

Adjacencies in Words

Jean-Marc Fedou

*Laboratoire Bordelais de Recherches en Informatique,
Université Bordeaux I, 33405 Talence, France*

and

Don Rawlings

*Mathematics Department, California Polytechnic State University,
San Luis Obispo, California 93407*

Based on two inversion formulas for enumerating words in the free monoid by adjacencies, we present a new approach to a class of permutation problems having Eulerian-type generating functions. We also show that a specialization of one of the inversion formulas gives Diekert's lifting to the free monoid of an inversion theorem due to Cartier and Foata.

1. INTRODUCTION

There are a number of powerful theories of inversion [9, 10, 13, 16] for dealing with combinatorial objects having generating functions of *Eulerian-type*

$$\frac{1}{1 + \sum_{n \geq 1} (-1)^n (1-t)^{n-1} c_n z^n}.$$

Using two such inversion formulas, we present new derivations of Stanley's [13] generating functions for generalized q -Eulerian and q -Euler polynomials on r -tuples of permutations. We further indicate how one of the inversion formulas gives Diekert's [5] lifting to the free monoid of an

inversion theorem of Cartier and Foata [4]. The inversion theorems we use enumerate words in the free monoid by adjacencies.

An *alphabet* X is a non-empty set whose elements are referred to as *letters*. A finite sequence (possibly empty) $w = x_1 x_2 \cdots x_n$ of n letters is said to be a *word* of *length* n . The empty word will be denoted 1. The set of all words formed with letters in X along with the concatenation product is known as the *free monoid* generated by X and is denoted by X^* . We let X^+ be the set of words having positive length.

From X , we construct the *adjacency alphabet* $A = \{a_{xy} : (x, y) \in X \times X\}$. The *adjacency monomial* and the *sieve polynomial* for $w = x_1 x_2 \cdots x_n \in X^*$ of length $n \geq 2$ are defined respectively as $a(w) = a_{x_1 x_2} a_{x_2 x_3} \cdots a_{x_{n-1} x_n}$ and $\bar{a}(w) = (a_{x_1 x_2} - 1)(a_{x_2 x_3} - 1) \cdots (a_{x_{n-1} x_n} - 1)$. For $0 \leq n \leq 1$, we set $a(w) = \bar{a}(w) = 1$. In $Z[A] \ll X \gg$, the algebra of formal series of words in X^* with coefficients from the commutative ring of polynomials in A having integer coefficients, the following inversion formulas hold:

THEOREM 1. *According to adjacencies, the words in X^* are generated by*

$$\sum_{w \in X^*} a(w)w = \left(1 - \sum_{w \in X^*} \bar{a}(w)w\right)^{-1}. \quad (1)$$

THEOREM 2. *For non-empty subsets $U, V \subseteq X$, the words according to adjacencies in $U^*V = \{uv : u \in U^*, v \in V\}$ are generated by*

$$\sum_{w \in U^*V} a(w)w = \left(1 - \sum_{w \in U^*} \bar{a}(w)w\right)^{-1} \left(\sum_{w \in U^*V} \bar{a}(w)w\right). \quad (2)$$

Theorem 1 may be deduced from Stanley's [14, p. 266] synthesis of an inversion formula on clusters due to Goulden and Jackson [10, p. 131] with a related result of Zeilberger's [16] that enumerates words by mistakes. Theorem 2 bears comparison to (but is not equivalent to either) Viennot's [15] formula that counts heaps of pieces with restricted maximal elements and with a theorem of Goulden and Jackson [10, p. 238] for strings with distinguished final string. Proofs of Theorems 1 and 2 are deferred to Section 6. In passing, we mention that Hutchinson and Wilf [11] have given a closed formula for counting words by adjacencies.

The applications we give rely on the fact that setting $a_{xy} = 1$ eliminates all words containing xy as a factor from the right-hand sides of (1) and (2).

For instance, suppose that $X = \{x, y, z\}$. Set $a_{xx} = a$, $a_{xy} = b$, and the remaining $a_{ij} = 1$. Theorem 1 yields

$$\begin{aligned} & \sum_{w \in (x, y, z)^*} a(w)w \\ &= \frac{1}{1 - y - z - \sum_{n \geq 1} (a - 1)^{n-1} x^n - \sum_{n \geq 1} (a - 1)^{n-1} (b - 1) x^n y} \\ &= (1 + x - ax)(1 - ax - y - z + (a - b)xy + (a - 1)xz)^{-1}. \end{aligned}$$

