

Adjacencies in Words

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Based on two inversion formulas for enumerating words in the free monoid by adjacencies, we present a new approach to a class of permutation problems having Eulerian-type generating functions. We also show that a specialization of one of the inversion formulas gives Diekert's lifting to the free monoid of an inversion theorem due to Cartier and Foata.

1. INTRODUCTION

There are a number of powerful theories of inversion [9, 10, 13, 16] for dealing with combinatorial objects having generating functions of *Eulerian-type*

$$\frac{1}{1 + \sum_{n \geq 1} (-1)^n (1-t)^{n-1} c_n z^n}.$$

Using two such inversion formulas, we present new derivations of Stanley's [13] generating functions for generalized q -Eulerian and q -Euler polynomials on r -tuples of permutations. We further indicate how one of the inversion formulas gives Diekert's [5] lifting to the free monoid of an

inversion theorem of Cartier and Foata [4]. The inversion theorems we use enumerate words in the free monoid by adjacencies.

An *alphabet* X is a non-empty set whose elements are referred to as *letters*. A finite sequence (possibly empty) $w = x_1 x_2 \cdots x_n$ of n letters is said to be a *word* of *length* n . The empty word will be denoted 1. The set of all words formed with letters in X along with the concatenation product is known as the *free monoid* generated by X and is denoted by X^* . We let X^+ be the set of words having positive length.

From X , we construct the *adjacency alphabet* $A = \{a_{xy} : (x, y) \in X \times X\}$. The *adjacency monomial* and the *sieve polynomial* for $w = x_1 x_2 \cdots x_n \in X^*$ of length $n \geq 2$ are defined respectively as $a(w) = a_{x_1 x_2} a_{x_2 x_3} \cdots a_{x_{n-1} x_n}$ and $\bar{a}(w) = (a_{x_1 x_2} - 1)(a_{x_2 x_3} - 1) \cdots (a_{x_{n-1} x_n} - 1)$. For $0 \leq n \leq 1$, we set $a(w) = \bar{a}(w) = 1$. In $Z[A] \ll X \gg$, the algebra of formal series of words in X^* with coefficients from the commutative ring of polynomials in A having integer coefficients, the following inversion formulas hold:

THEOREM 1. *According to adjacencies, the words in X^* are generated by*

$$\sum_{w \in X^*} a(w)w = \left(1 - \sum_{w \in X^*} \bar{a}(w)w\right)^{-1}. \quad (1)$$

THEOREM 2. *For non-empty subsets $U, V \subseteq X$, the words according to adjacencies in $U^*V = \{uv : u \in U^*, v \in V\}$ are generated by*

$$\sum_{w \in U^*V} a(w)w = \left(1 - \sum_{w \in U^*} \bar{a}(w)w\right)^{-1} \left(\sum_{w \in U^*V} \bar{a}(w)w\right). \quad (2)$$

Theorem 1 may be deduced from Stanley's [14, p. 266] synthesis of an inversion formula on clusters due to Goulden and Jackson [10, p. 131] with a related result of Zeilberger's [16] that enumerates words by mistakes. Theorem 2 bears comparison to (but is not equivalent to either) Viennot's [15] formula that counts heaps of pieces with restricted maximal elements and with a theorem of Goulden and Jackson [10, p. 238] for strings with distinguished final string. Proofs of Theorems 1 and 2 are deferred to Section 6. In passing, we mention that Hutchinson and Wilf [11] have given a closed formula for counting words by adjacencies.

The applications we give rely on the fact that setting $a_{xy} = 1$ eliminates all words containing xy as a factor from the right-hand sides of (1) and (2).

