Absorption Processes: Models for q-Identities

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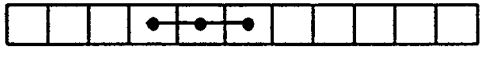
Several extensions of Blomqvist's absorption process are presented. Inherent in some of the associated distributions is a method for establishing q-identities ranging from properties of Gaussian polynomials to product expansions of basic hypergeometric series to extensions of results on Mahonian statistics. One process links the comajor index to Russian roulette. Also given are examples involving the Rogers–Ramanujan identities that demonstrate how q-expressions may be modeled with absorption processes.

1. INTRODUCTION

Blomqvist's [4] absorption process may be modified so as to provide a common experimental setting for a wide range of q-identities. The first innovation considered herein is the variation of particle size.

A sequence of j dots connected by a line will be referred to as a *j-particle*. The *j-absorption process* consists of sequentially propelling j-particles into a chamber l cells long according to the following Bernoulli trials scheme.

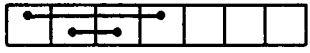
3-particle in an 11-cell chamber



To begin, a first j-particle $\mathscr{P}$ is placed in the leftmost j cells, one dot per cell. A coin with probability $q < 1$ of landing tails up is then tossed until (i) a heads occurs or (ii) $\mathscr{P}$ occupies the rightmost j cells and a tails occurs. In case (i), $\mathscr{P}$ is advanced one cell to the right each time tails appears and comes to rest when the heads occurs. In case (ii), $\mathscr{P}$ is removed from the chamber. Successive j-particles are similarly propelled into the chamber *as*

if occupied cells had been removed. When j exceeds the number of empty cells, subsequent j-particles are partially inserted and immediately removed without a single coin toss.

A j-particle that comes to rest in the chamber is deemed an *absorption*. If not absorbed, a j-particle is said to have *escaped*. To illustrate, suppose that three 2-particles are propelled into a chamber 7 cells long. If the three Bernoulli sequences are TH, H, and TT, then the final outcome is



The tally of absorptions is 2 and one 2-particle escaped (not shown). The 1-absorption process is the original one of Blomqvist (also see [6, 10, 21]).

The probability generating functions associated with the j-absorption process may be expressed in terms of basic hypergeometric series. For an integer $n \geq 1$, the *q-shifted factorial* of a is defined as

$$(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1}).$$

By convention, $(a;q)_0 = 1$. For integers $n, k \geq 0$, the *q-binomial coefficient* (also known as the *Gaussian polynomial*) is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q^n;q^{-1})_k}{(q;q)_k}.$$

The *basic hypergeometric series* ${}_r\phi_s$ with r upper and s lower parameters is

$$\begin{aligned} &{}_r\phi_s\left[\begin{matrix} a_1,\ldots,a_r \\ b_1,\ldots,b_s \end{matrix};q,z\right] \\ &\quad = \sum_{k\geq 0} \frac{(a_1;q)_k(a_2;q)_k\cdots(a_r;q)_k}{(q;q)_k(b_1;q)_k\cdots(b_s;q)_k}\left((-1)^k q^{\binom{k}{2}}\right)^{1+s-r} z^n, \end{aligned}$$

where $\binom{k}{2} = k(k-1)/2$. In section 2, the following results will be verified.

THEOREM 1. *Given that $n \geq 1$ j-particles are propelled into a chamber of length $l \geq 1$, let* Ab *denote the number of absorptions. If $0 < q < 1$ and $l \geq jn$ or $l \equiv -1 \bmod j$, then the probability of k absorptions is*

$$P_{j,l,n}\{\mathrm{Ab} = k\} = q^{(n-k)(l-jk-j+1)}\left(q^{l-j+1};q^{-j}\right)_k \begin{bmatrix} n \\ k \end{bmatrix}_{q^j}. \tag{1}$$

Moreover, when $l \geq jn$, the probability generating function of Ab *is*

$$P_{j,l,n}(z) = (q^{l-j+1}; q^{-j})_n \, z^n \, {}_1\phi_1\left[\begin{matrix} q^{-jn} \\ q^{l-jn+1} \end{matrix}; q^j, \frac{q^{l+1}}{z}\right]. \tag{2}$$

THEOREM 2 (Negative q-binomial distribution). *For a chamber of length $l \geq j$ and a positive integer $k \leq l/j$, let N denote the number of j-particles required to achieve k absorptions. If $0 < q < 1$, then the probability of the nth j-particle being the kth one absorbed is*

$$Q_{j,l,k}\{N = n\} = q^{(n-k)(l-jk+1)}(q^{l-j+1}; q^{-j})_k \begin{bmatrix} n-1 \\ n-k \end{bmatrix}_{q^j}. \tag{3}$$

Moreover, the probability generating function of N is

$$Q_{j,l,k}(z) = (q^{l-j+1}; q^{-j})_k \, z^k \, {}_1\phi_0\left[\begin{matrix} q^{jk} \\ - \end{matrix}; q^j, zq^{l-jk+1}\right]. \tag{4}$$

Theorem 1 for $j = 1$ is due to Blomqvist [4] and Theorem 2 for $j = 1$ was discovered by Dunkl [6]. Other modifications of Blomqvist's absorption process are considered in Sections 5 through 8.