2. A KEY BIJECTION

In applying Theorems 1 and 2 to the enumeration of permutations, we make repeated use of a bijection that associates a pair (σ, λ) , where σ is a permutation and λ is a partition, to a finite sequence w of non-negative integers. Let $N = \{0, 1, 2, \dots\}$ and N^n be the set of words of length n in N^* . The *rise set*, *rise number*, *inversion number*, and *norm* of $w = i_1 i_2 \cdots i_n \in N^n$ are respectively defined to be

$$\begin{aligned} \text{Ris } w &= \{k : 1 \leq k < n, i_k \leq i_{k+1}\}, & \text{ris } w &= |\text{Ris } w|, \\ \text{inv } w &= |\{(k, m) : 1 \leq k < m \leq n, i_k > i_m\}|, & \|w\| &= i_1 + \cdots + i_n. \end{aligned}$$

The set of non-decreasing words in N^n (i.e., partitions with at most n parts) will be denoted by P_n . A permutation σ in the symmetric group S_n on $\{1, 2, \dots, n\}$ will be viewed as the word $\sigma(1)\sigma(2)\cdots\sigma(n)$. The key bijection used in Sections 3 and 4 may be described as follows.

LEMMA 1. *For $n \geq 1$, there exists a bijection $f_n: S_n \times P_n \rightarrow N^n$ such that $\text{Ris } \sigma = \text{Ris } w$ and $\text{inv } \sigma + \|\lambda\| = \|w\|$ whenever $f_n(\sigma, \lambda) = w$.*

Proof. First, for $\sigma \in S_n$ and $1 \leq k \leq n$, let c_k be the cardinality of the set $\{j : k + 1 \leq j \leq n, \sigma(k) > \sigma(j)\}$. The number c_k counts the inversions in σ due to $\sigma(k)$. The word $c = c_1 c_2 \cdots c_n$ is known as the *Lehmer code* [12] of σ . Note that $\text{inv } \sigma = c_1 + \cdots + c_n = \|c\|$ and that $\text{Ris } \sigma = \text{Ris } c$. As an illustration, the Lehmer code of $\sigma = 51342 \in S_5$ is $c = 40110$. Also, $\text{inv } \sigma = 6 = \|c\|$ and $\text{Ris } \sigma = \{2, 3\} = \text{Ris } c$.

Next, for $(\sigma, \lambda) = (\sigma(1)\sigma(2)\cdots\sigma(n), \lambda_1 \lambda_2 \cdots \lambda_n) \in S_n \times P_n$, define $f_n(\sigma, \lambda)$ to be the word $w = i_1 i_2 \cdots i_n \in N^n$, where $i_k = c_k + \lambda_{\sigma(k)}$ for $1 \leq k \leq n$. When $f_n(\sigma, \lambda) = w$, we clearly have the properties

$$\begin{aligned} k \in \text{Ris } \sigma & \text{ iff } c_k + \lambda_{\sigma(k)} \leq c_{k+1} + \lambda_{\sigma(k+1)} \text{ iff } k \in \text{Ris } w, \\ \text{inv } \sigma + \|\lambda\| &= c_1 + \cdots + c_n + \lambda_1 + \cdots + \lambda_n = \|w\|. \end{aligned}$$

For example, the map f_5 sends the pair $(\sigma, \lambda) = (51342, 11112) \in S_5 \times P_5$ to the word $w = 61221 \in N^5$. Note that $\text{Ris } \sigma = \{2, 3\} = \text{Ris } w$ and that $\text{inv } \sigma + \|\lambda\| = 6 + 6 = \|w\|$.

The inverse of f_n may be realized by applying the *insertion-shift* bijection presented in [6] to the word w to obtain (σ^{-1}, λ) . The description of f_n given above was suggested by Foata (personal communication).

3. q -EULERIAN POLYNOMIALS

As the first application of Theorem 1, we derive a generating function for the sequence

$$A_n(t, q) = \sum_{\sigma \in S_n} t^{\text{ris } \sigma} q^{\text{inv } \sigma}.$$

The polynomial $A_n(t, 1)$ is the n th *Eulerian polynomial*. We further obtain the generating function for Stanley's [13] generalized q -Eulerian polynomials on r -tuples of permutations.

