A Generalized Mahonian Statistic on Absorption Ring Mappings

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Based on a coin-tossing scheme, a generalized Mahonian statistic is defined on absorption ring mappings and applied in obtaining combinatorial interpretations of the coefficient of q^j in the expansion of $\prod_{i=1}^{k}(1+q+q^2+\cdots+q^{m_i})$. In the permutation case, the statistic coincides with one studied by Han that specializes many known Mahonian statistics.

1. INTRODUCTION

Let S_n be the symmetric group on $\{1, 2, ..., n\}$. The *inversion number* and the *major index* of a permutation $\sigma = \sigma(1)\,\sigma(2) \cdots \sigma(n) \in S_n$ are defined as

$$\text{inv}\,\sigma = \#\{(i, j)\colon 1 \leqslant i < j \leqslant n, \sigma(i) > \sigma(j)\} \qquad \text{and} \qquad \text{maj}\,\sigma = \sum i,$$

where $\# A$ denotes the cardinality of set A and the sum is over the *descent set* $\{i\colon 1 \leqslant i < n, \sigma(i) > \sigma(i+1)\}$ of σ. It is well known that

$$\sum_{\sigma \in S_n} q^{\text{inv}\,\sigma} = [n]! = \sum_{\sigma \in S_n} q^{\text{maj}\,\sigma} \tag{1}$$

where $[i] = (1 + q + q^2 + \cdots + q^{i-1})$ and $[n]! = [1][2] \cdots [n]$ are the *q-analog* of i and the *q-factorial* of n, respectively. The first equality in (1) is due to Rodriguez [23]. MacMahon [14, 15] obtained a result more general than (1).

A statistic $s: S_n \to \{1, 2, ..., n(n-1)/2\}$ is said to be *Mahonian* if $\sum_{\sigma \in S_n} q^{s(\sigma)} = [n]!$. Besides inv and maj, many new Mahonian statistics have recently been discovered (see Foata and Zeilberger [6], Galovich and White [7], Han [8, 9], Kadell [10], Liang and Wachs [12], Rawlings [17], and Zeilberger and Bressoud [24]).

Using a scheme based on Bernoulli trials, a generalized Mahonian statistic is herein defined on a set of functions called *absorption ring mappings*.

The coin-tossing scheme, dubbed the *absorption ring process*, is as follows. A ring of n cells, one distinguished from the rest, is said to be an *absorption ring* of length n:

				→				
cell labels →	7	5	3	1	8	6	4	2

EXAMPLE 1

For convenience, the ring is layed out as a strip with the distinguished cell at the extreme left. The leftmost and rightmost cells are to be viewed as being attached at their outer edges. Also, in a one-to-one manner, each cell is assigned a label from $\{1, 2, ..., n\}$. Above, the cells are labeled according to the permutation $l = 7\,5\,3\,1\,8\,6\,4\,2 \in S_8$. The arrow in cell 1 indicates the direction in which the ring is to be *traversed*.

A sequence of $j \geqslant 1$ distinct integers is said to be a j-particle. A 0-*particle* is an underlined integer. For instance, 4–6–2 is a 3-particle and $\underline{3}$ is a 0-particle. For integers $j_1, j_2, ..., j_k \geqslant 0$, let $\mathbf{J} = (J_1, J_2, ..., J_k)$ denote a fixed k-tuple of particles where J_v is a j_v-particle and no integer appears in more than one particle. The integers appearing in such tuples will be restricted to the set $\{1, 2, ..., n\}$. Thus, we require that $j_1 + j_2 + \dots + j_k \leqslant n$.

The absorption ring process of *type* $\mathbf{J}$ begins by inserting J_1 into j_1 cells, one integer per cell, according to a *placement rule* Pl. A coin with probability $q < 1$ of landing tails up is then tossed until heads occurs. For each tails, J_1 moves one cell in the direction in which the ring is traversed. The particle J_1 comes to rest (is *absorbed*) when a head occurs. For $2 \leqslant v \leqslant n$, J_v is similarly propelled into the ring as if the cells occupied by integers belonging to $J_1, J_2, ..., J_{v-1}$ had been removed. Underlined integers are viewed as "not occupying space" and cells in which only they appear are not removed from consideration. For instance, suppose $n = 8$, $\mathbf{J} = (4–6–2, 5, \underline{8}, 7–3)$, and that J_v is initially placed in the leftmost j_v empty cells for $1 \leqslant v \leqslant 4$. If the sequences of Bernoulli trials for the four particles are TTTTTTHTHTHTTTTH (written as a single sequence), then the outcome is

		7–	——	–3				
	–2						4–	–6–
			5	$\underline{8}$				
cell labels →	7	5	3	1	8	6	4	2

EXAMPLE 2

Note that 7–3 traces out one *orbit* before coming to rest.