Beyond their intrinsic appeal, such distributions afford a method for establishing q-identities. To illustrate, Theorems 1 and 2 are used in Section 3 to give probabilistic proofs of known properties of Gaussian polynomials and of known product expansions for ${}_1\phi_0$ and ${}_1\phi_1$. In Section 9, distributions on an absorption ring are used to verify and extend two of MacMahon's [14] fundamental results concerning the inversion number and the comajor index. One absorption ring distribution incidentally reveals some information about Russian roulette.

Absorption processes also serve to model a variety of q-expressions. Included in Sections 5 and 6 are processes for the Rogers–Ramanujan identities.

Although not of primary interest here, two asymptotic distributions involving q-analogs of the Bessel function J_0 and the cosine function are derived in Section 4 from Theorem 1. These are referred to respectively as the *q-Bessel* and *Cauchy distributions*, the former being discovered by Griffin [9].

2. PROOF OF THEOREMS 1 AND 2

To prove (1), fix $n \geq 1$ and suppose that k of the n j-particles are absorbed. Note that, after the $(i-1)$st absorption occurs where $1 \leq i \leq (k+1)$, the probability associated with the next absorption is $(1 - q^{l-ji+1})$.

Also, the probability of an escape occurring between the $(i-1)$st and ith absorptions is $q^{l-ji+1} = q^{l-jk-j+1}q^{j(k+1-i)}$. As the $(n-k)$ escapes may be distributed in any manner whatsoever between the absorptions, it follows that

$$P_{j,l,n}\{\mathrm{Ab} = k\}$$
$$= \prod_{i=1}^{k}\left(1 - q^{l-ji+1}\right) q^{(n-k)(l-jk-j+1)} \sum_{0 \le m_1 \le m_2 \le \cdots \le m_{n-k} \le k} \left(q^j\right)^{s(m)},$$

where $s(m) = m_1 + m_2 + \cdots + m_{n-k}$. Since the sum on the above right is equal to $\left[{n \atop k}\right]_{q^j}$ (see Theorem 3.1 in [1]), the proof of (1) is complete.

To verify (2), first use the identity $\left[{n \atop k}\right]_q = (q^n; q^{-1})_{n-k}/(q;q)_{n-k}$ and the fact that $(a; q^{-1})_k = (a^{-1}; q)_k(-a)^k q^{-\binom{k}{2}}$ to rewrite formula (1) as

$$P_{j,l,n}\{\mathrm{Ab} = k\} = q^{(n-k)(l-jk-j+1)} \frac{\left(q^{l-j+1}; q^{-j}\right)_n}{\left(q^{l-jn+1}; q^j\right)_{n-k}} \frac{\left(q^{jn}; q^{-j}\right)_{n-k}}{\left(q^j; q^j\right)_{n-k}}$$
$$= \frac{\left(q^{l-j+1}; q^{-j}\right)_n}{\left(q^{l-jn+1}; q^j\right)_{n-k}} \frac{\left(q^{-jn}; q^j\right)_{n-k}}{\left(q^j; q^j\right)_{n-k}} (-1)^{n-k} q^{(l+1)(n-k)+j\binom{n-k}{2}}.$$

Then,

$$P_{j,l,n}(z) = \sum_{k=0}^{n} P_{j,l,n}\{\mathrm{Ab} = k\} z^k$$
$$= \left(q^{l-j+1}; q^{-j}\right)_n z^n \sum_{k=0}^{n} \frac{\left(q^{-jn}; q^j\right)_{n-k}(-1)^{n-k} q^{j\binom{n-k}{2}}}{\left(q^j; q^j\right)_{n-k}\left(q^{l-jn+1}; q^j\right)_{n-k}} \left(\frac{q^{l+1}}{z}\right)^{n-k}$$
$$= \left(q^{l-j+1}; q^{-j}\right)_n z^n \, {}_1\phi_1\left[\begin{matrix} q^{-jn} \\ q^{l-jn+1} \end{matrix}; q^j, \frac{q^{l+1}}{z}\right].$$

Formula (3) also follows from (1). For a fixed positive integer $k \le l/j$, the probability of the nth j-particle being the kth absorption is clearly

$$Q_{j,l,k}\{N = n\} = P_{j,l,n-1}\{\mathrm{Ab} = k-1\}\left(1 - q^{l-jk+1}\right)$$
$$= q^{(n-k)(l-jk+1)}\left(q^{l-j+1}; q^{-j}\right)_{k-1} \left[{n-1 \atop k-1}\right]_{q^j} \left(1 - q^{l-jk+1}\right)$$
$$= q^{(n-k)(l-jk+1)}\left(q^{l-j+1}; q^{-j}\right)_k \left[{n-1 \atop n-k}\right]_{q^j}.$$

The generating function in (4) may then be easily derived from (3).

