Restricted words by adjacencies

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Abstract

A recurrence, a determinant formula, and generating functions are presented for enumerating words with restricted letters by adjacencies. The main theorem leads to refinements (with up to two additional parameters) of known results on compositions, polyominoes, and permutations. Among the examples considered are (1) the introduction of the ascent variation on compositions, (2) the enumeration of directed vertically convex polyominoes by upper descents, area, perimeter, relative height, and column number, (3) a tri-variate extension of MacMahon's determinant formula for permutations with prescribed descent set, and (4) a combinatorial setting for an entire sequence of bibasic Bessel functions. .

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1. The main theorem: preliminary examples and an overview

An *alphabet* X is a non-empty set whose elements are referred to as *letters*. A finite sequence (possibly empty) of letters is said to be a *word*. The length of a word w, denoted by $l(w)$, is the number of letters in w. The set of all words X^* formed with letters in X along with the concatenation product is known as the *free monoid* generated by X. The set of words of positive length is denoted by X^+.

The concatenation product of non-empty subsets $W_1, W_2, \ldots, W_m$ in X^* is defined by $W_1W_2\cdots W_m = \{w_1w_2\cdots w_m\colon w_i \in W_i \text{ for } 1 \leqslant i \leqslant m\}$. The n-fold product of a non-empty subset W with itself is denoted by W^n. In particular, X^n is the set of words in X^* of length n. See [24] for more detail.

An *adjacency alphabet* $a = \{a_{i,xy}\colon x, y \in X,\ i \geqslant 1\}$ is associated to X. In $\mathbf{C}\langle a\rangle$, the ring of non-commutative polynomials in a with complex coefficients, the *adjacency* *monomial* and the *sieve polynomial* with *index* i for a word $w=x_1x_2\dots x_n\in X^n$ of length $n\geqslant 2$ are defined, respectively, by

$$a_{i,w}=a_{i,x_1x_2}a_{i+1,x_2x_3}\cdots a_{i+n-2,x_{n-1}x_n},$$

$$\bar{a}_{i,w}=(a_{i,x_1x_2}-1)(a_{i+1,x_2x_3}-1)\cdots(a_{i+n-2,x_{n-1}x_n}-1).$$

For a word w of length $n\leqslant 1$, $a_{i,w}=\bar{a}_{i,w}=1$. The index is motivated by the major index of MacMahon [26, Vol. I, p. 135].

Example 1. Let $X=\{u,v\}$ and $U=\{u\}$. Then UX^2 is the set of words in X^* of length 3 that begin with the letter $u: UX^2=\{uuu,uuv,uvu,uvv\}$. Consider the adjacency alphabet a for X with

$$a_{i,xy}=\begin{cases} tq^i & \text{if } x=y=u,\\ 1 & \text{otherwise,}\end{cases}$$

where $i\geqslant 1$, $x,y\in X$, and t and q are commuting indeterminates. Then $a_{i,uuu}=t^2q^{2i+1}$, $\bar{a}_{i,uuu}=(tq^i-1)(tq^{i+1}-1)$, and $\bar{a}_{i,uuv}=(tq^i-1)(1-1)=0$.

Fix n non-empty subsets $L_1,L_2,\dots,L_n\subseteq X$. In the algebra of formal power series of words $\mathbf{C}\langle a\rangle\ll X\gg$, let

$$A_{i,j}=\sum_{w\in L_iL_{i+1}\cdots L_j} a_{i,w}w$$

for $1\leqslant i\leqslant j\leqslant n$. By convention, $A_{i,j}=1$ if $i>j$. The elements in $\mathbf{C}\langle a\rangle$ are stipulated to commute with words in X^*. Note that $A_{1,n}$ enumerates words restricted to $L_1L_2\cdots L_n$ by adjacencies. In Example 1, if we select $L_1=U$ and $L_2=X=L_3$, then $A_{1,3}=t^2q^3uuu+tquuv+uvu+uvv$. In Section 2, the following recurrence and determinant for $A_{1,n}$ will be established.

Theorem 1. *For* $n\geqslant 1$,

$$(a)\ A_{1,n}=\sum_{i=1}^{n}\bar{A}_{1,i}A_{i+1,n}\quad \textit{and}\quad (b)\ A_{1,n}=\det(\bar{A}_{i,j})_{i,j=1}^{n}$$

where

$$\bar{A}_{i,j}=\begin{cases}\displaystyle\sum_{w\in L_iL_{i+1}\cdots L_j}\bar{a}_{i,w}w & \text{if } 1\leqslant i\leqslant j\leqslant n,\\ -1 & \text{if } 1\leqslant i=j+1\leqslant n,\\ 0 & \text{otherwise.}\end{cases}$$

In brief, the proof of (a) involves removal of first letters from words. Part (b) is then deduced from (a) using a non-commutative version of Cramer's rule. As the concatenation product is in general non-commutative, the terms that occur in the expansion of the determinant must be written in the form $\bar{A}_{1,\sigma(1)}\bar{A}_{2,\sigma(2)}\cdots\bar{A}_{n,\sigma(n)}$ where σ denotes a permutation of $\{1,2,\dots,n\}$.

Example 2 (*Fibonacci polynomials*). Let X and a be as in Example 1. Define the set of *u-adjacencies* in $w = x_1x_2\cdots x_n \in X^n$ as

$$\operatorname{Adj}_u w = \{k\colon x_k = x_{k+1} = u, 1 \leqslant k < n\}.$$

Further, define the *u-adjacency number* and the *u-adjacency index* of w respectively by $\operatorname{adj}_u W = |\operatorname{Adj}_u w|$ and $\operatorname{ind}_u w = \sum_{k\in \operatorname{Adj}_u w} k$. For $n \geqslant 3$, put

$$F_n(t,q) = \sum_{w\in X^{n-2}} t^{\operatorname{adj}_u w} q^{\operatorname{ind}_u w}.$$

Set $F_1(t,q) = F_2(t,q) = 1$. The number $F_n(0,1)$ is the nth Fibonacci number.

Formulas for $F_n(t,q)$ are readily deduced from Theorem 1. Let $L_i = X$ for $i \geqslant 1$. For $w \in L_i \cdots L_j$, note that

$$a_{i,w} = (tq^{i-1})^{\operatorname{adj}_u w} q^{\operatorname{ind}_u w} \quad \text{and} \quad \bar{a}_{i,w} = \begin{cases} (-1)^{j-i}(tq^i;q)_{j-i} & \text{if } w = uu\cdots u, \\ 0 & \text{otherwise,} \end{cases}$$

where $(t;q)_k = (1-t)(1-tq)\cdots(1-tq^{k-1})$ is the *q-shifted factorial.* By convention, $(t;q)_0 = 1$. Under the map $w \mapsto 1$, we have

$$A_{i,n-2} = \sum_{w\in X^{n-i-1}} a_{i,w} = F_{n-i+1}(tq^{i-1},q)$$

and, for $1 \leqslant i \leqslant j \leqslant n$,

$$\bar{A}_{i,j} = \sum_{w\in X^{j-i+1}} \bar{a}_{i,w} = \begin{cases} (-1)^{j-i}(tq^i;q)_{j-i} & \text{if } 1 \leqslant i < j \leqslant n, \\ 2 & \text{if } 1 \leqslant i = j \leqslant n. \end{cases}$$

For $n \geqslant 3$, Theorem 1 then implies that

$$F_n(t,q) = 2F_{n-1}(tq,q) + \sum_{i=2}^{n-2} (-1)^{i-1}(tq;q)_{i-1}F_{n-i}(tq^i,q) = \det(\bar{A}_{i,j})_{i,j=1}^{n-2}.$$

Theorem 1 is an extension of the work of Fedou and Rawlings [16]. Other theories that deal with adjacencies include the pattern algebra of Goulden and Jackson [20] and the theory of Möbius inversion as developed by Rota [30] and Stanley [31]. A closed formula for rearrangements by adjacency type was obtained by Hutchinson and Wilf [21].

