-Eulerian polynomial considered by Carlitz [3]. The following result is proven in section 6.

Theorem 2.1 *Let*

$$a_{n,k} = \sum_{j=0}^{n-k} \sum_{1 \le i_1 \le i_2 \le \ldots \le i_j \le k} q_{i_1} q_{i_2} \cdots q_{i_j},$$
$$b_{n,k,j} = \sum_{k-j \le i_1 \le i_2 \le \ldots \le i_{n-j} \le k} q_{i_1} q_{i_2} \cdots q_{i_{n-j}}, \quad \text{and}$$
$$(z)_{n,k,j} = \prod_{i=1}^{j} (1 - z q_{k-j+i}^{n-j+i})$$

with $a_{n,n} = 1 = (z)_{n,k,0}$ and $b_{n,k,0} = q_k^n$. The incomplete **q**-*Eulerian polynomials may be computed using the recurrence relationship*

$$E_{n,k}(z) = (z)_{n-1,k-1,k-1} a_{n,k} + z \sum_{j=1}^{k-1} (z)_{n-1,k-1,j-1} b_{n,k,j} E_{n-j,k-j}(z) \tag{5}$$

or the determinant formula $E_{n,k}(z) = \det(c_{i,j})_{1\le i,j\le k}$ *with entries*

$$c_{i,j} = \begin{cases} 1 & \text{if } 1 < i = j \le k \\ (z)_{n-i,k-i,k-i}a_{n-i+1,k-i+1} & \text{if } 1 = j \le i \le k \\ -z(z)_{n-i,k-i,j-i-1}b_{n-i+1,k-i+1,j-i} & \text{if } 1 < i < j \le k \\ 0 & \text{otherwise.} \end{cases}$$

Note that $a_{n,k}$ is the sum of the homogeneous symmetric polynomials of degrees 0 through $n-k$ in the variables $q_k,\ldots,q_1$ and that $b_{n,k,j}$ is the homogeneous symmetric polynomial of degree $n-j$ in $q_k,\ldots,q_{k-j}$. For instance, $a_{4,2} = 1+(q_2+q_1)+(q_2^2+q_2q_1+q_1^2)$ and $b_{4,4,2} = q_4^2+q_3^2+q_2^2+q_4q_3+q_4q_2+q_3q_2$. Also, observe that if $q_k = q_{k-1} = \ldots = q_1 = q$, then

$$a_{n,k} = \sum_{j=0}^{n-k}\binom{k+j-1}{j}q^j \text{ and } b_{n,k,j} = \binom{n}{j}q^{n-j}.$$

As an illustration of Theorem 2, $E_{4,1}(z) = |a_{4,1}| = 1+q_1+q_1^2+q_1^3$,

$$\begin{aligned} E_{4,2}(z) &= \begin{vmatrix} (z)_{3,1,1}a_{4,2} & -zb_{4,2,1} \\ a_{3,1} & 1 \end{vmatrix} = 1+q_1+q_2+q_1^2+q_1q_2+q_2^2 \\ &\qquad +z(q_1q_2^2+q_1q_2^3+q_1^2q_2+q_1^2q_2^2+q_1^2q_2^3+q_2^3), \end{aligned}$$

$$\begin{aligned} E_{4,3}(z) &= \begin{vmatrix} (z)_{3,2,2}a_{4,3} & -zb_{4,3,1} & -z(z)_{3,2,1}b_{4,3,2} \\ (z)_{2,1,1}a_{3,2} & 1 & -zb_{3,2,1} \\ a_{2,1} & 0 & 1 \end{vmatrix} \\ &= 1+q_1+q_2+q_3+z^2(q_1q_2q_3^3+q_1q_2^2q_3^2+q_1q_2^2q_3^3+q_2^2q_3^3) \\ &\quad +z\left(\begin{array}{l} q_1q_2+q_1q_3+q_1q_2^2+q_1q_3^2+q_1q_3^3+q_1q_2q_3+q_1q_2^2q_3+q_1q_2q_3^2 \\ +q_2^2+q_2q_3+q_2q_3^2+q_2q_3^3+q_2^2q_3+q_2^2q_3^2+q_3^2+q_3^3 \end{array}\right), \end{aligned}$$