The first step in obtaining a generating function for the distribution of (ris, inv) on S_n is to appropriately define the adjacency monomial and sieve polynomial for the alphabet N . Toward this end, we set $a_{ij} = t$ if $i \leq j$ and $a_{ij} = 1$ otherwise. For $w = i_1 i_2 \cdots i_n$, note that $a(w) = t^{\text{ris } w}$ and that

$$\bar{a}(w) = \begin{cases} (t-1)^{n-1} & \text{if } i_1 \leq i_2 \leq \cdots \leq i_n \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 1 reduces to

$$\sum_{w \in N^*} t^{\text{ris } w} w = \frac{1}{1 - \sum_{n \geq 1} (t-1)^{n-1} \sum_{0 \leq i_1 \leq i_2 \leq \cdots \leq i_n} i_1 i_2 \cdots i_n}. \quad (3)$$

Next, we assign the weight $W(i) = zq^i$ to each $i \in N$ and extend W to a multiplicative homomorphism on N^* . Let $(q; q)_0 = 1$ and, for $n \geq 1$, set $(q; q)_n = (1-q)(1-q^2) \cdots (1-q^n)$. Then, Lemma 1 and (3) justify the calculation

$$\begin{aligned} \sum_{n \geq 0} \frac{A_n(t, q) z^n}{(q; q)_n} &= \sum_{n \geq 0} z^n \sum_{(\sigma, \lambda) \in S_n \times P_n} t^{\text{ris } \sigma} q^{\text{inv } \sigma + \|\lambda\|} = \sum_{w \in N^*} t^{\text{ris } w} W(w) \\ &= \frac{1}{1 - \sum_{n \geq 1} (t-1)^{n-1} z^n \sum_{0 \leq i_1 \leq \cdots \leq i_n} q^{i_1 + \cdots + i_n}} \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{1 - \sum_{n \geq 1} (t-1)^{n-1} z^n / (q; q)_n} \\
 &= \frac{1-t}{e(-z(1-t), q) - t}, \tag{4}
 \end{aligned}$$

where $e(z, q) = \sum_{n \geq 0} z^n / (q; q)_n$ is a well-known q -analog of e^z .

The *common rise number* of an r -tuple $(\sigma_1, \sigma_2, \dots, \sigma_r)$ of permutations in $S'_n = S_n \times \dots \times S_n$ is defined to be $\text{cris}(\sigma_1, \sigma_2, \dots, \sigma_r) = |\cap_{j=1}^r \text{Ris } \sigma_j|$. The argument in (4) is readily adapted to deriving Stanley's [13] generating function for the polynomials

$$A_{n,r}(t, q_1, q_2, \dots, q_r) = \sum_{(\sigma_1, \sigma_2, \dots, \sigma_r) \in S'_n} t^{\text{cris}(\sigma_1, \sigma_2, \dots, \sigma_r)} q_1^{\text{inv } \sigma_1} q_2^{\text{inv } \sigma_2} \dots q_r^{\text{inv } \sigma_r}. \tag{5}$$

We sketch the details for $r = 2$ and then state the general result.

For letters $\mathbf{i} = (i_1, i_2)$ and $\mathbf{j} = (j_1, j_2)$ in the alphabet $N \times N$, we define

$$a_{\mathbf{i}\mathbf{j}} = \begin{cases} t & \text{if } i_1 \leq i_2 \text{ and } j_1 \leq j_2 \\ 1 & \text{otherwise.} \end{cases}$$

For $(v, w) = (i_1 i_2 \dots i_n, j_1 j_2 \dots j_n) \in (N \times N)^*$, we have $a(v, w) = t^{\text{cris}(v, w)}$, where $\text{cris}(v, w) = |\text{Ris } v \cap \text{Ris } w|$. Also,

$$\bar{a}(v, w) = \begin{cases} (t-1)^{n-1} & \text{if } i_1 \leq i_2 \leq \dots \leq i_n \text{ and } j_1 \leq j_2 \leq \dots \leq j_n \\ 0 & \text{otherwise.} \end{cases}$$

The map of Lemma 1 applied component-wise to $(S_n \times P_n) \times (S_n \times P_n)$,

$$f_n \times f_n(\sigma_1, \lambda; \sigma_2, \mu) = (f_n(\sigma_1, \lambda), f_n(\sigma_2, \mu)) = (v, w),$$

is a bijection to $N^n \times N^n$ with $\text{cris}(\sigma_1, \sigma_2) = \text{cris}(v, w)$, $\text{inv } \sigma_1 + \|\lambda\| = \|v\|$, and $\text{inv } \sigma_2 + \|\mu\| = \|w\|$. Repeating (4) with appropriate modifications gives

$$\sum_{n \geq 0} \frac{A_{n,2}(t, q_1, q_2) z^n}{(q_1; q_1)_n (q_2; q_2)_n} = \frac{1-t}{J(z(1-t), q_1, q_2) - t},$$

where $J(z, q_1, q_2) = \sum_{n \geq 0} (-1)^n z^n / (q_1; q_1)_n (q_2; q_2)_n$ is a bibasic Bessel function. We note that replacing z by $z(1-q_1)(1-q_2)$ and letting $q_1, q_2 \rightarrow 1^-$ give the original result of Carlitz, Scoville, and Vaughan [3] that initiated the study of statistics on r -tuples of permutations.