In increasing order, let $i_1, i_2, \ldots, i_m$ be the integers (underlined or not) in $\mathbf{J}$. The outcome may be encoded as a function $f: \{i_1, i_2, \ldots, i_m\} \to \{1, 2, \ldots, n\}$; if i_k is absorbed in cell c_k, then define $f(i_k) = c_k$. Referred to as an *absorption ring mapping*, f will be represented by the list $f(i_1)\, f(i_2) \cdots f(i_m)$ of its range values. The set of such mappings is denoted by $AR_n(\mathbf{J})$. In Example 2, $\{i_1, i_2, \ldots, i_7\} = \{2, 3, 4, 5, 6, 7, \underline{8}\}$ and $f = 7\,1\,4\,3\,2\,5\,1 \in AR_8(4\text{–}6\text{–}2, 5, \underline{8}, 7\text{–}3)$.

The *minimal flipping sequence* of $f \in AR_n(\mathbf{J})$, denoted by mfs f, is defined to be the shortest sequence of coin tosses that generates f. Further, let $|f|$ be the number of tails in mfs f. For $f = 7\,1\,4\,3\,2\,5\,1$ in Example 2, mfs $f = \mathrm{TTTTTTHTHTHH}$ and $|f| = 8$. Theorem 1 is proved in Section 2.

THEOREM 1. *Let $l \in S_n$ be an absorption ring labeling, $\mathbf{J} = (J_1, J_2, \ldots, J_k)$ a k-tuple of particles, and Pl a rule that specifies the initial placement of J_v into the cells left unoccupied by the absorptions of $J_1, J_2, \ldots, J_{v-1}$. The probability of $f \in AR_n(\mathbf{J})$ being generated by the absorption ring process is*

$$M_n(f) = \frac{q^{|f|}}{[n][n-j_1] \cdots [n-j_1-j_2- \cdots -j_{k-1}]}.$$

The main result of this article follows as an immediate corollary: Since M_n is a measure, $\sum_{f \in AR_n(\mathbf{J})} M_n(f) = 1$. Thus, Theorem 1 implies

$$\sum_{f \in AR_n(\mathbf{J})} q^{|f|} = [n][n-j_1] \cdots [n-j_1-j_2- \cdots -j_{k-1}]. \tag{2}$$

Specializations of (2) are shown in Section 3 to agree with the multinomial theorem and with the usual expansion of the product $[n]!$.

When $\mathbf{J}$ is an n-tuple of 1-particles, $AR_n(\mathbf{J}) = S_n$ and (2) reduces to

$$\sum_{\sigma \in S_n} q^{|\sigma|} = [n]!\,.$$

Thus, $|\ |$ is Mahonian for any l and Pl. In this case, $|\ |$ is equivalent to the generalized Mahonian statistic considered by Han [8, p. 41] and as such extends many known Mahonian statistics. In Section 4, the choices of l and Pl are given for which $|\ |$ reduces to the inversion number, major index, r-major index, and Denert's statistic. Also presented in Section 4 is an illustration of how (2) gives "Mahonian" interpretations for "partial" q-factorials such as $[2][4] \cdots [2n]$.

In Section 5, a modification of the absorption ring process is used to obtain MacMahon's aforementioned generalization of (1) on *rearrangements*. The modification, previously discussed in less generality in [19, 21], is achieved by allowing the absorption capacity to vary by cell.

Some remarks are in order. The absorption ring process generalizes a coin-tossing game considered by Moritz and Williams [16]. The consideration of an abstract placement rule was motivated by Knuth's [11, solution to exercise 24 of 5.1.1] generalized shooting order for Russian roulette (which corresponds to Han's [9, p. 41] "future-suite"). Stripped of probabilistic considerations, the absorption ring is equivalent to the cyclic intervals employed by Han [8, 9]. The adjective absorption was coined by Johnson and Kotz [13] in connection with a related process introduced by Blomqvist [2]. Similar to the derivation of (2), variations on Blomqvist's process were exploited in [21, 22] to deduce several classical q-identities in the theory of partitions.