In contrast, Theorem 1 has two unique features. First, it enumerates words with restrictions on all letters. This feature is used in Section 3 to derive a generating function for words ending with a finite sequence of restricted letters and is further exploited in Section 4 (Example 8) to obtain a combinatorial interpretation for an entire sequence of bibasic Bessel functions. Previous treatments only consider restrictions on one or two letters.

Second, the index is not explicit in other treatments. Tracking it leads to natural refinements (with up to two additional parameters) of known results on compositions, directed vertically convex polyominoes, and permutations. The primary example in this regard, a tri-variate extension of MacMahon's [26, Vol. I, p. 190] determinant formula for permutations with prescribed descent set, is presented in Section 6.

Several other examples illustrating and extending the approach of Fedou and Rawlings [15,16] are also included in Sections 4–6. In Example 5, the ascent variation of a composition is introduced.

2. Proof of Theorem 1

With a few changes, the proof given for Theorem 2 in Fedou and Rawlings [16] may be used to establish Theorem 1(a). For $n = 1$, Theorem 1(a) is obviously true. Assume $n \geqslant 2$. From Section 1,

$$A_{1,n} = \sum_{w} a_{1,w} w$$

where the sum is over words $w = x_1 x_2 \cdots x_n \in L_1 L_2 \cdots L_n$. Since $\bar{a}_{1,x_1} = 1$ and $\bar{a}_{1,x_1x_2} = a_{1,x_1x_2} - 1$, we see that

$$A_{1,n} = \sum_{w} \bar{a}_{1,x_1} a_{2,x_2 \ldots x_n} w + \sum_{w} \bar{a}_{1,x_1x_2} a_{2,x_2 \ldots x_n} w.$$

Similarly, the second sum on the above right splits as

$$\sum_{w} \bar{a}_{1,x_1x_2} a_{3,x_3 \ldots x_n} w + \sum_{w} \bar{a}_{1,x_1x_2x_3} a_{3,x_3 \ldots x_n} w.$$

Thus,

$$A_{1,n} = \sum_{i=1}^{2} \sum_{w} \bar{a}_{1,x_1 \ldots x_i} a_{i+1,x_{i+1} \ldots x_n} w + \sum_{w} \bar{a}_{1,x_1x_2x_3} a_{3,x_3 \ldots x_n} w.$$

Iterating and then factoring completes the proof of Theorem 1(a):

$$A_{1,n} = \sum_{i=1}^{n-1} \sum_{w} \bar{a}_{1,x_1 \ldots x_i} a_{i+1,x_{i+1} \ldots x_n} w + \sum_{w} \bar{a}_{1,w} w = \sum_{i=1}^{n} \bar{A}_{1,i} A_{i+1,n}.$$

For part (b), observe that Theorem 1(a) implies that

$$A_{k,n} = \sum_{i=k}^{n} \bar{A}_{k,i} A_{i+1,n}. \tag{1}$$

Writing (1) out for $k = 1, 2, \ldots, n$ leads to the system of equations

$$\begin{aligned} A_{1,n} - \bar{A}_{1,1} A_{2,n} - \bar{A}_{1,2} A_{3,n} - \cdots - \bar{A}_{1,n-1} A_{n,n} &= \bar{A}_{1,n}, \\ A_{2,n} - \bar{A}_{2,2} A_{3,n} - \cdots - \bar{A}_{2,n-1} A_{n,n} &= \bar{A}_{2,n}, \\ &\vdots \\ A_{n,n} &= \bar{A}_{n,n}. \end{aligned}$$

Solving for $A_{1,n}$ (by applying a non-commutative version of Cramer's rule or by simply backsolving) gives Theorem 1(b).

3. Generating functions

Theorem 1 may be used to obtain generating functions for words with specified restrictions. To this end, the index is dropped: For $w = x_1x_2\cdots x_n \in X^n$, let $a_w = a_{x_1x_2}a_{x_2x_3}\cdots a_{x_{n-1}x_n}$ and $\bar{a}_w = (a_{x_1x_2}-1)(a_{x_2x_3}-1)\cdots(a_{x_{n-1}x_n}-1)$. With $a_{i,w}$ replaced by a_w, Theorem 1 remains valid and implies that the generating function for words by adjacencies ending in a finite sequence of restricted letters is given by

Corollary 1. *For non-empty subsets $U, L_1, L_2, \ldots, L_m \subseteq X$,*

$$\sum_{w\in U^*L_1\cdots L_m} a_w w = \left(1 - \sum_{w\in U^+} \bar{a}_w w\right)^{-1} \sum_{k=1}^{m} \left[\left(\sum_{w\in U^*L_1\cdots L_k} \bar{a}_w w\right) \det(\bar{B}_{i,j})_{i,j=k+1}^{m}\right]$$

where the empty determinant is 1 *and*

$$\bar{B}_{i,j} = \begin{cases} \displaystyle\sum_{w\in L_i\cdots L_j} \bar{a}_w w & \text{if } 1\leqslant i\leqslant j\leqslant m, \\ -1 & \text{if } 1\leqslant i = j+1\leqslant m, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. For $n\geqslant 0$, Theorem 1(a) implies that

$$\begin{aligned}\sum_{w\in U^nL_1\cdots L_m} a_w w = &\sum_{k=1}^{n}\left(\sum_{w\in U^k} \bar{a}_w w\right)\left(\sum_{w\in U^{n-k}L_1\cdots L_m} a_w w\right) \\ &+\sum_{k=1}^{m}\left[\left(\sum_{w\in U^nL_1\cdots L_k} \bar{a}_w w\right)\left(\sum_{w\in L_{k+1}\cdots L_m} a_w w\right)\right].\end{aligned}$$

The desired result follows since the above and Theorem 1(b) together give

$$\begin{aligned}\sum_{w\in U^*L_1\cdots L_m} a_w w = &\left(\sum_{w\in U^+} \bar{a}_w w\right)\left(\sum_{w\in U^*L_1\cdots L_m} a_w w\right) \\ &+\sum_{k=1}^{m}\left[\left(\sum_{w\in U^*L_1\cdots L_k} \bar{a}_w w\right)\det(\bar{B}_{i,j})_{i,j=k+1}^{m}\right]. \qquad \square\end{aligned}$$

Some special cases are noteworthy. For $m=1$ and $L=L_1$, Corollary 1 reduces to

$$\sum_{w\in U^*L} a_w w = \left(1 - \sum_{w\in U^+} \bar{a}_w w\right)^{-1}\left(\sum_{w\in U^*L} \bar{a}_w w\right). \tag{2}$$

Furthermore, if $L=U$, (2) becomes

$$\sum_{w\in U^+} a_w w = \left(1 - \sum_{w\in U^+} \bar{a}_w w\right)^{-1}\left(\sum_{w\in U^+} \bar{a}_w w\right).$$

Equivalently,

$$\sum_{w\in U^*} a_w w = \left(1 - \sum_{w\in U^+} \bar{a}_w w\right)^{-1}. \tag{3}$$

Identities (2) and (3) are due to Fedou and Rawlings [16] and are more or less equivalent to several formulas in other algebraic settings (see [11, pp. 96–99]; [13, pp. 96–99]; [20, pp. 131, 238], [31–33]). An explicit connection with Diekert's work is given in [16].