If we let $\mathbf{q} = (q_1, q_2, \dots, q_r)$ and $(\mathbf{q}; \mathbf{q})_{n,r} = (q_1; q_1)_n (q_2; q_2)_n \cdots (q_r; q_r)_n$, it follows in general that

THEOREM 3 (Stanley). *For $r \geq 1$, the sequence $\{A_{n,r}(t, \mathbf{q})\}_{n \geq 0}$ is generated by*

$$\sum_{n \geq 0} \frac{A_{n,r}(t, \mathbf{q}) z^n}{(\mathbf{q}; \mathbf{q})_{n,r}} = \frac{1-t}{F_r(z(1-t), \mathbf{q}) - t},$$

where $F_r(z, \mathbf{q}) = \sum_{n \geq 0} (-1)^n z^n / (\mathbf{q}; \mathbf{q})_{n,r}$.

Further consideration of statistics on r -tuples of permutations is given in [7, 8]. In [7], we extend the technique of Carlitz *et al.* [3] and present recurrence relationships that refine Theorem 3. We also discuss several related distributions. In [8], we obtain a stronger version of Theorem 3 by using Theorems 1 and 2 in combination with a map that carries more information than does the bijection of Lemma 1.

4. q -EULER POLYNOMIALS

André [1] shows that if E_n is the number of up-down alternating permutations in S_n (that is, $\sigma \in S_n$ such that $\sigma(1) < \sigma(2) > \sigma(3) < \sigma(4) > \cdots$), then

$$\sum_{n \geq 0} \frac{E_n z^n}{n!} = \frac{1 + \sin z}{\cos z}. \quad (6)$$

The number E_n is known as the n th *Euler number*.

We now apply Theorems 1 and 2 to the more general problem of counting the set of *odd-up permutations*

$$\mathcal{O}_n = \{\sigma \in S_n : \sigma(1) < \sigma(2), \sigma(3) < \sigma(4), \dots\}$$

by inversion number and by the *number of even indexed rises*

$$\text{ris}_2 \sigma = |\{k \in \text{Ris } \sigma : k \text{ is even}\}|.$$

Toward this end, let

$$E_n(t, q) = \sum_{\sigma \in \mathcal{O}_n} t^{\text{ris}_2 \sigma} q^{\text{inv } \sigma}.$$

Note that $E_n(0, 1) = E_n$. The analysis is split into two cases: n odd and n even. We only present the odd case, which requires use of Theorem 2.

Let $U = \{\mathbf{i} = i_1 i_2 : i_1, i_2 \in N \text{ with } i_1 \leq i_2\}$, $V = N$, and X be the union of U and V . For $\mathbf{i} = i_1 i_2$, $\mathbf{j} = j_1 j_2 \in U$, and $k \in V$, we set

$$a_{\mathbf{ij}} = \begin{cases} t & \text{if } i_2 \leq j_1 \\ 1 & \text{otherwise} \end{cases} \quad \text{and} \quad a_{ik} = \begin{cases} t & \text{if } i_2 \leq k \\ 1 & \text{otherwise.} \end{cases}$$

Viewing a word $w \in U^*V$ as being in N^* , let $\text{ris}_2 w$ denote the number of rises in w having even index. Theorem 2 implies that

$$\sum_{w \in U^*V} t^{\text{ris}_2 w} = \frac{\sum_{m \geq 0} (t-1)^m \sum_{0 \leq i_1 \leq i_2 \leq \dots \leq i_{2m+1}} i_1 i_2 \dots i_{2m+1}}{1 - \sum_{m \geq 1} (t-1)^{m-1} \sum_{0 \leq i_1 \leq i_2 \leq \dots \leq i_{2m}} i_1 i_2 \dots i_{2m}}. \quad (7)$$