2. PROOF OF THEOREM 1

For $f \in AR_n(\mathbf{J})$, let $|f|_v$ be equal to the number of tails applied to J_v in mfs f. In other terms, $|f|_v$ is the minimum number of tails required in the generation of f for J_v to reach its rest position (determined by f) from its initial placement. Clearly, $|f| = |f|_1 + |f|_2 + \cdots + |f|_k$.

For a given $f \in AR_n(\mathbf{J})$, suppose that $J_1, J_2, ..., J_{v-1}$ have been absorbed in the positions that lead to the generation of f. As the number of unoccupied cells is $(n - j_1 - j_2 - \cdots - j_{v-1})$ and as a particle may sweep through any number of orbits before coming to rest, the probability of J_v being absorbed in the position required for the outcome to be f is

$$\sum_{\mu \geqslant 0} q^{|f|_v + \mu(n - j_1 - j_2 - \cdots - j_{v-1})}(1-q)$$

$$= q^{|f|_v}(1-q) \sum_{\mu \geqslant 0} q^{\mu(n - j_1 - j_2 - \cdots - j_{v-1})}$$

$$= \frac{q^{|f|_v}(1-q)}{1 - q^{n - j_1 - j_2 - \cdots - j_{v-1}}} = \frac{q^{|f|_v}}{[n - j_1 - j_2 - \cdots - j_{v-1}]}.$$

The desired result follows from the independence of Bernoulli trials:

$$M_n(f) = \frac{q^{|f|_1}}{[n]} \frac{q^{|f|_2}}{[n-j_1]} \cdots \frac{q^{|f|_k}}{[n - j_1 - j_2 - \cdots - j_{k-1}]}.$$

3. SOME PRODUCT EXPANSIONS

Let $\langle {n \atop i} \rangle_{\mathbf{J}}$ denote the number of mappings $f \in AR_n(\mathbf{J})$ with $|f| = i$. Formula (2) may then be rewritten as

$$[n][n-j_1]\cdots[n-j_1-j_2-\cdots-j_{k-1}] = \sum_{i \geqslant 0} \left\langle {n \atop i} \right\rangle_{\mathbf{J}} q^i. \tag{3}$$

Note that $\langle {n \atop i} \rangle_{\mathbf{J}} = 0$ for $i > (n-1) + (n-j_1-1) + \cdots + (n-j_1-\cdots-j_{k-1}-1)$.

Three cases of (3) are considered below. The first two show that (3) agrees with the usual combinatorial expansions of $[n]^k$ and of $[n]!$. For each, Pl is taken as the rule that calls for J_v to be initially inserted in the leftmost available j_v cells.

The Multinomial Expansion of $[n]^k$. For $l = 1\ 2 \ldots n \in S_n$ and the k-tuple $\mathbf{J} = (\underline{1}, \underline{2}, \ldots, \underline{k})$ of 0-particles, (3) reduces to

$$[n]^k = \sum_{i=0}^{k(n-1)} \left\langle {n \atop i} \right\rangle_{\mathbf{J}} q^i. \tag{4}$$

The sum in (4) may be regrouped in more familar terms. As 0-particles occupy no space, placement in the leftmost available cell means that each 0-particle is initially put in cell 1. Note that the minimum number of tails required in generating a function that takes on the value μ, $1 \leqslant \mu \leqslant n$, exactly m_μ times is $m_2 + 2m_3 + \cdots + (n-1)m_n$. Thus,

$$\left\langle {n \atop i} \right\rangle_{\mathbf{J}} = \sum_{\substack{m_1+m_2+\cdots+m_n=k \\ m_2+2m_3+\cdots+(n-1)m_n=i}} \binom{k}{m_1\ m_2 \cdots m_n}.$$

Formula (4) may then be rewritten so as to reveal the multinomial expansion of $[n]^k$, namely

$$(1+q+\cdots+q^{n-1})^k = \sum_{m_1+m_2+\cdots+m_n=k} \binom{k}{m_1\, m_2 \cdots m_n} q^{m_2+2m_3+\cdots+(n-1)m_n}.$$