Generating functions with other restrictions may also be derived from Theorem 1. For instance, to obtain the generating function for words by adjacencies with restricted first and last letters, let F, U, L be nonempty subsets of X. Theorem 1(a) implies that

$$\sum_{w\in FU^nL} a_w w = \sum_{k=1}^{n+1} \left(\sum_{w\in FU^{k-1}} \bar{a}_w w\right)\left(\sum_{w\in U^{n-k+1}L} a_w w\right) + \sum_{w\in FU^nL} \bar{a}_w w.$$

Thus,

$$\sum_{w\in FU^*L} a_w w = \left(\sum_{w\in FU^*} \bar{a}_w w\right)\left(\sum_{w\in U^*L} a_w w\right) + \sum_{w\in FU^*L} \bar{a}_w w.$$

The last equality and (2) together give a result

$$\sum_{w\in FU^*L} a_w w = \left(\sum_{w\in FU^*} \bar{a}_w w\right)\left(1 - \sum_{w\in U^+} \bar{a}_w w\right)^{-1}\left(\sum_{w\in U^*L} \bar{a}_w w\right) + \sum_{w\in FU^*L} \bar{a}_w w$$

that is comparable to one of Bousquet-Mélou's [5].

Example 3 (*A generating function for the Fibonacci polynomials*). Let X, a, and $F_n(t,q)$ be as in Example 2. Replace $a_{i,w}$ by a_w and set $q = 1$. As the map sending $w \mapsto z^{l(w)}$ extends to a continuous homomorphism from $\mathbf{C}\langle a\rangle\langle\langle X\rangle\rangle$ to $\mathbf{C}[t][[z]]$, it follows from (3) that

$$\sum_{n\geqslant 2} F_n(t,1)z^n = z^2 \sum_{w\in X^*} a_w z^{l(w)} = z^2\left(1 - \sum_{w\in X^+} \bar{a}_w z^{l(w)}\right)^{-1}$$
$$= \frac{z^2}{1 - 2z - \sum_{n\geqslant 2}(t-1)^{n-1}z^n} = \frac{z^2(1+z-tz)}{1 - z(1+t) + z^2(t-1)}.$$

Adding z to both sides above and setting $t=0$ gives the well-known generating function $\sum_{n\geqslant 1} F_n(0,1)z^n = z/(1-z-z^2)$ for the Fibonacci numbers.

4. Compositions

Let $N = \{0, 1, 2, \ldots\}$ and $N_+ = \{1, 2, 3, \ldots\}$. The *weak rise set*, *weak rise number*, *descent set*, *descent number*, *level set*, and *number of levels* of a word $w = x_1 x_2 \cdots x_n \in N^n$ are respectively defined by

$$\operatorname{WRis} w = \{k\colon 1 \leqslant k < n,\ x_k \leqslant x_{k+1}\}, \quad \operatorname{wris} w = |\operatorname{WRis} w|,$$

$$\operatorname{Des} w = \{k\colon 1 \leqslant k < n,\ x_k > x_{k+1}\}, \quad \operatorname{des} w = |\operatorname{Des} w|,$$

$$\operatorname{Lev} w = \{k\colon 1 \leqslant k < n,\ x_k = x_{k+1}\}, \quad \text{and} \quad \operatorname{lev} w = |\operatorname{Lev} w|.$$

Also, put $\|w\| = x_1 + x_2 + \cdots + x_n$.

A word $w \in N_+^n$ is said to be a *composition* of k with n *parts* if $\|w\| = k$. By convention, 0 has a single composition with no parts. In Examples 4–8, Corollary 1 and its special cases are used to enumerate compositions by number of parts (bounded and unbounded), rises, levels, and descents. In Example 5, a new statistic referred to as the ascent variation is considered. The associated recurrences and determinant formulas implied by Theorem 1 are not stated; only generating functions are presented.

The *q-binomial coefficient* is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{cases} \dfrac{(q;q)_n}{(q;q)_k (q;q)_{n-k}} & \text{if } 0 \leqslant k \leqslant n, \\ 0 & \text{otherwise.} \end{cases}$$

Repeated use will be made of the partition identities (see [2])

$$\sum_{0 \leqslant i_1 \leqslant i_2 \leqslant \cdots \leqslant i_k} q^{i_1 + i_2 + \cdots + i_k} = \frac{1}{(q;q)_k},$$

$$\sum_{0 \leqslant i_1 \leqslant i_2 \leqslant \cdots \leqslant i_k \leqslant m} q^{i_1 + i_2 + \cdots + i_k} = \begin{bmatrix} m+k \\ k \end{bmatrix}. \tag{4}$$

In all that follows, s, t, q, p, z, and z_x denote commuting indeterminates.

Example 4 (*Compositions with bounded parts by weak rises*). Let $N_{+,m} = \{1, 2, \ldots, m\}$. The number of compositions of k with i weak rises and n parts, each bounded by m, is equal to the coefficient of $t^i q^k$ in

$$c_{n,m}(t,q) = \sum_{w \in N_{+,m}^n} t^{\operatorname{wris} w} q^{\|w\|}.$$

Identity (3) furnishes a generating function for $c_{n,m}$. For integers $x, y \in N_+$, set $a_{xy} = t$ if $x \leqslant y$ and $a_{xy} = 1$ otherwise. For $w = x_1 x_2 \cdots x_n \in N_{+,m}^n$, note that $a_w = t^{\operatorname{wris} w}$ and that

$$\bar{a}_w = \begin{cases} (t-1)^{n-1} & \text{if } 1 \leqslant x_1 \leqslant x_2 \leqslant \cdots \leqslant x_n, \\ 0 & \text{otherwise.} \end{cases}$$

The definition of a_w, (3), and (4) imply that

$$\begin{aligned}\sum_{n\geqslant 0} c_{n,m}(t,q)z^n &= \sum_{w\in N_{+,m}^*} a_w q^{\|w\|} z^{l(w)} = \frac{1}{1-\sum_{w\in N_{+,m}^+} \bar{a}_w q^{\|w\|} z^{l(w)}} \\ &= \frac{1}{1-\sum_{n\geqslant 1}(t-1)^{n-1} z^n \sum_{1\leqslant x_1\leqslant\cdots\leqslant x_n\leqslant m} q^{x_1+\cdots+x_n}} \\ &= \frac{1-t}{\sum_{n\geqslant 0}(q(t-1)z)^n \left[{m+n-1 \atop n}\right] - t} \\ &= \frac{1-t}{(q(t-1)z;q)_m^{-1} - t} = \frac{(1-t)(q(t-1)z;q)_m}{1-t(q(t-1)z;q)_m}. \end{aligned} \tag{5}$$

The q-binomial series (see [2, p. 36]) was used in the next to last step.