Again set $W(i) = zq^i$ for $i \in N$ and multiplicatively extend W to N^* . Let $U^m V = \{uv : u \in U^* \text{ is of length } m, v \in V\}$. From Lemma 1, the bijection $f_{2m+1} : \mathcal{C}_{2m+1} \times P_{2m+1} \rightarrow U^m V$ satisfies the properties $\text{ris}_2 \sigma = \text{ris}_2 w$ and $\text{inv } \sigma + \|\lambda\| = \|w\|$ whenever $f_{2m+1}(\sigma, \lambda) = w$. It then follows from (7) that

$$\begin{aligned} & \sum_{m \geq 0} \frac{E_{2m+1}(t, q) z^{2m+1}}{(q; q)_{2m+1}} \\ &= \sum_{m \geq 0} z^{2m+1} \sum_{(\sigma, \lambda) \in \mathcal{C}_{2m+1} \times P_{2m+1}} t^{\text{ris}_2 \sigma} q^{\text{inv } \sigma + \|\lambda\|} \\ &= \sum_{w \in U^*V} t^{\text{ris}_2 w} W(w) \\ &= \frac{\sum_{m \geq 0} (t-1)^m z^{2m+1} \sum_{0 \leq i_1 \leq \dots \leq i_{2m+1}} q^{i_1 + \dots + i_{2m+1}}}{1 - \sum_{m \geq 1} (t-1)^{m-1} z^{2m} \sum_{0 \leq i_1 \leq \dots \leq i_{2m}} q^{i_1 + \dots + i_{2m}}} \\ &= \frac{\sum_{m \geq 0} (t-1)^m z^{2m+1} / (q; q)_{2m+1}}{1 - \sum_{m \geq 1} (t-1)^{m-1} z^{2m} / (q; q)_{2m}} \\ &= \frac{(1-t)^{1/2} \sin_q(z\sqrt{1-t})}{\cos_q(z\sqrt{1-t}) - t}, \end{aligned} \quad (8)$$

where $\cos_q z = \sum_{n \geq 0} (-1)^n z^{2n} / (q; q)_{2n}$ and $\sin_q z = \sum_{n \geq 0} (-1)^n z^{2n+1} / (q; q)_{2n+1}$. As the even case is essentially contained in the analysis above, we have

$$\sum_{n \geq 0} \frac{E_n(t, q) z^n}{(q; q)_n} = \frac{(1-t) \left(1 + (1-t)^{-1/2} \sin_q(z\sqrt{1-t}) \right)}{\cos_q(z\sqrt{1-t}) - t}.$$

Setting $t = 0$, replacing z by $z(1 - q)$, and letting $q \rightarrow 1^-$ give (6).

Generalization to r -tuples of m -permutations is relatively straightforward. Let $S_{n,m}$ denote the set of $\sigma \in S_n$ satisfying the property that $\sigma(k) > \sigma(k + 1)$ implies k is a multiple of m . Note that $S_{n,2} = \mathcal{O}_n$. For $(\sigma_1, \sigma_2, \dots, \sigma_r) \in S_{n,m}^r$, define $\text{cris}_m(\sigma_1, \sigma_2, \dots, \sigma_r)$ to be the number of $k \in \bigcap_{j=1}^r \text{Ris } \sigma_j$ such that k is a multiple of m . Combining the ideas behind Theorem 3 and (8) gives

THEOREM 4. *For $m, r \geq 1$, the sequence of polynomials*

$$E_{n,m,r}(t, \mathbf{q}) = \sum_{(\sigma_1, \sigma_2, \dots, \sigma_r) \in S_{n,m}^r} t^{\text{cris}_m(\sigma_1, \sigma_2, \dots, \sigma_r)} q_1^{\text{inv } \sigma_1} q_2^{\text{inv } \sigma_2} \dots q_r^{\text{inv } \sigma_r}$$

is generated by

$$\sum_{n \geq 0} \frac{E_{n,m,r}(t, \mathbf{q}) z^n}{(\mathbf{q}; \mathbf{q})_{n,r}} = \frac{(1-t) \left(1 + \sum_{\rho=1}^{m-1} (1-t)^{-\rho/m} \Phi_{m,\rho,r}(z \sqrt[m]{1-t}, \mathbf{q}) \right)}{\Phi_{m,0,r}(z \sqrt[m]{1-t}, \mathbf{q}) - t},$$

where $\Phi_{m,\rho,r}(z, \mathbf{q}) = \sum_{\nu \geq 0} (-1)^\nu z^{\nu m + \rho} / (\mathbf{q}; \mathbf{q})_{\nu m + \rho, r}$.