3. THE PROBABILISTIC APPROACH TO q-IDENTITIES

Absorption distributions may be used to translate facts of probability into q-identities. First, let us consider Blomqvist's distribution $P_{1,l,n}$. Suppose k of $(r+s)$ 1-particles are absorbed. Evidently, ν of the first r 1-particles and $(k-\nu)$ of the remaining s 1-particles are absorbed where $0 \le \nu \le k$. Thus,

$$P_{1,l,r+s}\{\mathrm{Ab} = k\} = \sum_{v=0}^{k} P_{1,l,r}\{\mathrm{Ab} = \nu\} P_{1,l-\nu,s}\{\mathrm{Ab} = k - v\}.$$

Replacing the probabilities above by the right-hand side of (1) and canceling like terms gives the q-analog of the Chu–Vandermonde identity [1, p. 37]:

$$\begin{bmatrix} r+s \\ k \end{bmatrix}_q = \sum_{\nu=0}^{k} q^{(r-\nu)(k-\nu)} \begin{bmatrix} r \\ \nu \end{bmatrix}_q \begin{bmatrix} s \\ k-\nu \end{bmatrix}_q.$$

As an application of Dunkl's distribution $Q_{1,l,k}$, fix $k \ge 1$ and let $\mathscr{A}_k(n)$ denote the event that the nth 1-particle is the kth absorption. For $(k-1) \le m < n$, the conditional probability of $\mathscr{A}_{k-1}(m)$ occurring given that $\mathrm{A}_k(n)$ occurs may be computed with the aid of (3):

$$\begin{aligned} \mathrm{Prob}\{\mathrm{A}_{k-1}(m) | \mathrm{A}_k(n)\} &= \frac{\mathrm{Prob}\{\mathscr{A}_{k-1}(m) \wedge \mathscr{A}_k(n)\}}{\mathrm{Prob}\{\mathscr{A}_{k(n)}\}} \\ &= \frac{Q_{1,l,k-1}\{N = m\} Q_{1,l-k+1,1}\{N = n-m\}}{Q_{1,l,k}\{N = n\}} \\ &= q^{m-k+1} \begin{bmatrix} m-1 \\ m-k+1 \end{bmatrix}_q \begin{bmatrix} n-1 \\ n-k \end{bmatrix}_q^{-1}. \end{aligned}$$

Combining the observation that $\sum_{m=k-1}^{n-1} \mathrm{Prob}\{\mathrm{A}_{k-1}(m) | \mathrm{A}_k(n)\} = 1$ with the above result implies that

$$\sum_{m=k-1}^{n-1} q^{m-k+1} \begin{bmatrix} m-1 \\ m-k+1 \end{bmatrix}_q = \begin{bmatrix} n-1 \\ n-k \end{bmatrix}_q.$$

A few cosmetic changes then reveal a standard property [1, p. 37] of the Gaussian polynomials:

$$\sum_{v=0}^{r} q^{\nu} \begin{bmatrix} \nu+k \\ \nu \end{bmatrix}_q = \begin{bmatrix} r+k+1 \\ k+1 \end{bmatrix}_q.$$

Theorems 1 and 2 also imply product expansions for ${}_1\phi_1$ and ${}_1\phi_0$. From the trivial observation that $1 = P_{j,l,n}(1)$, we have

$$ {}_1\phi_1\left[\begin{matrix} q^{-jn} \\ q^{l-jn+1} \end{matrix}; q^j, q^{l+1}\right] = \frac{1}{\left(q^{l-j+1}; q^{-j}\right)_n}, \tag{5} $$

where $0 < q < 1$. Similarly, (4) implies

$$ {}_1\phi_0\left[\begin{matrix} q^{jk} \\ - \end{matrix}; q^j, q^{l-jk+1}\right] = \frac{1}{\left(q^{l-j+1}; q^{-j}\right)_k}. $$

More general results for ${}_1\phi_1$ and ${}_1\phi_0$ may be found in [8, p. 236].

Evident in Dunkl's [6] remark that the identity $1 = Q_{1,l,j}(1)$ "expresses the known (Heine) sum of the ${}_1\phi_0$ series," the probabilistic method provides an experimental setting for establising q-identities. Further examples are given in [18] and in sections 7, 8, and 9. A probabilistic derivation of a formula for Ramanujan's ${}_1\psi_1$ sum was given by Kadell [11].

For later reference, two cases of the expansion for ${}_1\phi_1$ are singled out. Setting $(j, l) = (1, n)$ and letting $n \to \infty$ in (5) yields the identity

$$ \sum_{k \geq 0} \frac{q^{k^2}}{(q;q)_k^2} = \prod_{i \geq 1} \frac{1}{(1 - q^i)}, \tag{6} $$

which has been attributed to Euler [1, p. 20]. Selecting $(j, l) = (2, 2n)$ and then allowing $n \to \infty$ in (5) leads to an identity due to Cauchy [1, p. 20]:

$$ \sum_{k \geq 0} \frac{q^{2k^2-k}}{(q;q)_{2k}} = \prod_{i \geq 1} \frac{1}{(1 - q^{2i-1})}. \tag{7} $$

4. THE q-BESSEL AND CAUCHY DISTRIBUTIONS

Before further modifying Blomqvist's absorption process, a pause is taken to extract two asymptotic distributions from Theorem 1. For the j-absorption process, let Es denote the number of escapes. For $(j, l) = (1, n)$, define $B\{\mathrm{Es} = k\}$ to be the probability of having k escapes as $n \to \infty$. From (1),

$$ \begin{aligned} B\{\mathrm{Es} = k\} &= \lim_{n \to \infty} P_{1,n,n}\{\mathrm{Ab} = n - k\} = \lim_{n \to \infty} q^{k^2} \left(q^n; q^{-1}\right)_{n-k} \begin{bmatrix} n \\ k \end{bmatrix}_q \\ &= \frac{q^{k^2}}{(q;q)_k} \lim_{n \to \infty} \prod_{i=k+1}^{n} (1 - q^i) = \frac{q^{k^2}}{(q;q)_k^2} \prod_{i \geq 1} (1 - q^i). \end{aligned} $$