and

$$\begin{aligned} E_{4,4}(z) &= \begin{vmatrix} (z)_{3,3,3} & -zb_{4,4,1} & -z(z)_{3,3,1}b_{4,4,2} & -z(z)_{3,3,2}b_{4,4,3} \\ (z)_{2,2,2} & 1 & -zb_{3,3,1} & -z(z)_{2,2,1}b_{3,3,2} \\ (z)_{1,1,1} & 0 & 1 & -zb_{2,2,1} \\ 1 & 0 & 0 & 1 \end{vmatrix} \\ &= 1+z(q_2+q_2q_3+q_2q_4+q_3+q_3^2+q_3q_4+q_3q_4^2+q_3^2q_4+q_4+q_4^2+q_4^3) \\ &\quad +z^2\left(\begin{array}{l} q_2q_3^2+q_2q_4^2+q_2q_4^3+q_2q_3q_4+q_2q_3q_4^2+q_2q_3q_4^3 \\ +q_2q_3^2q_4+q_2q_3^2q_4^2+q_3q_4^3+q_3^2q_4^2+q_3^2q_4^3 \end{array}\right)+z^3(q_2q_3^2q_4^3). \end{aligned}$$

When $q_4 = \ldots = q_1 = q$, $E_{4,4}(z) = 1+q(3+5q+3q^2)z+q^3(3+5q+3q^2)z^2+q^6z^3$ is the ordinary 4^{th} q-Eulerian polynomial.

3 Russian Roulette

Theorem 1 provides a solution to a question regarding an n-player game of Russian roulette: Given that after the j^{th} discharge the gun fires with probability $1-q_{n-j}$ and that play continues until only one player remains, how many times does the survivor expect to pull the trigger? The case when the gun is reloaded after each discarge so that $q_n = q_{n-1} = \ldots = q_1 = q$ was considered by Knuth (personal communication). In the event that the gun has c chambers, initially contains $b \geq n-1$ bullets, and is not reloaded, then $1-q_n = b/c$, $1-q_{n-1} = (b-1)/c$, and $1-q_1 = (b-n+1)/c$.

The connection to Russian roulette is made by viewing each sweep through the playing order as a search and the discharge of the gun as a bullet finding its victim. In this light, the number of times the survivor expects to pull the trigger is the expected number of searches needed to find n of n objects minus the expected number of searches required to find 1 of 1 objects;

$$P'_{n,n}(1) - P'_{1,1}(1) = \frac{E'_{n,n}(1)}{[1]_1[2]_2\cdots[n]_n} + \sum_{j=2}^{n} \frac{q_j^j}{1-q_j^j}.$$

For $q_n = q_{n-1} = \ldots = q_1 = q$, Corollary 1 gives the simpler formula

$$P'_{n,n}(1) - P'_{1,1}(1) = -\frac{q}{1-q} + \sum_{j=1}^{n} (-1)^{j-1} \binom{n}{j} \frac{q^j}{1-q^j}.$$

The probability of the i^{th} player being the survivor is deduced at the end of the next section.

4 Proof of Theorem 1

Let $\Omega_{n,k}$ be the sample space for a sequential search for k of n objects. For $\omega \in \Omega_{n,k}$, define $F(\omega)$ to be the list of the k objects found in the order in which they are discovered. So $F : \Omega_{n,k} \to \mathcal{I}_{n,k}$. For ω in (1) and (2),

$$F(w) = 2435 \in \mathcal{I}_{6,4}.$$

For $f \in \mathcal{I}_{n,k}$, let $\Omega_{n,k}(f) = \{\omega \in \Omega_{n,k} : F(w) = f\}$. The element in $\Omega_{n,k}(f)$ consisting of the shortest sequence of Bernoulli trials will be denoted by ω_f. If $f = 2435 \in \mathcal{I}_{6,4}$, then

$$w_f = THTHTT\ THH. \tag{6}$$

Note that the i^{th} cyclic run in f coincides to the i^{th} run of tails between the $(i-1)^{st}$ and i^{th} heads. As before, each block in (6) coincides to a search.