Theorem 4 is essentially due to Stanley [13]. Note that $E_{n,1,r}(t, \mathbf{q})$ is equal to the generalized q -Eulerian polynomial defined in (5). Thus, taking $m = 1$ in Theorem 4 gives Theorem 3 as a corollary. We further remark that $\Phi_{m,\rho,1}(z, q)$ is a q -Olivier function. When $r = 1$ and $t = s = 0$, replacing z by $z(1 - q)$ and letting $q \rightarrow 1^-$ give the initial result of Carlitz [2] on m -permutations.

5. FROM THE TRACE TO THE FREE MONOID

As the final application, we use Theorem 1 to obtain Diekert's [5, pp. 96–99] lifting to the free monoid of an inversion formula due to Cartier and Foata [4] from a partially commutative monoid (or trace monoid) in which the defining binary relation admits a transitive orientation.

Let θ be an irreflexive symmetric binary relation on X . Define \equiv_θ to be the binary relation (induced by θ) on X^* consisting of the set of pairs (w, v) of words such that there is a sequence $w = w_0, w_1, \dots, w_m = v$, where each w_i is obtained by transposing a pair of letters in w_{i-1} that are consecutive and contained in θ . For instance, if $X = \{x, y, z\}$ and $\theta = \{(x, y), (y, x)\}$, then the sequence $zyyx, zyxy, zxyy$ implies that $zyyx \equiv_\theta zxyy$.

Clearly, \equiv_{θ} is an equivalence relation on X^* . The quotient of X^* by \equiv_{θ} gives the *partially commutative monoid* induced by θ and is denoted by $M(X, \theta)$. The equivalence class \hat{w} of $w \in X^*$ is referred to as the *trace* of w .

A word $w = x_1 x_2 \cdots x_n \in X^*$ is said to be a *basic monomial* if $x_i \theta x_j$ for all $i \neq j$. A trace \hat{w} is said to be *θ -trivial* if any one of its representatives is a basic monomial. If one lets $\mathcal{T}^+(X, \theta)$ be the set of θ -trivial traces, the inversion formula of Cartier and Foata reads as follows.

THEOREM 5 (Cartier and Foata). *For θ an irreflexive symmetric binary relation on X , the traces in $M(X, \theta)$ are generated by*

$$\sum_{\hat{w} \in M(X, \theta)} \hat{w} = \frac{1}{1 + \sum_{\hat{t} \in \mathcal{T}^+(X, \theta)} (-1)^{l(\hat{t})} \hat{t}},$$

where $l(\hat{t})$ denotes the length of any representative of \hat{t} .

A natural question to ask is whether \hat{w} and \hat{t} can be replaced by some canonical representatives so that Theorem 5 remains true as a formula in the free monoid X^* . As resolved by Diekert [5], such canonical representatives exist if and only if θ admits a transitive orientation.

To be precise, a subset $\vec{\theta}$ of θ is said to be an *orientation* of θ if θ is a disjoint union of $\vec{\theta}$ and $\{(x, y) : (y, x) \in \vec{\theta}\}$. The set of $t = t_1 t_2 \cdots t_n \in X^*$ satisfying $t_i \vec{\theta} t_{i+1}$ is denoted by $T^+(X, \vec{\theta})$. Note that $T^+(X, \vec{\theta})$ is a set of representatives for the θ -trivial traces $\mathcal{T}^+(X, \theta)$ whenever $\vec{\theta}$ is transitive. A word $w = x_1 x_2 \cdots x_n \in X^*$ is said to have a $\vec{\theta}$ -adjacency in position k if $x_k \vec{\theta} x_{k+1}$. We denote the number of $\vec{\theta}$ -adjacencies of w by $\vec{\theta}\text{adj} w$. Although Diekert did not explicitly introduce the notion of a $\vec{\theta}$ -adjacency, his lifting theorem may be paraphrased as follows.

THEOREM 6 (Diekert). *Let θ be an irreflexive symmetric binary relation on X and let $\vec{\theta}$ be an orientation of θ . Then, θ is transitive if and only if there exists a complete set W of representatives for the traces of $M(X, \theta)$ such that*

$$\sum_{w \in W} w = \frac{1}{1 + \sum_{t \in T^+(X, \vec{\theta})} (-1)^{l(t)} t}.$$

Moreover, $W = \{w \in X^* : \vec{\theta}\text{adj} w = 0\}$.

To see how Theorem 1 intervenes in the matter, suppose that $\vec{\theta}$ is an orientation of θ (not necessarily transitive for now). If for $x, y \in X$ we set $a_{xy} = a$ when $x \vec{\theta} y$ and $a_{xy} = 1$ otherwise, then Theorem 1 reduces to

$$\sum_{w \in X^*} a^{\vec{\theta}\text{adj} w} w = \frac{1}{1 + \sum_{t \in T^+(X, \vec{\theta})} (-1)^n (1-a)^{l(t)-1} t}. \quad (9)$$

When $\vec{\theta}$ is transitive, setting $a = 0$ in (9) gives the lifting of Theorem 5 to the free monoid as stated in Diekert's theorem. We close this section with two examples.