Making use of the definition

$$J_{0,q}(z) = \sum_{k \geq 0} (-1)^k \frac{q^{k^2}}{(q;q)_k^2} (z/2)^{2k}$$

given by Jackson [8, p. 25] for one of the q-analogs of the Bessel function $J_0(z)$ and also in view of (6), it follows that

$$B\{\mathrm{Es} = k\} = \frac{q^{k^2}/(q;q)_k^2}{J_{0,q}(2\sqrt{-1})}. \tag{8}$$

Note that $B\{\mathrm{Es} = 0\} = 1/J_{0,q}(2\sqrt{-1})$. Implicit in Blomqvist [4], (8) was rediscovered by Griffin [9] and used to prove (6). The connection with J_0 perhaps justifies referring to $B\{\mathrm{Es} = \cdot\}$ as the *q-Bessel distribution.*

For the second distribution, let

$$\cos_q z = \sum_{k \geq 0} \frac{(-1)^k q^{2k^2-k} z^{2k}}{(q;q)_{2k}}.$$

Note that $\cos_q(1-q)z \to \cos z$ as $q \to 1^-$. Other q-cosine functions have been considered by Jackson and Hahn [8, p. 23]. From (7), we have

$$\cos_q(\sqrt{-1}) = \prod_{i \geq 1} \frac{1}{(1-q^{2i-1})}.$$

For the case $(j, l) = (2, 2n)$ of the j-absorption process, let $C\{\mathrm{Es} = k\}$ be the asymptotic probability of having k escapes. By (1),

$$C\{\mathrm{Es} = k\} = \lim_{n \to \infty} P_{2,2n,n}\{\mathrm{Ab} = n-k\} = \frac{q^{2k^2-k}/(q;q)_{2k}}{\cos_q(\sqrt{-1})}.$$

Note that $C\{\mathrm{Es} = 0\} = 1/\cos_q(\sqrt{-1})$. In view of (7), $C\{\mathrm{Es} = \cdot\}$. may be reasonably referred to as *Cauchy's distribution.* A "modular" analog of $C\{\mathrm{Es} = \cdot\}$ is given in Theorem 5 of Section 8.

Bernoulli schemes for the reciprocals of a q-analog of the number e and of a q-analog of the Riemann-zeta function are presented in [17, 18]. Although not done so here, they may be easily reformulated in the context of the absorption ring considered in Section 7.

5. REBOUND ABSORPTION PROCESSES: MODELING q-SERIES

In a relatively straightforward manner, absorption processes may be used to model many q-series and q-products. In this section and the next, processes are described for both sides of the identity

$$\sum_{k \geq 0} \frac{q^{k^2}}{(q;q)_k} = \prod_{i \geq 1} \frac{1}{(1 - q^{5i-4})(1 - q^{5i-1})} \tag{9}$$

of Rogers and Ramanunjan [1, p. 104].

The *rebound absorption process* involves sequentially propelling 1-particles into a chamber, as in Section 1, with the following modifications. Upon reaching the rightmost cell, the "active" 1-particle $\mathscr{P}$ immediately reverses direction. If $\mathscr{P}$ returns to the leftmost empty cell and a tails is tossed, it is then removed. Furthermore, a new cell is adjoined to the right of the chamber in the event that $\mathscr{P}$ ever occupies the rightmost cell during the course of its run. For instance, suppose two 1-particles are sent into an empty chamber of length 3. If the sequences TH and TTT are tossed, then the result is

where one 1-particle escapes and the chamber length is increased to 4.

Let Ts denote the number of tails required of a newly inserted 1-particle to reach the rightmost cell (that is, Ts is the number of empty cells minus 1). The asymptotic probability of Ts being equal to k, denoted by $\mathscr{R}\{\text{Ts} = k\}$, may be computed using the theory of Markov chains.

Propelling the nth 1-particle is to be viewed as the nth step in the chain. Considering the number of tails required to reach the rightmost cell as the state, let $p_{k,m}$ be the transition probability of moving from state k to state m in one step. Since $p_{k,m} = 0$ whenever $|k - m| > 1$, the chain is a *birth-and-death* process. Note that $p_{k,k-1} = (1 - q^k)$ and $p_{k,k+1} = q^{2k+1}$. From the theory of Markov chains (e.g., see [5, p. 283]), it follows that

$$\begin{aligned}\mathscr{R}\{\text{Ts} = k\} &= \frac{p_{0,1}p_{1,2} \cdots p_{k-1,k}/p_{1,0}p_{2,1} \cdots p_{k,k-1}}{\sum_{k \geq 0} p_{0,1}p_{1,2} \cdots p_{k-1,k}/p_{1,0}p_{2,1} \cdots p_{k,k-1}} \\ &= \frac{q^{k^2}/(q;q)_k}{\sum_{k \geq 0} q^{k^2}/(q;q)_k}.\end{aligned}$$

By (9), $\mathscr{R}\{\text{Ts} = k\} = (q^{k^2}/(q;q)_k)\Pi_{i \geq 1}(1 - q^{5i-4})(1 - q^{5i-1})$.