A Combinatorial Expansion of $[n]!$. For $l = n \cdots 2\ 1 \in S_n$ and the k-tuple $\mathbf{J} = (k, \ldots, 2, 1)$ of 1-particles, (3) becomes

$$[n][n-1]\cdots[n-k+1] = \sum_{i=0}^{m} \left\langle {n \atop i} \right\rangle_{\mathbf{J}} q^i \tag{5}$$

where $m=(n-1)+(n-2)+\cdots+(n-k)$. Let $I_{n,i}=\#\{\sigma\in S_n : \operatorname{inv}\sigma=i\}$. From Table I in Section 4, it follows that $I_{n,i}=\langle{n \atop i}\rangle_{\mathbf{J}}$ for $k=n$. Thus, (5) implies

$$[n]!=\sum_{i=0}^{n(n-1)/2} I_{n,i}\, q^i.$$

The above expansion for $[n]!$ and the coefficients $I_{n,i}$ have been considered in some detail (see Comtet [3, p. 236–240] and Moritz and Williams [16]).

An Expansion of a Rogers–Ramanujan-Type Product. For an absorption ring with $(5n-1)$ cells, let $l=1\,2\cdots(5n-1)\in S_{5n-1}$ and take $\mathbf{J}=(1\text{–}2\text{–}3,\ 4\text{–}5,\ 6\text{–}7\text{–}8,\ 9\text{–}10, ..., (5n-4)\text{–}(5n-3)\text{–}(5n-2))$ to be an alternating $(2n-1)$-tuple of particles of sizes 3 and 2. Then (3) implies that

$$[5n-1][5n-4][5n-6][5n-9]\cdots[4][1]=\sum_{i\geqslant 0}\left\langle {5n-1 \atop i}\right\rangle_{\mathbf{J}} q^i.$$

As $[m]=(1-q^m)/(1-q)$, the preceding equality may be rewritten as

$$\begin{aligned}\prod_{i=1}^{n}(1-q^{5i-1})(1-q^{5i-4})&=(1-q)^{2n}\sum_{i\geqslant 0}\left\langle {5n-1 \atop i}\right\rangle_{\mathbf{J}} q^i\\ &=\sum_{i\geqslant 0}\sum_{m=0}^{i}(-1)^m\binom{2n}{m}\left\langle {5n-1 \atop i-m}\right\rangle_{\mathbf{J}} q^i.\end{aligned}$$

4. SOME SPECIALIZATIONS OF | | ON PERMUTATIONS

In the permutation case, many known Mahonian statistics coincide with specializations of | |. Several examples are summarized in Table I. Unless otherwise stated, the ring in this section has n cells, c_v is the label of the

TABLE I

| |	$l\in S_n$	$\mathbf{J}$	$Pl: J_v$ Is Inserted in the First Available Cell Encountered as the Ring Is Traversed Starting with
inv	$n\cdots 2\,1$	$(n, ..., 2, 1)$	cell n
maj	$n\cdots 2\,1$	$(n, ..., 2, 1)$	cell c_{v-1}
ind_r	$n\cdots 2\,1$	$(n, ..., 2, 1)$	cell n if $c_{v-1}+r-1\geqslant n$ and cell $c_{v-1}+r-1$ otherwise
comaj	$1\,2\cdots n$	$(1, 2, ..., n)$	cell c_{v-1}
den	$n\cdots 2\,1$	$(n, ..., 2, 1)$	cell $n+1-v$

cell in which J_v is absorbed, and each placement rule calls for J_1 to be inserted in the leftmost available cell. Also, the passage of the process from the rightmost to the leftmost empty cell is referred to as an *orbit*.

The r-Major Index. For $r \geqslant 1$, the *r-major index* of $\sigma \in S_n$ is defined in [17] to be

$$\text{ind}_r\, \sigma = \#\{(i, j)\colon 1 \leqslant i < j \leqslant n, \sigma(i) > \sigma(j) > \sigma(i) - r\} + \textstyle\sum i,$$

where the sum is over the set $\{i : 1 \leqslant i < n, \sigma(i) \geqslant \sigma(i+1) + r\}$ of *r-descents* of σ. For instance, $\text{ind}_3\ 1\,7\,2\,6\,3\,5\,4 = 5 + (2+4) = 11$. On S_n, note that $\text{ind}_1 = \text{maj}$ and $\text{ind}_n = \text{inv}$.