For instance, suppose that $X = \{x, y, z\}$. Set $a_{xx} = a$, $a_{xy} = b$, and the remaining $a_{ij} = 1$. Theorem 1 yields

$$\begin{aligned} & \sum_{w \in (x, y, z)^*} a(w)w \\ &= \frac{1}{1 - y - z - \sum_{n \geq 1} (a-1)^{n-1} x^n - \sum_{n \geq 1} (a-1)^{n-1} (b-1) x^n y} \\ &= (1 + x - ax)(1 - ax - y - z + (a-b)xy + (a-1)xz)^{-1}. \end{aligned}$$

2. A KEY BIJECTION

In applying Theorems 1 and 2 to the enumeration of permutations, we make repeated use of a bijection that associates a pair (σ, λ) , where σ is a permutation and λ is a partition, to a finite sequence w of non-negative integers. Let $N = \{0, 1, 2, \dots\}$ and N^n be the set of words of length n in N^* . The *rise set*, *rise number*, *inversion number*, and *norm* of $w = i_1 i_2 \cdots i_n \in N^n$ are respectively defined to be

$$\begin{aligned} \text{Ris } w &= \{k : 1 \leq k < n, i_k \leq i_{k+1}\}, & \text{ris } w &= |\text{Ris } w|, \\ \text{inv } w &= |\{(k, m) : 1 \leq k < m \leq n, i_k > i_m\}|, & \|w\| &= i_1 + \cdots + i_n. \end{aligned}$$

The set of non-decreasing words in N^n (i.e., partitions with at most n parts) will be denoted by P_n . A permutation σ in the symmetric group S_n on $\{1, 2, \dots, n\}$ will be viewed as the word $\sigma(1)\sigma(2)\cdots\sigma(n)$. The key bijection used in Sections 3 and 4 may be described as follows.

LEMMA 1. *For $n \geq 1$, there exists a bijection $f_n: S_n \times P_n \rightarrow N^n$ such that $\text{Ris } \sigma = \text{Ris } w$ and $\text{inv } \sigma + \|\lambda\| = \|w\|$ whenever $f_n(\sigma, \lambda) = w$.*

Proof. First, for $\sigma \in S_n$ and $1 \leq k \leq n$, let c_k be the cardinality of the set $\{j : k+1 \leq j \leq n, \sigma(k) > \sigma(j)\}$. The number c_k counts the inversions in σ due to $\sigma(k)$. The word $c = c_1 c_2 \cdots c_n$ is known as the *Lehmer code* [12] of σ . Note that $\text{inv } \sigma = c_1 + \cdots + c_n = \|c\|$ and that $\text{Ris } \sigma = \text{Ris } c$. As an illustration, the Lehmer code of $\sigma = 51342 \in S_5$ is $c = 40110$. Also, $\text{inv } \sigma = 6 = \|c\|$ and $\text{Ris } \sigma = \{2, 3\} = \text{Ris } c$.

Next, for $(\sigma, \lambda) = (\sigma(1)\sigma(2)\cdots\sigma(n), \lambda_1 \lambda_2 \cdots \lambda_n) \in S_n \times P_n$, define $f_n(\sigma, \lambda)$ to be the word $w = i_1 i_2 \cdots i_n \in N^n$, where $i_k = c_k + \lambda_{\sigma(k)}$ for $1 \leq k \leq n$. When $f_n(\sigma, \lambda) = w$, we clearly have the properties

$$\begin{aligned} k \in \text{Ris } \sigma & \text{ iff } c_k + \lambda_{\sigma(k)} \leq c_{k+1} + \lambda_{\sigma(k+1)} \text{ iff } k \in \text{Ris } w, \\ \text{inv } \sigma + \|\lambda\| &= c_1 + \cdots + c_n + \lambda_1 + \cdots + \lambda_n = \|w\|. \end{aligned}$$

For example, the map f_5 sends the pair $(\sigma, \lambda) = (51342, 11112) \in S_5 \times P_5$ to the word $w = 61221 \in N^5$. Note that $\text{Ris } \sigma = \{2, 3\} = \text{Ris } w$ and that $\text{inv } \sigma + \|\lambda\| = 6 + 6 = \|w\|$.

The inverse of f_n may be realized by applying the *insertion-shift* bijection presented in [6] to the word w to obtain (σ^{-1}, λ) . The description of f_n given above was suggested by Foata (personal communication).