A minor alteration of the rebound absorption process leads to a measure involving the identity

$$\sum_{k\geq 0} \frac{q^{k(k+1)}}{(q;q)_k} = \prod_{i\geq 1} \frac{1}{(1-q^{5i-3})(1-q^{5i-2})},$$

also of Rogers and Ramanujan [1, p. 104]: To wit, a direction change in the rightmost cell requires a tails (i.e., rebound is not immediate as time is allotted for "turning around"). The modified chain is still a birth-and-death process. The relevant transition probabilities are $p^*_{k,k-1} = (1-q^k)$ and $p^*_{k,k+1} = q^{2k+2}$. Letting $\mathscr{S}\{Ts = k\}$ be the stationary probability of k tails being required to reach the rightmost cell, we see that

$$\begin{aligned}\mathscr{S}\{\mathrm{Ts} = k\} &= \frac{q^{k(k+1)}/(q;q)_k}{\sum_{k\geq 0} q^{k(k+1)}/(q;q)_k} \\ &= \frac{q^{k(k+1)}}{(q;q)_k} \prod_{i\geq 1} (1-q^{5i-3})(1-q^{5i-2}).\end{aligned}$$

The Rogers–Ramanujan identities have appeared before in a probabilistic context, namely, the hard hexagon model developed by Baxter [2].

Similar absorption models may be given for the Heine–Euler distributions first considered by Benkherouf and Bather [3] (in connection with oil exploration) and further investigated by Kemp [12, 13].

6. ALTERNATELY SIZED PARTICLES: MODELING q-PRODUCTS

The q-product in (9) arises from a natural variation of the j-absorption process: Particles of different sizes are to be alternately sent into the chamber. Beginning with a 2-particle, suppose that n 2-particles and n 3-particles are alternately propelled into a chamber of length l as in Section 1. Let $R_{l,2n}\{\mathrm{Ab} = k\}$ denote the probability that k absorptions occur. For $l = 5n$, the probability of all $2n$ particles being absorbed is clearly

$$R_{5n,2n}\{\mathrm{Ab} = 2n\} = R_{5n,2n}\{\mathrm{Es} = 0\} = \prod_{i=1}^{n} (1-q^{5i-4})(1-q^{5i-1}).$$

Thus $\lim_{n\to\infty} R_{5n,2n}\{\mathrm{Es} = 0\}$ is the reciprocal of the right-hand side of (9).

The distribution $R_{l,2n}\{\mathrm{Ab} = \cdot\}$ may be recursively determined. We first extend the definition of $R_{l,m}\{\mathrm{Ab} = \cdot\}$ to include odd m. Beginning with a

3-particle, suppose that n 3-particles and $(n-1)$ 2-particles are alternately propelled into the chamber. We let $R_{l,2n-1}\{\mathrm{Ab}=k\}$ be the probability of having k absorptions.

In regards to $R_{l,2n}\{\mathrm{Ab}=k\}$, the first 2-particle is either absorbed or escapes. Thus,

$$R_{l,2n}\{\mathrm{Ab}=k\} = \left(1-q^{l-1}\right)R_{l-2,2n-1}\{\mathrm{Ab}=k-1\} + q^{l-1}R_{l,2n-1}\{\mathrm{Ab}=k\}$$

holds for $l \geq 2$ and $n \geq 1$. Similarly,

$$R_{l,2n-1}\{\mathrm{Ab}=k\} = \left(1-q^{l-2}\right)R_{l-3,2n-2}\{\mathrm{Ab}=k-1\} + q^{l-2}R_{l,2n-2}\{\mathrm{Ab}=k\}$$

is true for $l \geq 3$ and $n \geq 1$. The initial conditions are $R_{l,0}\{\mathrm{Ab}=0\}=1$, $R_{l,m}\{\mathrm{Ab}<0\}=0$, $R_{2,2n-1}\{\mathrm{Ab}=0\}=q^{n-1}$, $R_{2,2n-1}\{\mathrm{Ab}=1\}=(1-q^{n-1})$, and $R_{l,m}\{\mathrm{Ab}=k\}=0$ whenever $k>\min\{l/2,m\}$.

7. THE ABSORPTION RING: MAHONIAN DISTRIBUTIONS AND RUSSIAN ROULETTE

To further illustrate the probabilistic approach to q-identities, two processes from [16] are recast here in the context of an absorption ring. Generalizations of the two are sketched in Section 8. These processes are then used in Section 9 to prove and extend MacMahon's [14] results on the inversion number and the comajor index.

To set the stage, let $n_1, n_2, \ldots, n_l$ denote a sequence of non-negative integers. Put $n = n_1 + n_2 + \cdots + n_l$. The set of functions mapping $\{1,2,\ldots,n\}$ to $\{1,2,\ldots,l\}$ that take on the value i exactly n_i times will be denoted by $\mathscr{M}[1^{n_1}2^{n_2}\cdots l^{n_l}]$. Such a function f will be expressed as a list of its range elements: $f=f(1)f(2)\cdots f(n)$. The *descent set* of $f \in \mathscr{M}[1^{n_1}2^{n_2}\cdots l^{n_l}]$, denoted by Des f, consists of the indices k, $1 \leq k \leq (n-1)$, such that $f(k) > f(k+1)$. The *inversion number* and *comajor index* of f are

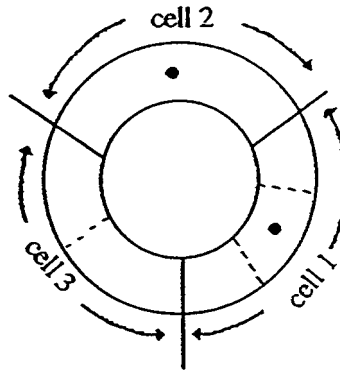
$$\mathrm{inv}\, f = |\{(k,m): k<m, f(k)>f(m)\}| \qquad \text{and} \qquad \mathrm{comaj}\, f = \sum_{k \in \mathrm{Des}\, f} (n-k$$