TRANSITIVE EXAMPLE. Let $X = \{x, y, z\}$ with $\theta = \{(x, y), (y, x), (x, z), (z, x)\}$. Among other possibilities, $\vec{\theta} = \{(y, x), (z, x)\}$ is a transitive orientation of θ . The $\vec{\theta}$ -adjacencies of a word correspond to factors yx and zx . Note that $T^+(X, \vec{\theta}) = \{x, y, z, yx, zx\}$ is a complete set of representatives for the θ -trivial traces $\mathcal{T}^+(X, \theta)$. Also, the only word in

$$\widehat{xzyxy} = \left\{ \begin{array}{ccccc} xzxyy, & xzyxy, & xzyyx, & xxzyy, & zxyyy, \\ zxyxy, & zxyyx, & zyxyx, & zyxyx, & zyxzx \end{array} \right\}$$

having no $\vec{\theta}$ -adjacencies is $xxzyy$. From (9), we have

$$\sum_{w \in \{x, y, z\}^*} a^{\vec{\theta} \text{adj} w} w = \frac{1}{1 - (x + y + z) + (1 - a)(yx + zx)}.$$

Setting $a = 0$ gives an identity that can be viewed as having been lifted from the trace monoid as in Theorem 6.

NON-TRANSITIVE EXAMPLE. Let X and θ be as in the previous example. The orientation $\vec{\theta} = \{(y, x), (x, z)\}$ is not transitive. Observe that the word yxz in $T^+(X, \vec{\theta}) = \{x, y, z, yx, xz, yxz\}$ is not a θ -trivial trace. Also, $\widehat{yxz} = \{yxz, xyz, yzx\}$ contains two words having no $\vec{\theta}$ -adjacencies. Nevertheless, (9) implies

$$\sum_{w \in \{x, y, z\}^*} a^{\vec{\theta} \text{adj} w} w = \frac{1}{1 - (x + y + z) + (1 - a)(yx + xz) - (1 - a)^2 yxz}.$$

6. PROOFS FOR THEOREMS 1 AND 2

To establish Theorem 1, we begin by noting that (1) is equivalent to

$$\sum_{w \in X^*} a(w)w - \sum_{w \in X^*} \left(\sum_{w=uv, v \neq 1} a(u)\bar{a}(v) \right) w = 1.$$

Thus, by equating coefficients, it suffices to show that

$$a(w) = \sum_{w=uv, v \neq 1} a(u)\bar{a}(v) \tag{10}$$

for all $w \in X^+$. We proceed by induction on the length $l(w)$ of w . For $l(w) = 1$, (10) is trivially true. Suppose $l(w) \geq 2$. Then w factorizes as $w = w_1xy$, where $w_1 \in X^*$ and $x, y \in X$. Assuming (10) holds for words of length smaller than w , it follows that

$$\begin{aligned}
 a(w) &= a_{xy}a(w_1x) = a(w_1x) + (a_{xy} - 1)a(w_1x) \\
 &= a(w_1x)\bar{a}(y) + \bar{a}(xy) \sum_{w_1x=uv_1x} a(u)\bar{a}(v_1x) \\
 &= a(w_1x)\bar{a}(y) + \sum_{w_1xy=uv_1xy} a(u)\bar{a}(v_1xy) \\
 &= \sum_{w=uv, v \neq 1} a(u)\bar{a}(v),
 \end{aligned}$$

and the proof is complete.

We use an alternate approach to prove Theorem 2. Let W denote the left-hand side of (2) and define

$$W_{n+1} = \sum_w a(w)w,$$

where the sum is over words $w = x_1x_2 \cdots x_{n+1} \in U^*V$ of length $(n+1)$. Note that $W = \sum_{n \geq 0} W_{n+1}$. Since $\bar{a}(x_1) = 1$ and $\bar{a}(x_1x_2) = a_{x_1x_2} - 1$, it is a triviality that

$$W_{n+1} = \sum_w \bar{a}(x_1)a(x_2, \dots, x_{n+1})w + \sum_w \bar{a}(x_1x_2)a(x_2 \cdots x_{n+1})w$$

for $n \geq 1$. Similarly, the second sum on the above right may be split as

$$\sum_w \bar{a}(x_1x_2)a(x_3 \cdots x_{n+1})w + \sum_w \bar{a}(x_1x_2x_3)a(x_3 \cdots x_{n+1})w$$

so that

$$\begin{aligned}
 W_{n+1} &= \sum_{k=1}^2 \sum_w \bar{a}(x_1 \cdots x_k)a(x_{k+1} \cdots x_{n+1})w \\
 &\quad + \sum_w \bar{a}(x_1x_2x_3)a(x_3 \cdots x_{n+1})w.
 \end{aligned}$$