The symbol $(w_f; j_1, j_2, \ldots, j_k)$ will signify the element in $\Omega_{n,k}(f)$ that results by inserting $j_i \geq 0$ copies of T^{n-i+1} between the $(i-1)^{st}$ and i^{th} heads in ω_f. The outcomes in (6) and (2) are related as follows.

$$THTHTT\ TTTT\ TTTT\ THTT\ TH = (THTHTT\ THH; 0, 0, 2, 1)$$

If $\Omega_{n,k,m}$ consists of the elements in $\Omega_{n,k}$ having m blocks and if $\mathcal{I}_{n,k,d}$ denotes the subset of $\mathcal{I}_{n,k}$ consisting of injections with d descents, then

$$\Omega_{n,k,m} = \bigcup_{d=0}^{m-1} \bigcup_{f \in \mathcal{I}_{n,k,d}} \bigcup_{d+1+j_1+\cdots+j_k=m} (w_f; j_1, j_2, \ldots, j_k).$$

So the probability of m searches being needed to find k of n objects is

$$p_{n,k}(m) = \sum_{\omega\in\Omega_{n,k,m}} Pr(\omega) = \sum_{d=0}^{m-1}\sum_{f\in I_{n,k,d}} Pr(\omega_f) \sum_{d+1+j_1+\cdots+j_k=m}\prod_{i=1}^{k} q_{k-i+1}^{(n-i+1)j_i}$$

$$= (1-q_1)\cdots(1-q_k)\sum_{d=0}^{m-1}\sum_{f\in I_{n,k,d}}\prod_{i=1}^{k} q_{k-i+1}^{\mathrm{run}(f,i)} \sum_{j_1+\cdots+j_k=m-1-d}\prod_{i=1}^{k} q_{k-i+1}^{(n-i+1)j_i}.$$

Multiplying the above by z^m, summing over $m\geq 1$, and noting (4) gives (3):

$$P_{n,k}(z) = z\prod_{i=1}^{k}(1-q_i)\sum_{m\geq 0} z^m \sum_{f\in I_{n,k,m}}\prod_{i=1}^{k} q_{k-i+1}^{\mathrm{run}(f,i)} \sum_{m\geq 0} z^m \sum_{j_1+\cdots+j_k=m}\prod_{i=1}^{k} q_{k-i+1}^{(n-i+1)j_i}$$

$$= \frac{z(1-q_1)(1-q_2)\cdots(1-q_k)E_{n,k}(z)}{(1-zq_1^{n-k+1})(1-zq_2^{n-k+2})\cdots(1-zq_k^{n})}.$$

To get at the formula for $P'_{n,k}(1)$, first note that (3) and the fact that $P_{n,k}(1)=1$ imply that

$$E_{n,k}(1) = [n-k+1]_1[n-k+2]_2\cdots[n]_k.$$

The above and application of logarithmic differentiation to (3) completes the proof of Theorem 1.

The matter left open at the end of section 3 may now be addressed. The probability of player i being the survivor in Russian roulette is clearly

$$\sum_{\omega\in\Omega_{n,n}} Pr(F(\omega)=f : f(n)=i) = \sum_{f\in I_{n,n}} Pr(w_f)\sum_{m\geq 0}\sum_{j_1+\cdots+j_k=m}\prod_{i=1}^{n} q_{k-i+1}^{(n-i+1)j_i}$$

$$= \frac{\sum_f q_n^{\mathrm{run}(f,1)} q_{n-1}^{\mathrm{run}(f,2)}\cdots q_1^{\mathrm{run}(f,n)}}{[1]_1[2]_2\cdots[n]_n}$$

where the last sum is over all $f\in I_{n,n}$ with $f(n)=i$. The case $q_n = q_{n-1} = \ldots = q_1 = q$ was given in Rawlings [8]. Other formulas for the probability of player i winning have been obtained by Blom, Englund, Sandell [1] and Sandell [10].