3. q -EULERIAN POLYNOMIALS

As the first application of Theorem 1, we derive a generating function for the sequence

$$A_n(t, q) = \sum_{\sigma \in S_n} t^{\text{ris } \sigma} q^{\text{inv } \sigma}.$$

The polynomial $A_n(t, 1)$ is the n th *Eulerian polynomial*. We further obtain the generating function for Stanley's [13] generalized q -Eulerian polynomials on r -tuples of permutations.

The first step in obtaining a generating function for the distribution of (ris, inv) on S_n is to appropriately define the adjacency monomial and sieve polynomial for the alphabet N . Toward this end, we set $a_{ij} = t$ if $i \leq j$ and $a_{ij} = 1$ otherwise. For $w = i_1 i_2 \cdots i_n$, note that $a(w) = t^{\text{ris } w}$ and that

$$\bar{a}(w) = \begin{cases} (t-1)^{n-1} & \text{if } i_1 \leq i_2 \leq \cdots \leq i_n \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 1 reduces to

$$\sum_{w \in N^*} t^{\text{ris } w} w = \frac{1}{1 - \sum_{n \geq 1} (t-1)^{n-1} \sum_{0 \leq i_1 \leq i_2 \leq \cdots \leq i_n} i_1 i_2 \cdots i_n}. \quad (3)$$

Next, we assign the weight $W(i) = zq^i$ to each $i \in N$ and extend W to a multiplicative homomorphism on N^* . Let $(q; q)_0 = 1$ and, for $n \geq 1$, set $(q; q)_n = (1-q)(1-q^2) \cdots (1-q^n)$. Then, Lemma 1 and (3) justify the calculation

$$\begin{aligned} \sum_{n \geq 0} \frac{A_n(t, q) z^n}{(q; q)_n} &= \sum_{n \geq 0} z^n \sum_{(\sigma, \lambda) \in S_n \times P_n} t^{\text{ris } \sigma} q^{\text{inv } \sigma + \|\lambda\|} = \sum_{w \in N^*} t^{\text{ris } w} W(w) \\ &= \frac{1}{1 - \sum_{n \geq 1} (t-1)^{n-1} z^n \sum_{0 \leq i_1 \leq \cdots \leq i_n} q^{i_1 + \cdots + i_n}} \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{1 - \sum_{n \geq 1} (t-1)^{n-1} z^n / (q; q)_n} \\
 &= \frac{1-t}{e(-z(1-t), q) - t}, \tag{4}
 \end{aligned}$$

where $e(z, q) = \sum_{n \geq 0} z^n / (q; q)_n$ is a well-known q -analog of e^z .

The *common rise number* of an r -tuple $(\sigma_1, \sigma_2, \dots, \sigma_r)$ of permutations in $S'_n = S_n \times \dots \times S_n$ is defined to be $\text{cris}(\sigma_1, \sigma_2, \dots, \sigma_r) = |\cap_{j=1}^r \text{Ris } \sigma_j|$. The argument in (4) is readily adapted to deriving Stanley's [13] generating function for the polynomials

$$A_{n,r}(t, q_1, q_2, \dots, q_r) = \sum_{(\sigma_1, \sigma_2, \dots, \sigma_r) \in S'_n} t^{\text{cris}(\sigma_1, \sigma_2, \dots, \sigma_r)} q_1^{\text{inv } \sigma_1} q_2^{\text{inv } \sigma_2} \dots q_r^{\text{inv } \sigma_r}. \tag{5}$$

We sketch the details for $r = 2$ and then state the general result.