Pick $l = n\ \cdots\ 2\,1 \in S_n$, let $\mathbf{J} = (n, \cdots, 2, 1)$, and suppose that Pl calls for $J_v = n + 1 - v$ to be placed in the first empty cell encountered as the ring is traversed from cell n if $c_{v-1} + r - 1 \geqslant n$ and from cell $c_{v-1} + r - 1$ otherwise. Roughly speaking, the process may fall to the left up to $r - 1$ cells each time a head occurs. The outcome for $n = 7$, $r = 3$, and the sequence TTTHTHTTHTTTHTHTHH is

	2	4	6	7	5	3	1 .
cell labels →	7	6	5	4	3	2	1

EXAMPLE 3

The mapping generated is a permutation $\sigma \in S_n = AR_n(\mathbf{J})$: $\sigma(i) = c$ where c is the cell in which i comes to rest. In Example 3, $\sigma = 1\,7\,2\,6\,3\,5\,4 \in S_7$. Note that $|\sigma| = 11 = \text{ind}_3\, \sigma$.

For $\sigma \in S_n$, let rmfs σ denote the minimal flipping sequence of σ written in reverse. The reason that $|\,| = \text{ind}_r$ may be seen by comparing σ with rmfs σ. Relative to Example 3, consider

$$\begin{array}{lllllllllll} \text{rmfs}\,\sigma = \text{H} & \text{H} & | & \text{T} & \text{HT} & \text{HT}_4 & | & \text{TT} & \text{HTT} & \text{HT}_6 & \text{HT}_7\text{T}_7\text{T}_6 \\ \quad\ \sigma = 1 & 7 & {}_3{>} & & 2 & 6 & {}_3{>} & & 3 & 5 & 4 \end{array} \tag{6}$$

where ${}_3{>}$'s highlight 3-descents in σ and each bar indicates the completion of an orbit. In general, there is a one-to-one correspondence between r-descents in σ and orbits generated by mfs σ. For $1 \leqslant j \leqslant n$, let

$$I_r(j) = \#\{i : i < j \leqslant n, \sigma(i) > \sigma(j) > \sigma(i) - r\}\ .$$

Note that $I_r(j)$ is equal to the number of empty cells that the process falls left when the head that generates $\sigma(j)$ is tossed. In (6), $I_3(7) = 2$ and the process falls left from cell 4 to cell 6. Also, the contribution of the two tails subscripted by 7 in (6) made towards generating the first orbit are "negated". For $1 \leqslant j \leqslant n$, subscript any tail by j that is negated by a fall associated with the head that generates $\sigma(j)$. The number of subscripted T's in rmfs σ is equal to the first term in the definition of $\mathrm{ind}_r \sigma$. Furthermore, note that the contribution to $\mathrm{ind}_r \sigma$ made by the ith r-descent counted from right to left in σ is equal to the number of nonsubscripted tails between the $(i-1)$st and ith bars counted from right to left in rmfs σ. It follows that $|\sigma| = \mathrm{ind}_r \sigma$.

In the case $r=1$, an equivalent process was considered by Moritz and Williams [16]. The connection of their process with the Mahonian statistic known as the comajor index was made by Rawlings and Treadway [20].

Denert's Statistic. For $\sigma \in S_n$, Denert's statistic [4], denoted by den σ and as defined by Foata and Zeilberger [6], is the number of ordered pairs (i, j), $1 \leqslant i < j \leqslant n$, satisfying

$$\text{(a)} \qquad \sigma(i) \leqslant j \text{ or } \sigma(i) > \sigma(j) \qquad \text{if} \quad \sigma(j) > j \quad \text{or}$$

$$\text{(b)} \qquad \sigma(j) < \sigma(i) \leqslant j \qquad \text{if} \quad \sigma(j) \leqslant j.$$

An index j such that $\sigma(j) > j$ is known as an exceedance. The permutation $\sigma = 7\,3\,2\,6\,1\,5\,4 \in S_7$ has three exceedances $(j = 1, 2, 4)$ and den $\sigma = 11$.