The generating function for compositions with *no* bound on part size follows from (5) by letting $m\to\infty$: Define

$$(t;q)_\infty = \prod_{i=0}^{\infty}(1-tq^i),$$

$c_n(t,q) = \sum_{w\in N_+^n} t^{\text{wris}\, w} q^{\|w\|}$, and $E_q(z) = \sum_{n=0}^{\infty} q^{\binom{n}{2}} z^n/(q;q)_n$. By Corollary 2.2 on p. 19 of [2], $E_q(z) = (-z;q)_\infty$. As $m\to\infty$, (5) becomes

$$\sum_{n\geqslant 0} c_n(t,q)z^n = \frac{(1-t)E_q(q(1-t)z)}{1-tE_q(q(1-t)z)}.$$

Incidently, the right-hand side of the last equality with zq replaced by z is the generating function for q-Eulerian polynomials (e.g., see [16]). To be precise, $q^{-n}(q;q)_n c_n(t;q)$ enumerates permutations of the symmetric group by rise and inversion numbers. In Examples 6–8, it will become clear that many of the generating functions for enumerating permutations also occur in the context of compositions.

Example 5 (*Compositions by weak rises and ascent variation*). The *ascent variation* of a composition $w = x_1x_2\cdots x_n \in N_+^n$ is defined to be

$$\operatorname{avar} w = \sum_{k\in \text{WRis}\, w} (x_{k+1}-x_k).$$

For $w = 5\,2\,5\,6\,1 \in N_+^5$, $\operatorname{avar} w = 3+1 = 4$. The ascent variation is motivated by the perimeter of a directed vertically convex polyomino (Example 9).

To enumerate compositions by weak rises and ascent variation, first note that

$$\sum_{1\leqslant x_1\leqslant x_2\leqslant\cdots\leqslant x_n} q^{x_n} \prod_{k=1}^{n-1} q^{x_k}(tp^{x_{k+1}-x_k}-1) = \frac{q^n}{(q;q)_n} \prod_{k=1}^{n-1} \left(\frac{(1-q^k)t}{1-pq^k} - 1\right). \tag{6}$$

To see why, rewrite the sum in (6) as

$$\sum_{x_1\geqslant 1} q^{nx_1} \sum_{x_2\geqslant x_1} (tp^{x_2-x_1}-1)q^{(n-1)(x_2-x_1)} \cdots \sum_{x_n\geqslant x_{n-1}} (tp^{x_n-x_{n-1}}-1)q^{x_n-x_{n-1}}.$$

Iteratively summing the sequence of geometric series gives the desired result.

Let

$$c_n(t,q,p) = \sum_{w\in N_+^n} t^{\mathrm{wris}\,w} q^{\|w\|} p^{\mathrm{avar}\,w}.$$

For integers $x, y \in N_+$, set $a_{xy} = tp^{y-x}$ if $x \leqslant y$ and $a_{xy} = 1$ otherwise. For $w = x_1x_2\cdots x_n \in N_+^n$, note that $a_w = t^{\mathrm{wris}\,w} p^{\mathrm{avar}\,w}$ and that

$$\bar{a}_w = \begin{cases} \prod_{k=1}^{n-1}(tp^{x_{k+1}-x_k} - 1) & \text{if } 1 \leqslant x_1 \leqslant x_2 \leqslant \cdots \leqslant x_n, \\ 0 & \text{otherwise.} \end{cases}$$

It therefore follows from (3) and (6) that

$$\begin{aligned}\sum_{n\geqslant 0} c_n(t,q,p)z^n &= \sum_{w\in N_+^*} a_w q^{\|w\|} z^{l(w)} = \left(1 - \sum_{w\in N_+^+} \bar{a}_w q^{\|w\|} z^{l(w)}\right)^{-1} \\ &= \left(1 - \sum_{n\geqslant 1} z^n \sum_{1\leqslant x_1\leqslant\cdots\leqslant x_n} q^{x_n} \prod_{k=1}^{n-1} q^{x_k}(tp^{x_{k+1}-x_k} - 1)\right)^{-1} \\ &= \left(1 - \sum_{n\geqslant 1} \frac{q^n z^n}{(q;q)_n} \prod_{k=1}^{n-1}\left(\frac{(1-q^k)t}{1-pq^k} - 1\right)\right)^{-1}.\end{aligned}$$

Setting $t = 1$ gives

$$\sum_{n\geqslant 0} c_n(1,q,p)z^n = \left(1 + \sum_{n\geqslant 1} \frac{(-1)^n q^{\binom{n}{2}} (q(1-p)z)^n}{(p;q)_n (q;q)_n}\right)^{-1} = \frac{1}{J_{0,q,q}(p, q(1-p)z)}$$

where $J_{v,q,p}$ is the vth bibasic Bessel function defined in (8). Further study of the ascent variation and related statistics is made in [29].

Example 6 (*Down-up alternating compositions with bounded parts*). A composition $w = x_1x_2\cdots x_n \in N_+^n$ such that $x_1 \geqslant x_2 \leqslant x_3 \geqslant x_4 \leqslant \cdots$ is said to be *down–up alternating*. Let $U = \{x_1x_2 \in N_{+,m}^2\colon x_1 \geqslant x_2\}$, $L = N_{+,m}$, and $X = U \cup L$. For $w = x_1x_2\cdots x_{2n} \in U^n$ or $w \in U^nL$, let $\mathrm{des}_e\, w$ denote the number of even indices k such that $x_k > x_{k+1}$. Put

$$d_{2n,m}(t,q) = \sum_{w\in U^n} t^{\mathrm{des}_e\, w} q^{\|w\|} \quad \text{and} \quad d_{2n+1,m}(t,q) = \sum_{w\in U^nL} t^{\mathrm{des}_e\, w} q^{\|w\|}.$$

The coefficient of q^k in $d_{n,m}(0,q)$ is equal to the number of down–up alternating compositions of k having n parts, each part bounded by m.

Identities (2) and (3) may be used to obtain a generating function for $d_{n,m}(t,q)$. Only the analysis associated with an odd number of parts is presented. For $x = x_1x_2$, $y = y_1y_2 \in U$, and $l \in L$, set

$$a_{xy} = \begin{cases} t & \text{if } x_2 > y_1 \\ 1 & \text{otherwise} \end{cases} \quad \text{and} \quad a_{xl} = \begin{cases} t & \text{if } x_2 > l, \\ 1 & \text{otherwise.} \end{cases}$$

For $w = x_1x_2\cdots x_{2n+1} \in U^nL$, note that $a_w = t^{\mathrm{des}_e\, w}$ and that

$$\bar{a}_w = \begin{cases} (t-1)^n & \text{if } m \geqslant x_1 \geqslant x_2 > x_3 \geqslant \cdots > x_{2n+1} \geqslant 1, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, $\sum_{n\geqslant 0} d_{2n+1,m}(t,q)z^{2n+1} = z^{-1}\sum_{w\in U^*L} a_w q^{\|w\|} z^{2l(w)}$. From (2) and (4),

$$\sum_{n\geqslant 0} d_{2n+1,m}(t,q)z^{2n+1} = \frac{\sum_{n\geqslant 0}(t-1)^n z^{2n+1}\sum_{m\geqslant x_1\geqslant x_2>x_3\geqslant\cdots>x_{2n+1}\geqslant 1} q^{x_1+\cdots+x_{2n+1}}}{1-\sum_{n\geqslant 1}(t-1)^{n-1}z^{2n}\sum_{m\geqslant x_1\geqslant x_2>x_3\geqslant\cdots\geqslant x_{2n}\geqslant 1} q^{x_1+\cdots+x_{2n}}}$$

$$= \frac{\sum_{n\geqslant 0} q^{n(n+1)+2n+1}(t-1)^n \begin{bmatrix} m+n \\ 2n+1 \end{bmatrix} z^{2n+1}}{1-\sum_{n\geqslant 1} q^{n(n+1)}(t-1)^{n-1}\begin{bmatrix} m+n \\ 2n \end{bmatrix} z^{2n}}.$$