For letters $\mathbf{i} = (i_1, i_2)$ and $\mathbf{j} = (j_1, j_2)$ in the alphabet $N \times N$, we define

$$a_{\mathbf{i}\mathbf{j}} = \begin{cases} t & \text{if } i_1 \leq i_2 \text{ and } j_1 \leq j_2 \\ 1 & \text{otherwise.} \end{cases}$$

For $(v, w) = (i_1 i_2 \dots i_n, j_1 j_2 \dots j_n) \in (N \times N)^*$, we have $a(v, w) = t^{\text{cris}(v, w)}$, where $\text{cris}(v, w) = |\text{Ris } v \cap \text{Ris } w|$. Also,

$$\bar{a}(v, w) = \begin{cases} (t-1)^{n-1} & \text{if } i_1 \leq i_2 \leq \dots \leq i_n \text{ and } j_1 \leq j_2 \leq \dots \leq j_n \\ 0 & \text{otherwise.} \end{cases}$$

The map of Lemma 1 applied component-wise to $(S_n \times P_n) \times (S_n \times P_n)$,

$$f_n \times f_n(\sigma_1, \lambda; \sigma_2, \mu) = (f_n(\sigma_1, \lambda), f_n(\sigma_2, \mu)) = (v, w),$$

is a bijection to $N^n \times N^n$ with $\text{cris}(\sigma_1, \sigma_2) = \text{cris}(v, w)$, $\text{inv } \sigma_1 + \|\lambda\| = \|v\|$, and $\text{inv } \sigma_2 + \|\mu\| = \|w\|$. Repeating (4) with appropriate modifications gives

$$\sum_{n \geq 0} \frac{A_{n,2}(t, q_1, q_2) z^n}{(q_1; q_1)_n (q_2; q_2)_n} = \frac{1-t}{J(z(1-t), q_1, q_2) - t},$$

where $J(z, q_1, q_2) = \sum_{n \geq 0} (-1)^n z^n / (q_1; q_1)_n (q_2; q_2)_n$ is a bibasic Bessel function. We note that replacing z by $z(1-q_1)(1-q_2)$ and letting $q_1, q_2 \rightarrow 1^-$ give the original result of Carlitz, Scoville, and Vaughan [3] that initiated the study of statistics on r -tuples of permutations.

If we let $\mathbf{q} = (q_1, q_2, \dots, q_r)$ and $(\mathbf{q}; \mathbf{q})_{n,r} = (q_1; q_1)_n (q_2; q_2)_n \cdots (q_r; q_r)_n$, it follows in general that

THEOREM 3 (Stanley). *For $r \geq 1$, the sequence $\{A_{n,r}(t, \mathbf{q})\}_{n \geq 0}$ is generated by*

$$\sum_{n \geq 0} \frac{A_{n,r}(t, \mathbf{q}) z^n}{(\mathbf{q}; \mathbf{q})_{n,r}} = \frac{1-t}{F_r(z(1-t), \mathbf{q}) - t},$$

where $F_r(z, \mathbf{q}) = \sum_{n \geq 0} (-1)^n z^n / (\mathbf{q}; \mathbf{q})_{n,r}$.

Further consideration of statistics on r -tuples of permutations is given in [7, 8]. In [7], we extend the technique of Carlitz *et al.* [3] and present recurrence relationships that refine Theorem 3. We also discuss several related distributions. In [8], we obtain a stronger version of Theorem 3 by using Theorems 1 and 2 in combination with a map that carries more information than does the bijection of Lemma 1.

4. q -EULER POLYNOMIALS

André [1] shows that if E_n is the number of up-down alternating permutations in S_n (that is, $\sigma \in S_n$ such that $\sigma(1) < \sigma(2) > \sigma(3) < \sigma(4) > \cdots$), then

$$\sum_{n \geq 0} \frac{E_n z^n}{n!} = \frac{1 + \sin z}{\cos z}. \quad (6)$$

The number E_n is known as the n th *Euler number*.

We now apply Theorems 1 and 2 to the more general problem of counting the set of *odd-up permutations*

$$\mathcal{O}_n = \{\sigma \in S_n : \sigma(1) < \sigma(2), \sigma(3) < \sigma(4), \dots\}$$

by inversion number and by the *number of even indexed rises*

$$\text{ris}_2 \sigma = |\{k \in \text{Ris } \sigma : k \text{ is even}\}|.$$

Toward this end, let

$$E_n(t, q) = \sum_{\sigma \in \mathcal{O}_n} t^{\text{ris}_2 \sigma} q^{\text{inv } \sigma}.$$

Note that $E_n(0, 1) = E_n$. The analysis is split into two cases: n odd and n even. We only present the odd case, which requires use of Theorem 2.