Iterating the above argument and then factoring give

$$\begin{aligned} W_{n+1} &= \sum_{k=1}^n \sum_w \bar{a}(x_1, \dots, x_k) a(x_{k+1} \cdots x_{n+1}) w + \sum_w \bar{a}(w) w \\ &= \sum_{k=1}^n \left(\sum_u \bar{a}(u) u \right) W_{n+1-k} + \sum_w \bar{a}(w) w, \end{aligned}$$

where the sum to the immediate left of W_{n+1-k} is over words $u = x_1 \cdots x_k \in U^+$ of length k . As the above recurrence relationship for W_{n+1} is valid for $n \geq 0$, it follows that

$$W = \left(\sum_{w \in U^+} \bar{a}(w) w \right) W + \sum_{w \in U^*V} \bar{a}(w) w,$$

which implies Theorem 2.

Either of the preceding arguments may be easily modified to give an inversion formula for words in X^* that end in a fixed word v . Without giving the details, we have

THEOREM 7. *According to adjacencies, words ending in a word $v = b_1 b_2 \cdots b_m \in X^*$ of length m are generated by*

$$\sum_{w \in X^*} a(wv) wv = \left(1 - \sum_{w \in X^+} \bar{a}(w) w \right)^{-1} \left(a(v) \sum_{w \in X^*} \bar{a}(wb_1) wv \right).$$

REFERENCES

1. D. André, Mémoire sur les permutations alternées, *J. Math.* **7** (1881), 167–184.
2. L. Carlitz, Generating functions for a special class of permutations, *Proc. Amer. Math. Soc.* **47** (1975), 251–256.
3. L. Carlitz, R. Scoville, and T. Vaughan, Enumeration of pairs of permutations, *Discrete Math.* **14** (1976), 215–239.
4. P. Cartier and D. Foata, “Problèmes combinatoires de commutation et de réarrangements,” *Lecture Notes in Mathematics*, Vol. 85, Springer-Verlag, Berlin/New York, 1969.
5. V. Diekert, “Combinatorics on Traces,” *Lecture Notes in Computer Science*, Vol. 454, Springer-Verlag, Berlin/New York, 1990.
6. J. M. Fedou, Fonctions de Bessel, empilements et tresses, in “Séries formelle et combinatoire algébrique” (P. Leroux and C. Reutenauer, Eds.), pp. 189–202, Publ. du LaCIM, Vol. 11, Université à Montréal, Québec, 1992.
7. J. M. Fedou and D. P. Rawlings, Statistics on pairs of permutations, *Discrete Math.*, to appear.
8. J. M. Fedou and D. P. Rawlings, More Statistics on Permutation Pairs, *Electron. J. Combin.* **1** (1994), R11.
9. I. M. Gessel, “Generating Functions and Enumeration of Sequences,” *Doctoral thesis*, Massachusetts Institute of Technology, Cambridge, MA, 1977.

10. I. P. Goulden and D. M. Jackson, "Combinatorial Enumeration," Wiley, New York, 1983.
11. J. P. Hutchinson and H. S. Wilf, On Eulerian circuits and words with prescribed adjacency patterns, *J. Combin. Theory Ser. A* **18** (1975), 80–87.
12. D. E. Knuth, "The Art of Computer Programming," Vol. 3, Addison-Wesley, Reading, MA, 1974.
13. R. P. Stanley, Binomial posets, Möbius inversion, and permutation enumeration, *J. Combin. Theory Ser. A* **20** (1976), 336–356.
14. R. P. Stanley, "Enumerative Combinatorics," Vol. I, Wadsworth & Brooks, Belmont, CA, 1986.
15. X. G. Viennot, Heaps of pieces. I. Basic definitions and combinatorial lemmas, in "Combinatoire énumérative" (G. Labelle and P. Leroux, Eds.), Lecture Notes in Mathematics, Vol. 1234, Springer-Verlag, Berlin, 1986.
16. D. Zeilberger, Enumeration of words by their number of mistakes, *Discrete Math.* **34** (1981), 89–92.