5 Proof of Corollary 1

In this section, fix $q_n = q_{n-1} = \cdots = q_1 = q$ and let $E_n(z,q)$ denote the ordinary n^{th} q-Eulerian polynomial. The first equality given in Carlitz [3] is equivalent to

$$E_n(z,q) = (1-z)(1-zq)\cdots(1-zq^n)\sum_{m\geq 0}[m+1]^n z^m \tag{7}$$

where $[j] = (1-q^j)/(1-q)$ denotes the q-analog of j. Identity (7) also follows from a more general result of MacMahon's [6, p. 211 of Vol. 2].

Together, (3) and (7) imply

$$P_{n,n}(z) = z(1-z)\sum_{m\geq 0}(1-q^{m+1})^n z^m = \sum_{m\geq 1}\left((1-q^m)^n-(1-q^{m-1})^n\right)z^m.$$

Equating coefficients of like powers of z then gives part one of Corollary 1. To complete the proof, note that

$$\begin{aligned}
P'_{n,n}(1) &= (1-q)^n + \sum_{m\geq 2} m\left(\sum_{j=1}^{n}(-1)^j\binom{n}{j}(q^j-1)q^{j(m-1)}\right)\\
&= (1-q)^n + \sum_{j=1}^{n}(-1)^{j-1}\binom{n}{j}(1-q^j)\left(\frac{1}{(1-q^j)^2}-1\right)\\
&= (1-q)^n + \sum_{j=1}^{n}(-1)^{j-1}\binom{n}{j}q^j\left(1+\frac{1}{1-q^j}\right).
\end{aligned}$$

The above is easily seen to be equivalent to the second half of Corollary 1.

6 Proof of Theorem 2

As $(1-q_k)\cdots(1-q_1)\sum_{1\leq i_1\leq\cdots\leq i_j\leq k} q_{i_1}\cdots q_{i_j}$ is clearly the probability of all k finds and exactly j tails occuring on the first search, we have

$$p_{n,k}(1) = (1-q_k)\cdots(1-q_1)\sum_{j=0}^{n-k}\sum_{1\leq i_1\leq\cdots\leq i_j\leq k} q_{i_i}\cdots q_{i_j} = a_{n,k}\prod_{i=1}^{k}(1-q_i). \tag{8}$$

Next, consider the case that 2 or more searches ($m\geq 2$) are needed to find k of n objects. The probability of j finds, $0\leq j\leq k-1$, occuring on the first search is

$$(1-q_k)\cdots(1-q_{k-j+1})\sum_{k-j\leq i_1\leq\cdots\leq i_{n-j}\leq k} q_{i_1}\cdots q_{i_{n-j}} = b_{n,k,j}\prod_{i=k-j+1}^{k}(1-q_i).$$

Therefore, for $m\geq 2$,

$$p_{n,k}(m) = \sum_{j=0}^{k-1} b_{n,k,j}(1-q_k)\cdots(1-q_{k-j+1})p_{n-j,k-j}(m-1).$$

Multiplying the preceding equality by z^m and summing over $m\geq 2$ leads to

$$P_{n,k}(z) - zp_{n,k}(1) = z\sum_{j=0}^{k-1} b_{n,k,j}(1-q_k)\cdots(1-q_{k-j+1})P_{n-j,k-j}(z).$$

In view of (8) and the fact that $b_{n,k,0} = q_k^n$, it follows that

$$P_{n,k}(z) = \frac{z(1-q_1)\cdots(1-q_k)}{(1-zq_k^n)} \left(a_{n,k} + \sum_{j=1}^{k-1} b_{n,k,j} \frac{P_{n-j,k-j}(z)}{(1-q_1)\cdots(1-q_{k-j})} \right). \tag{9}$$

Finally, as Theorem 1 implies that

$$E_{n,k}(z) = \frac{(1-zq_1^{n-k+1})(1-zq_2^{n-k+2})\cdots(1-zq_k^n)}{z(1-q_1)(1-q_2)\cdots(1-q_k)} P_{n,k}(z),$$

(5) is seen to be equivalent to (9).