Let $U = \{\mathbf{i} = i_1 i_2 : i_1, i_2 \in N \text{ with } i_1 \leq i_2\}$, $V = N$, and X be the union of U and V . For $\mathbf{i} = i_1 i_2$, $\mathbf{j} = j_1 j_2 \in U$, and $k \in V$, we set

$$a_{\mathbf{ij}} = \begin{cases} t & \text{if } i_2 \leq j_1 \\ 1 & \text{otherwise} \end{cases} \quad \text{and} \quad a_{ik} = \begin{cases} t & \text{if } i_2 \leq k \\ 1 & \text{otherwise.} \end{cases}$$

Viewing a word $w \in U^*V$ as being in N^* , let $\text{ris}_2 w$ denote the number of rises in w having even index. Theorem 2 implies that

$$\sum_{w \in U^*V} t^{\text{ris}_2 w} = \frac{\sum_{m \geq 0} (t-1)^m \sum_{0 \leq i_1 \leq i_2 \leq \dots \leq i_{2m+1}} i_1 i_2 \dots i_{2m+1}}{1 - \sum_{m \geq 1} (t-1)^{m-1} \sum_{0 \leq i_1 \leq i_2 \leq \dots \leq i_{2m}} i_1 i_2 \dots i_{2m}}. \quad (7)$$

Again set $W(i) = zq^i$ for $i \in N$ and multiplicatively extend W to N^* . Let $U^m V = \{uv : u \in U^* \text{ is of length } m, v \in V\}$. From Lemma 1, the bijection $f_{2m+1} : \mathcal{C}_{2m+1} \times P_{2m+1} \rightarrow U^m V$ satisfies the properties $\text{ris}_2 \sigma = \text{ris}_2 w$ and $\text{inv } \sigma + \|\lambda\| = \|w\|$ whenever $f_{2m+1}(\sigma, \lambda) = w$. It then follows from (7) that

$$\begin{aligned} & \sum_{m \geq 0} \frac{E_{2m+1}(t, q) z^{2m+1}}{(q; q)_{2m+1}} \\ &= \sum_{m \geq 0} z^{2m+1} \sum_{(\sigma, \lambda) \in \mathcal{C}_{2m+1} \times P_{2m+1}} t^{\text{ris}_2 \sigma} q^{\text{inv } \sigma + \|\lambda\|} \\ &= \sum_{w \in U^*V} t^{\text{ris}_2 w} W(w) \\ &= \frac{\sum_{m \geq 0} (t-1)^m z^{2m+1} \sum_{0 \leq i_1 \leq \dots \leq i_{2m+1}} q^{i_1 + \dots + i_{2m+1}}}{1 - \sum_{m \geq 1} (t-1)^{m-1} z^{2m} \sum_{0 \leq i_1 \leq \dots \leq i_{2m}} q^{i_1 + \dots + i_{2m}}} \\ &= \frac{\sum_{m \geq 0} (t-1)^m z^{2m+1} / (q; q)_{2m+1}}{1 - \sum_{m \geq 1} (t-1)^{m-1} z^{2m} / (q; q)_{2m}} \\ &= \frac{(1-t)^{1/2} \sin_q(z\sqrt{1-t})}{\cos_q(z\sqrt{1-t}) - t}, \end{aligned} \quad (8)$$

where $\cos_q z = \sum_{n \geq 0} (-1)^n z^{2n} / (q; q)_{2n}$ and $\sin_q z = \sum_{n \geq 0} (-1)^n z^{2n+1} / (q; q)_{2n+1}$. As the even case is essentially contained in the analysis above, we have

$$\sum_{n \geq 0} \frac{E_n(t, q) z^n}{(q; q)_n} = \frac{(1-t) \left(1 + (1-t)^{-1/2} \sin_q(z\sqrt{1-t}) \right)}{\cos_q(z\sqrt{1-t}) - t}.$$

Setting $t = 0$, replacing z by $z(1 - q)$, and letting $q \rightarrow 1^-$ give (6).