A similar argument for the even case and the last result give

$$\sum_{n\geqslant 0} d_{n,m}(t,q)z^n = \frac{1-t+\sqrt{1-t}S_{q,m}(z\sqrt{1-t})}{C_{q,m}(z\sqrt{1-t})-t} \tag{7}$$

where

$$C_{q,m}(z) = \sum_{n\geqslant 0}(-1)^n q^{n(n+1)}\begin{bmatrix} m+n \\ 2n \end{bmatrix} z^{2n},$$

$$S_{q,m}(z) = \sum_{n\geqslant 0}(-1)^n q^{n^2+3n+1}\begin{bmatrix} m+n \\ 2n+1 \end{bmatrix} z^{2n+1}.$$

Carlitz' [9] result for down–up alternating compositions with unbounded parts follows from (7): First set $t=0$. Then, as $m\to\infty$, note that $C_{q,m}(z)$ and $S_{q,m}(z)$ approach the q-cosine function $\sum_{n\geqslant 0}(-1)^n q^{n(n+1)}z^{2n}/(q;q)_{2n}$ and the q-sine function $\sum_{n\geqslant 0}(-1)^n q^{n^2+3n+1}z^{2n+1}/(q;q)_{2n+1}$ respectively. Carlitz' result parallels the work of André [1] on alternating permutations.

Example 7 (*Compositions by rises, levels, and descents*). For $w=x_1x_2\cdots x_n\in N^n$, an index k such that $x_k<x_{k+1}$ is said to be a *rise* in w. The coefficient of $s^it^jq^k$ in

$$g_n(s,t,q) = \sum_{w\in N_+^n} s^{\operatorname{lev} w}t^{\operatorname{des} w}q^{\|w\|}$$

equals the number of compositions of k with n parts, i levels, j descents, and $n-1-i-j$ rises.

For $x,y\in N_+$, set $a_{xy}=1$ if $x<y$, $a_{xy}=s$ if $x=y$, and $a_{xy}=t$ if $x>y$. If $w=x_1x_2\cdots x_n\in N_+^n$, then $a_w=s^{\operatorname{lev} w}t^{\operatorname{des} w}$. Observe that

$$\sum_{n\geqslant 0} g_n(s,t,q)z^n = \sum_{w\in N_+^*} a_w q^{\|w\|}z^{l(w)}.$$

In view of (3), we have

$$\sum_{n\geqslant 0} g_n(s,t,q)z^n = \left[1-\sum_{w\in N_+^+}\bar{a}_w q^{\|w\|}z^{l(w)}\right]^{-1}.$$

By noting for $w=x_1x_2\cdots x_n\in N_+^n$ that $\bar{a}_w=(t-1)^{n-1-i}(s-1)^i$ if $x_1\geqslant\cdots\geqslant x_n\geqslant 1$ with equality holding exactly i times and that $\bar{a}_w=0$ otherwise, it is not difficult to see that the series

$$S=(t-1)\sum_{w\in N_+^+}\bar{a}_w q^{\|w\|}z^{l(w)}$$

counts non-empty partitions by their weight (q), number of parts (z), number of distinct parts ($t-1$), and repeated parts ($s-1$). Constructing these partitions by concatenating a block of parts equal to 1, then a block of parts equal to 2, and so on, we further see that

$$S=\prod_{i\geqslant 1}\left(1+\frac{(t-1)zq^i}{1-(s-1)zq^i}\right)-1=\frac{((s-t)zq;q)_\infty}{((s-1)zq;q)_\infty}-1.$$

Putting the pieces together gives a result due to Carlitz [8]:

$$\sum_{n\geqslant 0}g_n(s,t,q)z^n=\frac{(1-t)((s-1)zq;q)_\infty}{((s-t)zq;q)_\infty-t((s-1)zq;q)_\infty}.$$

Example 8 (*Pairs of compositions and* (q,p)*-Bessel functions*). For $v\in\{0,1\}$, the bibasic Bessel functions defined by

$$J_{v,q,p}(s,z)=\sum_{n\geqslant 0}(-1)^n\frac{q^{\binom{n+v}{2}}z^{n+v}}{(s;q)_{n+v}(p;p)_n}\tag{8}$$

appear in several combinatorial contexts (see [10,12,14–16,18]). An interpretation involving the entire sequence $\{J_{v,q,p}\}_{v\geqslant 0}$ may be given in terms of pairs of compositions.

To see how, consider the alphabet $X=\{\binom{x}{y}\colon x,y\in N_+\}$. Elements of X will be referred to as *biletters* and are not be be confused with binomial coefficients. Let $U=X$ and $L=\{\binom{1}{y}\colon y\in N_+\}$. A biword

$$\binom{v}{w}=\binom{x_1x_2\cdots x_n}{y_1y_2\cdots y_n}\in X^n$$

may be viewed as a pair of compositions $v,w\in N_+^n$. The *mixed weak descent and strict rise number* of $\binom{v}{w}\in X^n$ is defined as

$$dr\binom{v}{w}=|\{k\colon 1\leqslant k<n,\ x_k\geqslant x_{k+1},\ y_k<y_{k+1}\}|.$$

For $m\geqslant 0$, let $b_{n,m}=\sum_{\binom{v}{w}\in U^nL^m}t^{dr\binom{v}{w}}q^{\|w\|}p^{\|v\|}$. Identity (3) and Corollary 1 imply that

$$\sum_{n\geqslant 0}b_{n,m}z^{n+m}=\begin{cases}\dfrac{1-t}{J_{0,q,p}(q,qp(1-t)z)-t} & \text{if } m=0,\\[2ex] \dfrac{-\sum_{k=1}^m(-1)^k\det(\bar{B}_{i,j})_{i,j=k+1}^m J_{k,q,p}(q,qp(1-t)z)}{J_{0,q,p}(q,qp(1-t)z)-t} & \text{if } m>0,\end{cases}\tag{9}$$

where

$$\bar{B}_{i,j} = \begin{cases} (t-1)^{j-i}(qpz)^{j-i+1}q^{\binom{j-i+1}{2}}/(q;q)_{j-i+1} & \text{if } 1 \leqslant i \leqslant j \leqslant m, \\ -1 & \text{if } 1 \leqslant i = j+1 \leqslant m, \\ 0 & \text{otherwise.} \end{cases}$$

The proof of (9) goes as follows. For biletters $\binom{x}{y}, \binom{l}{k} \in X$, set $a_{\binom{xl}{yk}} = t$ if $x \geqslant l$ and $y < k$. Otherwise, let $a_{\binom{xl}{yk}} = 1$. If

$$\binom{v}{w} = \binom{x_1 x_2 \cdots x_n}{y_1 y_2 \cdots y_n} \in X^n,$$

then

$$\bar{a}_{\binom{v}{w}} = \begin{cases} (t-1)^{n-1} & \text{if } x_1 \geqslant x_2 \geqslant \cdots \geqslant x_n \geqslant 1 \text{ and } 1 \leqslant y_1 < y_2 < \cdots < y_n, \\ 0 & \text{otherwise.} \end{cases}$$

Note that

$$\sum_{n \geqslant 0} b_{n,m} z^{n+m} = \sum_{\binom{v}{w} \in U^* L^m} a_{\binom{v}{w}} q^{\|w\|} p^{\|v\|} z^{l\binom{v}{w}}.$$

The result in (9) then follows routinely from (3), Corollary 1, and (4).

Other mixtures of descents and rises lead to different generalized Bessel functions. Also, bounding the parts of the composition provides another combinatorial context for the finite bibasic Bessel functions in [15,18].