The statistic $|\;|$ is equal to den when $l = n \cdots 2\,1 \in S_n$, $\mathbf{J} = (n, ..., 2, 1)$, and *Pl* calls for $J_v = n+1-v$ to be placed in the first empty cell encountered in traversing the ring from cell $n+1-v$ (*i.e.*, the vth cell from the left). For $n=7$, the sequence TTTHTHTTHTTTHTHTHH generates

	1	4	6	7	2	3	5
cell labels →	7	6	5	4	3	2	1

EXAMPLE 4

The associated permutation $\sigma = 7\,3\,2\,6\,1\,5\,4 \in S_7$ satisfies $|\sigma| = 11 = \text{den}\,\sigma$.

Towards proving that $|\;| = \text{den}$, compare σ of Example 4 with rmfs σ:

$$\begin{array}{lcccccccc} \text{rmfs}\,\sigma = \text{H} & | & \text{HT}_2 & | & \text{HT} \;\; \text{HT}_4 & | & \text{T}_4\text{T}_4 \;\; \text{HTT} & \text{HT} & \text{HTTT} \\ \sigma = \hat{7} & & \hat{3} & & 2 \quad \hat{6} & & 1 & 5 & 4 \end{array} \tag{7}$$

where exceedances are marked by hats and bars indicate orbits. There is a one-to-one correspondence between exceedances and orbits generated by mfs σ. For an exceedance j, subscript by j each T occuring in the string of consecutive tails in rmfs σ to the right of the head that generates $\sigma(j)$. The number of subscripted T's in rmfs σ is equal to the number of i's satisfying

(a) in the definition of den. In (7), the three T_4's coincide to the contribution made by the exceedance $j=4$ to den via part (a): $\sigma(1)=7>6=\sigma(4)$, $\sigma(2)=3\leqslant 4$, and $\sigma(3)=2\leqslant 4$. Similarly, the number of consecutive plain T's in rmfs σ on the right of the head that generates a nonexceedance j is equal to the number of i's satisfying (b). Thus, $|\sigma|=\text{den}\,\sigma$. This argument is equivalent to one used by Foata and Zeilberger [6].

Remark 1. For a ring of n cells and a fixed sequence $\mathbf{J}=(J_1, J_2, ..., J_n)$ of 1-particles, there are n distinct placement rules: For $1\leqslant v\leqslant n$, J_v may be inserted into any of the $n+1-v$ cells that remain empty. For a fixed $\mathbf{J}$ and a fixed labeling $l\in S_n$, the $n!$ placement rules induce $n!$ distinct Mahonian statistics on S_n.

Remark 2. The placement rules for the inversion number, major index, and Denert's statistic possess a certain natural geometry: From Table I, the *Pl*'s of these three statistics respectively call for $\mathbf{J}_v$ to be placed in the leftmost empty cell, continue from where $\mathbf{J}_{v-1}$ stopped, and to be inserted in the first empty cell on or above the "diagonal" (vth cell from the left). Another example of a geometrically motivated *Pl* is the "reflection" placement rule that calls for

(a) $\mathbf{J}_1$ to be inserted in the leftmost cell and,

(b) if $\mathbf{J}_{v-1}$ stops in the μth leftmost cell, then $\mathbf{J}_v$ is to be placed in the first empty cell encountered starting from the μ^{th} rightmost cell.

For *Pl* as above, $l=n\cdots 2\ 1\in S_n$, and $\mathbf{J}=(n, ..., 2, 1)$, let $\text{refl}\,\sigma=|\sigma|$. To illustrate, note that mfs 5 2 1 3 4 = THTTTHTHTHH. Thus, refl 5 2 1 3 4 = 6.

Remark 3. The absorption ring process also provides interpretations (in terms of permutations) for "partial" q-factorials such as $[2][4]\cdots[2n]$. For instance, for a ring with $2n$ cells, one possibility is to take $\mathbf{J}$ to be the sequence $(2n\text{-}(2n-1), ..., 4\text{–}3, 2\text{–}1)$ of n 2-particles, the cell labeling as $l=2n\cdots 2\ 1\in S_{2n}$, and *Pl* as the rule that calls for J_1 to be placed in the leftmost two cells and J_v to be placed in the first two empty cells encountered in traversing the ring from where J_{v-1} was absorbed. Then $AR_{2n}(\mathbf{J})$ is a subset of S_{2n}, $|\sigma|=\text{maj}\,\sigma$ for all $\sigma\in AR_{2n}(\mathbf{J})$, and we have

$$\sum_{\sigma\in AR_{2n}(\mathbf{J})} q^{\text{maj}\,\sigma}=[2][4]\cdots[2n].$$

By taking $\mathbf{J}$ to be an alternating sequence of 2-particles and 3-particles (n of each), "Mahonian" interpretations for $[3][5][8][10]\cdots[5n-2][5n]$ may similarly be obtained on subsets of S_{5n}.