For example, the function $f = 2\ 2\ 1\ 2\ 1 \in \mathscr{M}[1^2 2^3]$ has Des $f=\{2,4\}$, inv $f=5$, and comaj $f=4$. Also needed is the *q-multinomial coefficent*:

$$\left[\begin{matrix} n \\ n_1 n_2 \cdots n_l \end{matrix}\right]_q = \frac{(q;q)_n}{(q;q)_{n_1}(q;q)_{n_2}\cdots(q;q)_{n_l}}.$$

An *absorption ring of type* $\vec{n} = (n_1, n_2, \ldots, n_l)$ is constructed by first stretching a chamber having l cells, consecutively numbered from 1 to l, into a ring so that the 1st and lth cells are joined at their outer edges (making escape impossible). Cell i is then partitioned into n_i subcells, thereby expanding its absorption capacity to n_i 1-particles.

The *inversion number process of type* $\vec{n} = (n_1, n_2, \ldots, n_l)$ consists of sequentially propelling $n = (n_1 + n_2 + \cdots + n_l)$ 1-particles from the first empty subcell. For each 1-particle, a coin with probability $q < 1$ of landing tails up is tossed until heads occurs. The active 1-particle advances to the next empty subcell each time tails occurs and comes to rest (is absorbed) when heads appears. For expedience, the Bernoulli sequences of the n 1-particles will be strung together into a single sequence. As a partial illustration, suppose that the two 1-particles are accordingly propelled into an absorption ring of type $\vec{n} = (3, 1, 2)$. If the string TTTHTTTTTTH occurs, then the result is



The outcome of the inversion number process of type $\vec{n} = (n_1, n_2, \ldots, n_l)$ may be partially encoded by a function $f \in \mathscr{M}[1^{n_1} 2^{n_2} \cdots l^{n_l}]$; just set $f(i)$ equal to the number of the cell (containing the subcell) in which the ith 1-particle comes to rest. For instance, if $l = 3$ and $\vec{n} = (2, 1, 2)$, then the function associated with the sequence HTTHTHTTHH is $f = 1\ 3\ 2\ 1\ 3$.

Let mbs f denote the Bernoulli sequence of minimal length that generates f. In general, inv f is equal to the number of tails occurring in mbs f. The following result was established in [16]. As the proof given therein contains a minor flaw, a corrected version is inlcuded below. For permutations ($\vec{n} = (1, 1, \ldots, 1)$), (10) was first obtained by Rawlings and Treadway [20].

THEOREM 3. *For the inversion number process of type* $\vec{n} = (n_1, n_2, \ldots, n_l)$, *let* $F(i)$ *denote the number of the cell in which the ith* 1-*particle is absorbed. For* $0 \le q \le 1$, *the probability of* $f \in \mathscr{M}[1^{n_1}2^{n_2} \cdots l^{n_l}]$ *occurring is*

$$\mathscr{I}_{\vec{n}}\{F = f\} = q^{\operatorname{inv} f} \begin{bmatrix} n \\ n_1\, n_2 \ldots n_l \end{bmatrix}_q^{-1}. \tag{10}$$

Proof. Clearly, (10) is true for $l = 1$. For $l > 1$ and $f \in \mathscr{M}[1^{n_1}2^{n_2} \cdots l^{n_l}]$, the first heads to appear in a Bernoulli sequence that generates f occurs either (i) after the nth toss or (ii) on or before the nth toss. In case (i), the process may be viewed as being restarted on the $(n+1)$st toss. In case (ii), if ν denotes the toss on which the first heads occurs, then

$$n_1 + n_2 + \cdots + n_{f(1)-1} < \nu \le n_1 + n_2 + \cdots + n_{f(1)}. \tag{11}$$

Let $g = f(2)f(3)\cdots f(n)$ and say that $f(1) = i$. We then have

$$\mathscr{I}_{\vec{n}}\{F = f\} = q^n \mathscr{I}_{\vec{n}}\{F = f\} + \sum_{\nu} q^{\nu-1}(1-q)\mathscr{I}_{\vec{n}^*}\{F = g\}, \tag{12}$$

where $\vec{n}^* = (n_1, \ldots, n_i - 1, \ldots, n_l)$ and the sum is over all ν satisfying (11). Since mbs $f =$ TT...TH mbs g, where the sequence immediately preceding mbs g contains $(n_1 + n_2 + \cdots + n_{i-1})$ tails, it is clear that

$$\operatorname{inv} f = \text{ the number of tails in mbs } f = n_1 + n_2 + \cdots + n_{i-1} + \operatorname{inv} g.$$

Theorem 3 then follows from (12) by induction. ∎

The second process considered in this section, referred to as the *comajor index process of type* $\vec{n} = (n_1, n_2, \ldots, n_l)$, proceeds as does the inversion number process with one exception: For $2 \le i \le n$, the ith 1-particle is inserted into the first empty subcell measured from the beginning of the cell in which the $(i-1)$st 1-particle has been absorbed. The proof of (10) may be modified to establish the first half of