To complete the proof of Theorem 2, note that repeated use of (5) gives the following system of equations for $E_{n,k}(z), E_{n-1,k-1}(z), \ldots, E_{n-k+1,1}(z)$:

$$\begin{aligned}
E_{n,k} - \sum_{j=0}^{k-1} z(z)_{n-1,k-1,j-1} b_{n,k,j} E_{n-j,k-j} &= (z)_{n-1,k-1,k-1} a_{n,k} \\
E_{n-1,k-1} - \sum_{j=0}^{k-2} z(z)_{n-1,k-1,j} b_{n-1,k-1,j} E_{n-1-j,k-1-j} &= (z)_{n-2,k-2,k-2} a_{n-1,k-1} \\
&\vdots \\
E_{n-k+1,1} &= (z)_{n-k,0,0} a_{n-k+1,1}\,.
\end{aligned}$$

The determinant formula in Theorem 2 is then obtained by applying Cramer's rule to solve for $E_{n,k}(z)$.

7 Dunkl's Inverse Absorption Distribution

Under the condition that only one find is allowed per search, let $d_{n,k}(m)$ denote the probability that m searches are needed to find k of n objects. For $q_k = \ldots = q_1 = q$, Dunkl [4] derived an explicit formula for $d_{n,k}(m)$ from Blomqvist's [2] absorption distribution. Dunkl's result may also be deduced from Theorems 1 and 2.

First, it is clear that

$$d_{n,k}(m) = \sum_{m_1+m_2+\cdots+m_k=m} p_{n,1}(m_1) p_{n-1,1}(m_2) \cdots p_{n-k+1,1}(m_k).$$

Multiplying the above by z^m, summing over $m \geq k$, and applying Theorems 1 and 2 gives

$$\begin{aligned}
D_{n,k}(z) &\doteq \sum_{m\geq k} d_{n,k}(m) z^m = P_{n,1}(z) P_{n-1,1}(z) \cdots P_{n-k+1,1}(z) \\
&= \frac{z^k(1-q_k)\cdots(1-q_1)[n]_k \cdots [n-k+1]_1}{(1-zq_k^n)\cdots(1-zq_1^{n-k+1})}.
\end{aligned} \tag{10}$$

In the case that $q_k = \ldots = q_1 = q$, the q-binomial series may be applied in (10) to obtain

$$D_{n,k}(z) = z^k (1-q^n)\cdots(1-q^{n-k+1}) \sum_{j\geq 0} \begin{bmatrix} k+j-1 \\ j \end{bmatrix} (zq^{n-k+1})^j$$

where $\begin{bmatrix} n \\ k \end{bmatrix} = \dfrac{(1-q^n)(1-q^{n-1})\cdots(1-q^{n-k+1})}{(1-q^k)(1-q^{k-1})\cdots(1-q)}$ denotes the q-binomial coefficient. Thus, for $q_k = \ldots = q_1 = q$, it follows that

$$d_{n,k}(m) = q^{(m-k)(n-k+1)}(1-q^n)(1-q^{n-1})\cdots(1-q^{n-k+1}) \begin{bmatrix} m-1 \\ k-1 \end{bmatrix}.$$

Acknowledgement and Closing Remarks This work and a subsequent article [9] were motivated and inspired by Donald Knuth. Besides many valuable and enlightening suggestions, he independently obtained the second part of Corollary 1 using an inclusion-exclusion argument.

In [9], it is assumed that the probability of object j being found under the conditions and restrictions of the i^{th} search is $1-q_{ij}$ where q_{ij} is independent $q_{i'j'}$ for $(i,j) \neq (i',j')$. Besides a simple and direct explanation of Corollary 1, the methods in [9] lead to a probabilistic proof of (7) and to the discovery of a new formula for the **q**-Eulerian polynomials.

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