5. THE STATISTIC | | ON REARRANGEMENTS

Let $\mathbf{n}=(n_1, n_2, ..., n_k)$ be a sequence of non-negative integers and put $n=n_1+n_2+\cdots+n_k$. The set of mappings from $\{1, 2, ..., n\}$ to $\{1, 2, ..., k\}$ that take on the value i exactly n_i times will be denoted by $R_{\mathbf{n}}$. Often, $R_{\mathbf{n}}$ is characterized as the set of *rearrangements* of the nondecreasing sequence containing n_i i's for $1\leqslant i\leqslant k$. Note that $R_{\mathbf{n}}=S_n$ when $n_i=1$ for $1\leqslant i\leqslant k$.

The statistics maj and inv are defined on $f\in R_{\mathbf{n}}$ by

$$\operatorname{inv} f=\#\{(i,j)\colon 1\leqslant i<j\leqslant n, f(i)>f(j)\} \qquad \text{and} \quad \operatorname{maj} f=\sum i$$

where the sum is over the *descent set* $\{i: 1\leqslant i<n, f(i)>f(i+1)\}$ of f. The more general version of (1) obtained by MacMahon [14, 15] is

$$\sum_{f\in R_{\mathbf{n}}} q^{\operatorname{inv} f}=\begin{bmatrix} n \\ n_1 n_2 \cdots n_k \end{bmatrix}=\sum_{f\in R_{\mathbf{n}}} q^{\operatorname{maj} f}$$

where the middle expression denotes the *q-multinomial coefficient* defined by

$$\begin{bmatrix} n_1 \\ n_1 n_2 \cdots n_k \end{bmatrix}=\frac{[n]!}{[n_1]!\,[n_2]!\cdots[n_k]!}.$$

A statistic s on $R_{\mathbf{n}}$ with distribution given by the q-multinomial coefficient is said to be *Mahonian*.

The absorption ring process may be slightly modified so that | | coincides with Han's [9, p. 42] generalized Mahonian statistic on $R_{\mathbf{n}}$. The modification consists essentially of varying the absorption capacity by cell: In an absorption ring with k cells labeled by $l\in S_k$, cell i will be allowed to absorb n_i 1-particles. To obtain a tractable result, the particles in $\mathbf{J}$ will be limited to size 1. Furthermore, for the process to traverse cell i, a consecutive run of tails must occur equal in length to n_i minus the number of 1-particles previously absorbed in cell i. For example, suppose there are $n=3$ cells labeled by $l=1\ 2\ 3\in S_3$ with respective capacities $n_1=2=n_2$ and $n_3=1$. If $\mathbf{J}=(1, 2, 3, 4, 5)$ and Pl specifies the initial placement $\mathbf{J}_v$ to be in the leftmost available (not filled to capacity) cell, then the sequence THTTTHTHTTHH generates the outcome

cell capacities →	2	2	1
	1, 4	3, 5	2
cell labels →	1	2	3

EXAMPLE 5

The generated function is $f = 1\ 3\ 2\ 1\ 2 \in R_{(2,2,1)}$. As mfs $f = \text{HTTTHTHHH}$, $|f| = 4$. The following result holds.