THEOREM 4. *For the comajor index process of type* $\vec{n} = (n_1, n_2, \ldots, n_l)$, *let* $F(i)$ *denote the number of the cell in which the ith* 1-*particle is absorbed. For* $0 \le q < 1$, *the probability of* $f \in \mathscr{M}[1^{n_1}2^{n_2} \cdots l^{n_l}]$ *occurring is*

$$\mathscr{C}_{\vec{n}}\{F = f\} = q^{\operatorname{comaj} f} \begin{bmatrix} n \\ n_1\, n_2 \ldots n_l \end{bmatrix}_q^{-1}. \tag{13}$$

Moreover, if des f *denotes the number of descents in* f, *then the probability generating function of* des *relative to the measure in* (13) *is*

$$\mathscr{C}_{\vec{n}}(z) = (z;q)_{n+1} \begin{bmatrix} n \\ n_1\, n_2 \cdots n_l \end{bmatrix}_q^{-1} \sum_{k \ge 0} z^k \prod_{i=1}^{l} \begin{bmatrix} k + n_i \\ n_i \end{bmatrix}_q. \tag{14}$$

A key point for proving (13) and for Section 9 is that comaj f is equal to the number of tails in the shortest Bernoulli sequence that generates f relative to the comajor index process. Formula (14) follows from MacMahon's [14, Vol. 2, p. 211] solution to the q-Simon Newcomb problem. For permutations, (13) was first deduced in other terms by Moritz and Williams [15]; the connection with the comajor index was made by Rawlings and Treadway [20].

As the comajor index process for $\vec{n} = (1, 1, \ldots, 1)$ is clearly equivalent to a game of Russian roulette with n participants (even assuming the winner does not take another turn after the runner-up's death), some macabre information may be extracted from Theorem 4. Let $S_n = \mathscr{M}[1^1 2^1 \ldots n^1]$(i.e., the set of permutations of $\{1, 2, \ldots, n\}$). For $\vec{n} = (1, 1, \ldots, 1)$, (13) is the probability that a given "order of death" occurs; letting $[n] = (1 + q)(1 + q + q^2) \cdots (1 + q + \cdots + q^{n-1})$, we have $\mathscr{C}_{\vec{n}}\{F = \alpha\} = q^{\operatorname{comaj} \alpha}/[n]_q!$ for all $\alpha \in S_n$. Although computationally impractical for large n, the probability of participant k winning at Russian roulette is therefore

$$\operatorname{Prob}\{k \text{ wins}\} = \frac{1}{[n]_q!} \sum_{\alpha} q^{\operatorname{comaj} \alpha},$$

where the sum is over all $\alpha \in S_n$ satisfying $\alpha(n) = k$. From (14), a measure of the courage needed to win may be computed. As des α is equal to the minimum number of times that the comajor index process must "wrap around the ring" in order to generate α, $\mathscr{C}'_{\vec{n}}(1)$ gives the minimum number of times the winner may expect ot pull the trigger. Incidentally, the undertaker's waiting time is just the negative binomial distribution of order $(n - 1)$.

8. EXPERIMENTING WITH THE ABSORPTION RING

The absorption ring provides the context for a range of experimentation. Some possibilities include sending in particles of different sizes, varying the initial placement of particles, and stipulating that each tails advances the active particle by $c > 1$ cells (which may be viewed as a variation on the problem of Josephus). However, experimentation here will be limited to the introduction of 2-particles into an absorption ring.

Some preliminaries are needed. Relative to a sequence $i_1 < i_2 < \cdots < i_k$ of integers, (i_k, i_1) and pairs of the form (i_m, i_{m+1}) are said to be *ring consecutive*. A permutation $\gamma = \gamma(1)\gamma(2) \cdots \gamma(2n) \in S_{2n}$ is said to be *pairwise ring consecutive* if, for $1 \le i \le k$, the pair $(\gamma(2k - 1), \gamma(2k))$ is ring consecutive relative to the sequence obtained by deleting $\gamma(1), \gamma(2), \ldots, \gamma(2k - 2)$ from $1 < 2 < \cdots < 2n$. The set of such permu-

tations is denoted by $\mathscr{PRC}_{2n}$. To illustrate, $\gamma = 2\ 3\ 8\ 1\ 7\ 4\ 5\ 6 \in \mathscr{PRC}_8$. The *pairwise inversion number* of $\gamma \in \mathscr{PRC}_{2n}$, denoted by pinv γ, is defined to be the number of pairs (k, m) with $1 \le k \le n$ and $2k \le m \le 2n$ that satisfy the condition $\max\{\gamma(2k-1), \gamma(2k)\} > \gamma(m)$. Observe that pinv $2\ 3\ 8\ 1\ 7\ 4\ 5\ 6 = 9$.

As the first experiment (Exp Y), let us consider sequentially propelling n 2-particles into a ring of $2n$ cells consecutively numbered from 1 to $2n$. A first 2-particle $\mathscr{P}$ is placed in cells 1 and 2, one dot per cell. A coin with probability of $q < 1$ of landing tails up is tossed until heads occurs. Particle $\mathscr{P}$ advances a cell each time tails occurs and comes to rest when heads appears. Subsequently, the remaining $(n-1)$ 2-particles are propelled around the ring as if any occupied cells had been removed.