5. Directed vertically convex polyominoes

A *polyomino* P is a finite union of unit squares in the plane such that P is connected, P has no finite cut set, and the vertex coordinates of each square are integers. Two polyominoes are equivalent if there exists a translation that maps one onto the other. Generating functions and asymptotic estimates have been given for many subclasses of polyominoes [4,6,7,17,23,28]. Bousquet-Mélou [6] provides a survey. In Example 9, directed vertically convex polyominoes are enumerated by upper descents, perimeter, area, relative height, and column number.

A polyomino P is *vertically convex* (VC) if the intersection of any vertical line with P is connected. Both polyominoes in Fig. 1 are VC. Henceforth, the ith column from the left in a VC polyomino P will be labeled c_i.

An index k is said to be an *upper* (resp. *lower*) *descent* of a VC polyomino P if the top (resp. bottom) square in column c_k lies above the top (resp. bottom) square of column c_{k+1}. The sets of upper and lower descents of P are respectively denoted by $\mathrm{UDes}\,P$ and $\mathrm{LDes}\,P$. In Fig. 1, $\mathrm{UDes}\,P = \{2,4\}$ and $\mathrm{LDes}\,P = \{2\}$. A VC polyomino P with $\mathrm{LDes}\,P = \emptyset$ is said to be a *directed vertically convex* (DVC) polyomino. If we also have $\mathrm{UDes}\,P = \emptyset$, then P is said to be a *staircase* or *parallelogram* polyomino.

Let DVC_n denote the set of directed vertically convex polyominoes with n columns. The *upper descent number* of $P \in \mathrm{DVC}_n$ is $\mathrm{udes}\,P = |\mathrm{UDes}\,P|$. The *area*, *perimeter*, and

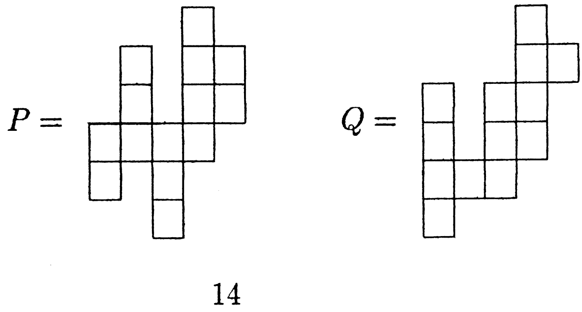


Fig. 1.

relative height of P are, respectively, defined to be the number of squares contained by P, the length of the boundary of P, and the change in the y-coordinate from the lower left-hand vertex of c_1 to the the upper right-hand vertex of c_n. These statistics of P will be, respectively, denoted by area P, per P, and $h(P)$. For $Q \in \mathrm{DVC}_5$ of Fig. 1, udes $Q=2$, area $Q=13$, per $Q=26$, and $h(Q)=5$. If P is a staircase polyomino, then $h(P)$ is equal to the number of rows in P. The following identities will be useful:

$$\sum (sq)^{x_n-y_n} q^{y_n} \prod_{k=1}^{n-1} (sq)^{x_k-y_k} q^{y_k} (tp^{y_k-x_{k+1}}-1) = \frac{q^{\binom{n+1}{2}} \prod_{k=1}^{n-1} \left(\frac{(1-q^k)tp}{1-pq^k}-1\right)}{(sq;q)_n (q;q)_n}$$

where the sum is over $x_1 \geqslant y_1 > x_2 \geqslant y_2 > \cdots \geqslant y_n \geqslant 1$ and

$$\sum (sq)^{x_{n+1}-1} q \prod_{k=1}^{n} (sq)^{x_k-y_k} q^{y_k} (tp^{y_k-x_{k+1}}-1) = \frac{q^{\binom{n+2}{2}} \prod_{k=1}^{n} \left(\frac{(1-q^k)tp}{1-pq^k}-1\right)}{(sq;q)_{n+1} (q;q)_n} \tag{10}$$

where the sum is over $x_1 \geqslant y_1 > x_2 \geqslant y_2 > \cdots \geqslant y_{n+1} = 1$. They may be verified in the same way as was (6).

Essentially, each example in Section 4 could be redone in the context of DVC polyominoes. Only one example is considered here.

Example 9 (*DVC polyominoes by upper descents, perimeter, area, relative height and column number*). First, consider the alphabets of biletters $U = \{\binom{x}{y} : x, y \in N_+,\ x \geqslant y\}$, $L = \{\binom{x}{1} : x \in N_+\}$, and $X = U$. A bijection used in [7] may be easily extended to one from DVC_{n+1} to $U^n L$: For $Q \in \mathrm{DVC}_{n+1}$ with columns labeled $c_1, c_2, \ldots c_{n+1}$, let x_i be the number of squares in column c_i for $1 \leqslant i \leqslant n+1$, y_i be the number of squares in c_i at and above the square in c_i that is adjacent to the bottom square of c_{i+1} for $1 \leqslant i \leqslant n$, and $y_{n+1} = 1$. For instance, see Fig. 2.

The map $\phi_n : \mathrm{DVC}_{n+1} \to U^n L$ that associates Q to the biword

$$\begin{pmatrix} x_1 x_2 \cdots x_{n+1} \\ y_1 y_2 \cdots y_{n+1} \end{pmatrix} = \begin{pmatrix} v \\ w \end{pmatrix}$$

is a bijection.

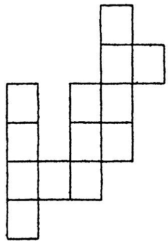
$Q =$  $\mapsto \begin{pmatrix} x_1 x_2 x_3 x_4 x_5 \\ y_1 y_2 y_3 y_4 y_5 \end{pmatrix} = \begin{pmatrix} 4\,1\,3\,4\,1 \\ 3\,1\,2\,2\,1 \end{pmatrix}.$

Fig. 2.

Let

$$g_n(s,t,q,p) = \sum_{Q \in \mathrm{DVC}_n} s^{h(Q)} t^{\mathrm{udes}\, Q} q^{\mathrm{area}\, Q} p^{\mathrm{per}\, Q}.$$

To determine g_n, we define the adjacency alphabet for X as follows. For biletters $\binom{x}{y}, \binom{j}{k} \in X$, set $a_{\binom{xj}{yk}} = tp^{2(y-j)}$ if $y > j$ and $a_{\binom{xj}{yk}} = 1$ otherwise. So, for

$$\binom{v}{w} = \begin{pmatrix} x_1 x_2 \cdots x_n \\ y_1 y_2 \cdots y_n \end{pmatrix} \in X^n,$$

$$\bar{a}_{\binom{v}{w}} = \begin{cases} \prod_{k=1}^{n-1} (tp^{2(y_k - x_{k+1})} - 1) & \text{if } x_1 \geqslant y_1 > x_2 \geqslant y_2 > \cdots \geqslant y_n \geqslant 1, \\ 0 & \text{otherwise.} \end{cases} \tag{11}$$

Also, if

$$\phi_n(Q) = \begin{pmatrix} x_1 x_2 \cdots x_{n+1} \\ y_1 y_2 \cdots y_{n+1} \end{pmatrix} = \binom{v}{w}, \text{ then } t^{\mathrm{udes}\, Q} p^{2 \sum_{k \in \mathrm{UDes}\, Q} (y_k - x_{k+1})} = a_{\binom{v}{w}},$$

area $Q = \|v\|$, $h(Q) = \|v\| - \|w\| + 1$, and

$$\mathrm{per}\, Q = 2 \left(n + 1 + h(Q) + \sum_{k \in \mathrm{UDes}\, Q} (y_k - x_{k+1}) \right).$$