THEOREM 2. *Let $l \in S_k$ be a labeling of an absorption ring with k cells and suppose that cell i has absorption capacity $n_i \geqslant 0$ for $1 \leqslant i \leqslant k$. Further, let $n = (n_1 + n_2 + \cdots + n_k)$, $\mathbf{J} = (J_1, J_2, ..., J_n)$ be an n-tuple of 1-particles, and Pl be a rule that specifies the initial placement of J_v into a cell not completely occupied by the absorptions of $J_1, J_2, ..., J_{v-1}$. The probability of $f \in R_{\mathbf{n}}$ being generated by the absorption ring process is*

$$M_n(f) = q^{|f|} \begin{bmatrix} n \\ n_1\, n_2 \cdots n_k \end{bmatrix}^{-1}.$$

Proof. Suppose the generation of f requires that J_v come to rest in c_v. As in the proof of Theorem 1, let $|f|_v$ denote the minimum number of tails needed in generating f for J_v to reach c_v. As it takes n_1 tails to traverse c_1, the probability of J_1 being absorbed in cell c_1 is

$$\sum_{\mu \geqslant 0} q^{|f|_1 + \mu n}(1 + q + \cdots + q^{n_1 - 1})(1 - q) = \frac{q^{|f|_1}[n_1]}{(1 - q^n)/(1 - q)} = \frac{q^{|f|_1}[n_1]}{[n]}.$$

The result follows from the independence of Bernoulli trials and induction.

As a corollary, we have that $|\ |$ is Mahonian for any l, Pl, and $\mathbf{J}$ an n-tuple of 1-particles: Since $\sum_{f \in R_{\mathbf{n}}} M_n(f) = 1$, Theorem 2 implies that

$$\sum_{f \in R_{\mathbf{n}}} q^{|f|} = \begin{bmatrix} n \\ n_1 n_2 \cdots n_k \end{bmatrix}.$$

The choices of l, $\mathbf{J}$, and Pl for which $|\ |$ reduces to inv, maj, ind_r, and den on $R_{\mathbf{n}}$ (see [18, 9] for definitions of ind_r and den on $R_{\mathbf{n}}$) are displayed in Table II. As before, c_v is the label of the cell in which J_v comes to rest and each Pl calls for the placement of J_1 in the leftmost cell.

TABLE II

$\|\ \|$	$l \in S_n$	$\mathbf{J}$	Pl: J_v Is Inserted in the First Available Cell Encountered as the Ring Is Traversed Starting with
inv	$k \cdots 2\ 1$	$(n, ..., 2, 1)$	cell k
maj	$k \cdots 2\ 1$	$(n, ..., 2, 1)$	cell c_{v-1}
ind_r	$k \cdots 2\ 1$	$(n, ..., 2, 1)$	cell k if $c_{v-1} + r - 1 \geqslant k$ and cell $c_{v-1} + r - 1$ otherwise
den	$k \cdots 2\ 1$	$(n, ..., 2, 1)$	cell $k + 1 - \mu$ where μ satisfies $\sum_{i=0}^{\mu-2} n_{k-i} < v \leqslant \sum_{i=0}^{\mu-1} n_{k-i}$

As a final example, the absorption ring process with the appropriate triple $(l, \mathbf{J}, Pl)$ from Table II is used to compute den f for $f = 2\ 1\ 3\ 2\ 3\ 1\ 4\ 3\ 1 \in R_{(3, 2, 3, 1)}$. The associated ring is

cell capacities →	1	3	2	3
cell labels →	4	3	2	1

As $\mathbf{J} = (9, ..., 2, 1)$ and $0 < 1 \leqslant n_4 = 1$, $J_1 = 9$ is inserted in cell 4. As J_1 is absorbed in cell 1 and as $1 = n_4 < 2, 3, 4 \leqslant n_4 + n_3 = 4$, it follows that $J_2 = 8$, $J_3 = 7$, and $J_4 = 6$ are initially placed in cell 3. The outcome corresponding to $f = 2\ 1\ 3\ 2\ 3\ 1\ 4\ 3\ 1$ is

cell capacities →	1	3	2	3
	7	3, 5, 8	1, 4	2, 6, 9
cell labels →	4	3	2	1

EXAMPLE 6

Since mfs $f = $ TTTTTTHHTTTTTTTHTTTTHTTTHHTHHH, we have that den $f = |f| = 20$.

6. CONCLUDING REMARK

For $f \in AR_n(\mathbf{J})$, let orb f be the minimum number of orbits needed to generate f. In the permutation case, orb is comparable to Han's [9, p. 41] generalized exceedance number and to Knuth's [11, exercise 24 of 5.1.1] generalized descent number. The joint distribution of (orb, $|\ |$) on $AR_n(\mathbf{J})$ is considered in [1].

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