Each outcome of Exp Y corresponds in a natural way to a permutation $\gamma \in \mathscr{PRC}_{2n}$: For $1 \le i \le n$, let $\gamma(2i-1)$ and $\gamma(2i)$, respectively, be the cell numbers where the trailing and leading dots of the ith 2-particle come to rest. For instance, if the ring has 8 cells and the sequence HTTTTTHTHH is tossed, then $\gamma = 1\ 2\ 8\ 3\ 5\ 6\ 4\ 7$. By observing that pinv γ is equal to the number of tails in the shortest Bernoulli sequence generating γ, a minor adaptation of the proof of (10) leads to the following modular analog of Cauchy's distribution.

THEOREM 5. *For* Exp Y, *let* $\Gamma(2i-1)$ *and* $\Gamma(2i)$, *respectively, denote the cell number in which the trailing and leading dots of the ith* 2*-particle come to rest. For* $0 \le q \le 1$, *the probability of* $\gamma \in \mathscr{PRC}_{2n}$ *occurring is*

$$Y_{2n}\{\Gamma = \gamma\} = \frac{q^{\operatorname{pinv}\gamma}}{[2]_q[4]_q \cdots [2n]_q}$$

where, for a positive integer i, $[i]_q = 1 + q + \cdots + q^{i-1}$.

As a second experiment (Exp Z), let us repeat Exp Y with one modification: The ith 2-particle for $2 \le i \le n$ is to be inserted in the first two empty cells in front (relative to the orientation of motion) of where the $(i-1)$st 2-particle has been absorbed. As comaj γ equals the number of tails in the shortest Bernoulli sequence generating γ, the proof of the following is routine.

THEOREM 6. *For* Exp z, *let* $\Gamma(2i-1)$ *and* $\Gamma(2i)$, *respectively denote the cell numbers in which the trailing and leading dots of the ith* 2*-particle come to rest. For* $0 \le q < 1$, *the probability of a given* $\gamma \in \mathscr{PRC}_{2n}$ *occurring is*

$$Z_{2n}\{\Gamma = \gamma\} = \frac{q^{\operatorname{comaj}\gamma}}{[2]_q[4]_q \cdots [2n]_q}.$$

9. MORE EXAMPLES OF THE PROBABILISTIC METHOD

As the measures $\mathscr{I}_{\vec{n}}\{F = \cdot\}$ and $\mathscr{C}_{\vec{n}}\{F = \cdot\}$ of Section 7 trivially satisfy $\sum_f \mathscr{I}_{\vec{n}}\{F = f\} = 1 = \sum_f \mathscr{C}_{\vec{n}}\{F = f\}$, Theorems 3 and 4 imply the advertised results of MacMahon [14]; namely,

$$\sum_{f \in \mathscr{M}[1^{n_1}2^{n_2}\cdots l^{n_l}]} q^{\operatorname{inv} f} = \begin{bmatrix} n \\ n_1 n_2 \cdots n_l \end{bmatrix}_q = \sum_{f \in \mathscr{M}[1^{n_1}2^{n_2}\cdots l^{n_l}]} q^{\operatorname{comaj} f}.$$

In the permutation case $\vec{n} = (1, 1, \ldots, 1)$, a bijection $\psi: S_n \to S_n$ satisfying

$$\operatorname{inv} \alpha = \operatorname{comaj} \psi(\alpha) \qquad \text{for all } \alpha \in S_n$$

may be easily extracted from the processes of Section 7. Just define $\psi(\alpha)$ to be the result of (i) first encoding $\alpha \in S_n$ as its Bernoulli sequence of minimal length mbs α under the rules of the inversion number process and (ii) then decoding mbs α according to the comajor index process. For instance,

$$\alpha = 4\,3\,1\,5\,2 \in S_5 \to \text{TTTHTTHHTHH} \to 4\,2\,3\,1\,5 = \psi(\alpha) \in S_5.$$

As may be verified, inv 4 3 1 5 2 = 6 = comaj 4 2 3 1 5. Foata [7] gave the first such bijection for general $\vec{n} = (n_1, n_2, \ldots, n_l)$.

Analogous information may be similarly obtained from the distributions of Section 8. In view of Theorems 5 and 6, we see that

$$\sum_{\gamma \in \mathscr{PRC}_{2n}} q^{\operatorname{pinv} \gamma} = [2]_q [4]_q \cdots [2n]_q = \sum_{\gamma \in \mathscr{PRC}_{2n}} q^{\operatorname{comaj} \gamma}.$$

Also, for $\alpha \in \mathscr{PRC}_{2n}$ with minimal Bernoulli sequence $\mathscr{B}$ relative to Exp Y, let $\Lambda(\alpha)$ be the permutation in $\mathscr{PRC}_{2n}$ having $\mathscr{B}$ as its minimal sequence relative to Exp Z. The map $\Lambda: \mathscr{PRC}_{2n} \to \mathscr{PRC}_{2n}$ is a bijection satisfying

$$\operatorname{pinv} \alpha = \operatorname{comaj} \Lambda(\alpha) \qquad \text{for all } \alpha \in \mathscr{PRC}_{2n}.$$

As an example,

$$\alpha = 3\,4\,2\,5\,8\,1\,6\,7 \to \text{TTHTHTTTHH} \to 3\,4\,6\,7\,5\,8\,1\,2 = \Lambda(\alpha)$$

and pinv 3 4 2 5 8 1 6 7 = 6 = comaj 3 4 6 7 5 8 1 2. Further consideration of statistics on ring consecutive permutations is given in [19].

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