Therefore, in view of (2), (10), and (11), we have

$$\begin{aligned} \sum_{n \geqslant 0} g_{n+1}(s,t,q,p) z^{n+1} &= sp^2 \sum_{\binom{v}{w} \in U^* L} a_{\binom{v}{w}} (sp^2)^{\|v\| - \|w\|} q^{\|v\|} (p^2 z)^{l\binom{v}{w}} \\ &= \frac{sp^2 \sum_{\binom{v}{w} \in U^* L} \bar{a}_{\binom{v}{w}} (sqp^2)^{\|v\| - \|w\|} q^{\|w\|} (p^2 z)^{l\binom{v}{w}}}{1 - \sum_{\binom{v}{w} \in U^+} \bar{a}_{\binom{v}{w}} (sqp^2)^{\|v\| - \|w\|} q^{\|w\|} (p^2 z)^{l\binom{v}{w}}} \\ &= \frac{sp^2 \sum_{n \geqslant 0} \frac{(-1)^n (qp^2 z)^{n+1} q^{\binom{n+1}{2}}}{(sqp^2;q)_{n+1} (q;q)_n} \prod_{k=1}^{n} \left(1 - \frac{(1-q^k) t p^2}{1 - p^2 q^k}\right)}{1 + \sum_{n \geqslant 1} \frac{(-1)^n (qp^2 z)^n q^{\binom{n}{2}}}{(sqp^2;q)_n (q;q)_n} \prod_{k=1}^{n-1} \left(1 - \frac{(1-q^k) t p^2}{1 - p^2 q^k}\right)}. \end{aligned}$$

The case $p = 1$ may be expressed in terms of the Bessel functions of (8):

$$\sum_{n \geqslant 0} g_n(s,t,q,1) z^{n+1} = \frac{sJ_{1,q,q}(sq, q(1-t)z)}{J_{0,q,q}(sq, q(1-t)z) - t}.$$

In 1965, Klarner [22] enumerated DVC polyominoes by area which corresponds to the above with $s = t = p = z = 1$. Barcucci et al. [3] added the column number in 1991 ($s = t = p = 1$). More recently, Bousquet-Mélou [6] enumerated various classes of DVC polyominoes by perimeter, number of columns, height, and area ($t = 1$). The case $t = 0$, $s = p = 1$ is due to Delest and Fedou [12] and counts staircase polyominoes by area and column number. The height statistic was added to their result by Bousquet-Mélou and Viennot [7]. With a little more work, Theorem 1(b) may be used to obtain a determinant formula for

$$g_n(r, s, t, q, p) = \sum_{Q \in \mathrm{DVC}_n} r^{\mathrm{uind}\, Q} s^{h(Q)} t^{\mathrm{udes}\, Q} q^{\mathrm{area}\, Q} p^{\mathrm{per}\, Q}$$

where the *upper index* of Q is $\mathrm{uind}\, Q = \sum_{k \in \mathrm{UDes}\, Q} k$.

6. The Simon Newcomb problem

For a sequence $k = \{k_x\}_{x \geqslant 0} \subset N$ having only a finite number of nonzero terms, the set of rearrangements of the word $0^{k_0} 1^{k_1} 2^{k_2} \cdots$ is denoted by R_k. In other terms, a word w is in R_k if and only if each letter $x \in N$ appears exactly k_x times in w. If $k_x = 1$ for $1 \leqslant x \leqslant n$ and if $k_x = 0$ otherwise, then R_k is just the set of permutations S_n of $\{1, 2, \ldots, n\}$.

The problem of Simon Newcomb is to determine the number of rearrangements with a prescribed descent set. MacMahon solved the problem, giving seperate treatments [26, Vol. I, pp. 190, 200] for S_n and R_k. Niven [27] rediscovered MacMahon's solution for S_n. A lattice path proof for the permutation case was given by Gessel and Viennot [19]. A q-analog of MacMahon's formula on S_n was obtained and generalized to sequences of permutations by Stanley [31]. As presented in Examples 10 and 11, Theorem 1(b) affords further extensions of MacMahon's solutions on S_n and R_k.

Some notation and observations are common to both cases. Throughout, fix a sequence $0 = v_0 < v_1 < \cdots < v_{d+1} = n$ of integers. For a word $w \in N^n$, let $\mathrm{WRis}_v\, w = \mathrm{WRis}\, w \cap \{v_1, v_2, \ldots, v_d\}$, and $\mathrm{wris}_v\, w = |\mathrm{WRis}_v\, w|$. Further, fix a map $\beta : N \to C$. The index of $w \in N^n$ relative to v and β is defined as

$$\mathrm{ind}_{v,\beta}\, w = \sum_{v_j \in \mathrm{WRis}_v\, w} \beta(j).$$

Two natural choices for β are (1) $\beta(j) = j$ for all $j \in N$ and (2) $\beta(j) = v_j$ if $0 \leqslant j \leqslant d+1$ and $\beta(j) = 0$ otherwise.

The appropriate alphabets to consider are

$$L_i = \{x_1 x_2 \cdots x_{v_i - v_{i-1}} \in N^{v_i - v_{i-1}} \colon x_1 \leqslant x_2 \leqslant \cdots \leqslant x_{v_i - v_{i-1}}\}$$

for $1 \leqslant i \leqslant d+1$ and $X = L_1 \cup \cdots \cup L_{d+1}$. For $x = x_1 x_2 \cdots x_r$, $y = y_1 y_2 \cdots y_s \in X$, define $a_{i,xy} = t p^{\beta(i)}$ if $x_r \leqslant y_1$ and $a_{i,xy} = 1$ otherwise. For a word $w = x_1 x_2 \cdots x_n \in L_1 \cdots L_{d+1}$, note that

$$a_{1,w} = t^{\mathrm{wris}_v\, w} p^{\mathrm{ind}_{v,\beta}\, w}.$$

Also, if $w = x_1 x_2 \cdots x_{v_j - v_{i-1}} \in L_i \cdots L_j$, then

$$\bar{a}_{i,w} = \begin{cases} (-1)^{j-i}(t;\, p)_{i,j,\beta} & \text{if } 0 \leqslant x_1 \leqslant x_2 \leqslant \cdots \leqslant x_{v_j - v_{i-1}}, \\ 0 & \text{otherwise}, \end{cases} \tag{12}$$

where $(t;\, p)_{i,j,\beta} = (1 - tp^{\beta(i)})(1 - tp^{\beta(i+1)}) \cdots (1 - tp^{\beta(j)})$. For the choice $\beta(j) = j$ for all $j \in N$, observe that $(t;\, p)_{i,j,\beta} = (tp^i;\, p)_{j-i}$.

Example 10. (*Permutations with prescribed descent set by inversion number*). The *inversion number* of a word $w = x_1 x_2 \cdots x_n \in N^n$ is

$$\operatorname{inv} w = |\{(k, m)\colon 1 \leqslant k < m \leqslant n,\, x_k > x_m\}|.$$

Let Π_n denote the set of non-decreasing words in N^n (i.e., the set of partitions with at most n parts). A permutation $\sigma \in S_n$ will be viewed as the word $\sigma(1)\sigma(2)\cdots\sigma(n)$.

As noted in [16], the map $\psi_n : S_n \times \Pi_n \to N^n$ defined by

$$\psi_n(\sigma(1)\sigma(2)\cdots\sigma(n), \lambda_1\lambda_2\cdots\lambda_n) = x_1 x_2 \cdots x_n$$

where $x_k = \lambda_{\sigma(k)} + |\{j\colon k + 1 \leqslant j \leqslant n,\, \sigma(k) > \sigma(j)\}|$ is a bijection. Moreover, if $\psi_n(\sigma, \lambda) = w$, then

$$\operatorname{WRis} \sigma = \operatorname{WRis} w \quad \text{and} \quad \operatorname{inv} \sigma + \|\lambda\| = \|w\|. \tag{13}$$

In a permutation, a weak rise is in fact a strict rise. As an example, note that ψ_5 maps the pair $(5\,1\,3\,4\,2,\ 1\,1\,1\,1\,2) \in S_5 \times \Pi_5$ to the word $6\,1\,2\,2\,1 \in N^5$. Further observe that $\operatorname{WRis} 5\,1\,3\,4\,2 = \{2, 3\} = \operatorname{WRis} 6\,1\,2\,2\,1$ and that $\operatorname{inv} 5\,1\,3\,4\,2 + \|1\,1\,1\,1\,2\| = 6 + 6 = \|6\,1\,2\,2\,1\|$. Fedou and Rawlings [16] used ψ_n, (2), and (3) to deduce some of Stanley's [31] extensions of the results of Carlitz et al. [10] for enumerating sequences of permutations by various statistics. Along the same lines, Foata and Han [18] obtained similar results on signed permutations. Although not done so here, this example may also be extended to sequences of signed permutations.

For integers $0 = v_0 < v_1 < \cdots < v_{d+1} = n$ and a fixed $\beta : N \to \mathbf{C}$, let

$$f_{n,v,\beta}(t, q, p) = \sum_{\sigma \in S_{n,v}} t^{\operatorname{wris}_v \sigma} q^{\operatorname{inv} \sigma} p^{\operatorname{ind}_{v,\beta} \sigma}$$

where $S_{n,v}$ denotes the set of permutations $\sigma \in S_n$ such that $\operatorname{Des} \sigma \subseteq \{v_1, v_2, \ldots, v_d\}$. Note that $f_{n,v,\beta}(0, q, 1)$ enumerates permutations $\sigma \in S_n$ with $\operatorname{Des} \sigma = \{v_1, v_2, \ldots, v_d\}$ by inversion number.

To obtain a determinant for $f_{n,v,\beta}(t, q, p)$, first note from (13) that, when restricted to $S_{n,v} \times \Pi_n$, ψ_n is a bijection onto $L_1 \cdots L_{d+1}$. Together, (13), (4), (12), and Theorem 1(b) give

$$\begin{aligned} f_{n,v,\beta}(t, q, p) &= (q; q)_n \sum_{(\sigma,\lambda) \in S_{n,v} \times \Pi_n} t^{\operatorname{wris}_v \sigma} q^{\operatorname{inv} \sigma + \|\lambda\|} p^{\operatorname{ind}_{v,\beta} \sigma} \\ &= (q; q)_n \sum_{w \in L_1 \cdots L_{d+1}} a_{1,w} q^{\|w\|} = (q; q)_n \det(\bar{A}_{i,j})_{i,j=1}^{d+1} \end{aligned}$$

where $\bar{A}_{i,j} = 0$ if $i > j+1$, $\bar{A}_{j+1,j} = -1$, and, for $1 \leqslant i \leqslant j \leqslant d+1$,

$$\bar{A}_{i,j} = \sum_{w \in L_i \ldots L_j} \bar{a}_{i,w} q^{\|w\|} = (-1)^{j-i} (t;\, p)_{i,j,\beta} \sum_{0 \leqslant x_1 \leqslant \cdots \leqslant x_{v_j - v_{i-1}}} q^{x_1 + \cdots + x_{v_j - v_{i-1}}}$$

$$= \frac{(-1)^{j-i} (t;\, p)_{i,j,\beta}}{(q;q)_{v_j - v_{i-1}}}.$$

As $\det(\bar{A}_{i,j}) = \det((-1)^{j-i} \bar{A}_{i,j})$, it further follows that

$$f_{n,v,\beta}(t,q,\,p) = \det\left((t;\, p)_{i,j,\beta} \begin{bmatrix} n - v_{i-1} \\ v_j - v_{i-1} \end{bmatrix} \right)_{i,j=1}^{d+1} \tag{14}$$

where, by convention, $(t;\, p)_{i,j,\beta} = 1$ for $j - i \leqslant 0$. Setting $t = 0$ and $p = 1$ in (14) gives Stanley's [31] q-analog of MacMahon's formula [26, Vol. I, p. 190].

Example 11. (*Rearrangements with prescribed descent set*). For a sequence $k = \{k_x\}_{x \geqslant 0} \subset N$ having only a finite number of nonzero terms, put $\|k\| = \sum_{x \geqslant 0} k_x$ and $Z^k = \prod_{x \geqslant 0} z_x^{k_x}$. Let $R_{k,v}$ be the set of words $w \in R_k$ such that $\operatorname{Des} w \subseteq \{v_1, v_2, \ldots, v_d\}$. The coefficient of $t^0 p^0 Z^k$ in

$$g_{n,v,\beta}(t,\,p,Z) = \sum_{\|k\| = n} \sum_{w \in R_{k,v}} t^{\mathrm{wris}_v\, w} p^{\mathrm{ind}_{v,\beta}\, w} Z^k$$

is the number of $w \in R_k$ of length $\|k\| = n$ with $\operatorname{Des} w = \{v_1, v_2, \ldots, v_d\}$.

For $w = x_1 x_2 \cdots x_r \in N^*$, define $\eta(w) = z_{x_1} z_{x_2} \cdots z_{x_r}$. Denoting the complete symmetric polynomial by h_k (see [25, p. 14]), it follows from Theorem 1(b) and (12) that

$$g_{n,v,\beta}(t,\,p,Z) = \sum_{w \in L_1 \cdots L_{d+1}} a_{1,w} \eta(w) = \det(\bar{A}_{i,j})_{i,j=1}^{d+1}$$

where $\bar{A}_{i,j} = 0$ if $i > j+1$, $\bar{A}_{j+1,j} = -1$, and, for $1 \leqslant i \leqslant j \leqslant d+1$,

$$\bar{A}_{i,j} = \sum_{w \in L_i \cdots L_j} \bar{a}_{i,w} \eta(w) = (-1)^{j-i} (t;\, p)_{i,j,\beta} \sum_{0 \leqslant x_1 \leqslant \cdots \leqslant x_{v_j - v_{i-1}}} z_{x_1} z_{x_2} \cdots z_{x_{v_j - v_{i-1}}}$$

$$= (-1)^{j-i} (t;\, p)_{i,j,\beta} h_{v_j - v_{i-1}}(z_0, z_1, \ldots).$$

The formula for $g_{n,v,\beta}$ may be rewritten as

$$g_{n,v}(t,\,p,Z) = \det\big((t;\, p)_{i,j,\beta} h_{v_j - v_{i-1}}(z_0, z_1, \ldots) \big)_{i,j=1}^{d+1} \tag{15}$$

with the conventions that $(t;\, p)_{i,j,\beta} = 1$ if $j - i \leqslant 0$ and $h_k = 0$ if $k < 0$. For $p = 1$, Stanley [31] gives another solution involving Möbius inversion. With $t = 0$ and $p = 1$, (15) reduces to MacMahon's result [26, Vol. I, p. 200].

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