Generalization to r -tuples of m -permutations is relatively straightforward. Let $S_{n,m}$ denote the set of $\sigma \in S_n$ satisfying the property that $\sigma(k) > \sigma(k + 1)$ implies k is a multiple of m . Note that $S_{n,2} = \mathcal{O}_n$. For $(\sigma_1, \sigma_2, \dots, \sigma_r) \in S_{n,m}^r$, define $\text{cris}_m(\sigma_1, \sigma_2, \dots, \sigma_r)$ to be the number of $k \in \bigcap_{j=1}^r \text{Ris } \sigma_j$ such that k is a multiple of m . Combining the ideas behind Theorem 3 and (8) gives

THEOREM 4. *For $m, r \geq 1$, the sequence of polynomials*

$$E_{n,m,r}(t, \mathbf{q}) = \sum_{(\sigma_1, \sigma_2, \dots, \sigma_r) \in S_{n,m}^r} t^{\text{cris}_m(\sigma_1, \sigma_2, \dots, \sigma_r)} q_1^{\text{inv } \sigma_1} q_2^{\text{inv } \sigma_2} \dots q_r^{\text{inv } \sigma_r}$$

is generated by

$$\sum_{n \geq 0} \frac{E_{n,m,r}(t, \mathbf{q}) z^n}{(\mathbf{q}; \mathbf{q})_{n,r}} = \frac{(1-t) \left(1 + \sum_{\rho=1}^{m-1} (1-t)^{-\rho/m} \Phi_{m,\rho,r}(z \sqrt[m]{1-t}, \mathbf{q}) \right)}{\Phi_{m,0,r}(z \sqrt[m]{1-t}, \mathbf{q}) - t},$$

where $\Phi_{m,\rho,r}(z, \mathbf{q}) = \sum_{\nu \geq 0} (-1)^\nu z^{\nu m + \rho} / (\mathbf{q}; \mathbf{q})_{\nu m + \rho, r}$.

Theorem 4 is essentially due to Stanley [13]. Note that $E_{n,1,r}(t, \mathbf{q})$ is equal to the generalized q -Eulerian polynomial defined in (5). Thus, taking $m = 1$ in Theorem 4 gives Theorem 3 as a corollary. We further remark that $\Phi_{m,\rho,1}(z, q)$ is a q -Olivier function. When $r = 1$ and $t = s = 0$, replacing z by $z(1 - q)$ and letting $q \rightarrow 1^-$ give the initial result of Carlitz [2] on m -permutations.

5. FROM THE TRACE TO THE FREE MONOID

As the final application, we use Theorem 1 to obtain Diekert's [5, pp. 96–99] lifting to the free monoid of an inversion formula due to Cartier and Foata [4] from a partially commutative monoid (or trace monoid) in which the defining binary relation admits a transitive orientation.

Let θ be an irreflexive symmetric binary relation on X . Define \equiv_θ to be the binary relation (induced by θ) on X^* consisting of the set of pairs (w, v) of words such that there is a sequence $w = w_0, w_1, \dots, w_m = v$, where each w_i is obtained by transposing a pair of letters in w_{i-1} that are consecutive and contained in θ . For instance, if $X = \{x, y, z\}$ and $\theta = \{(x, y), (y, x)\}$, then the sequence $zyyx, zyxy, zxyy$ implies that $zyyx \equiv_\theta zxyy$.

Clearly, \equiv_{θ} is an equivalence relation on X^* . The quotient of X^* by \equiv_{θ} gives the *partially commutative monoid* induced by θ and is denoted by $M(X, \theta)$. The equivalence class \hat{w} of $w \in X^*$ is referred to as the *trace* of w .

A word $w = x_1 x_2 \cdots x_n \in X^*$ is said to be a *basic monomial* if $x_i \theta x_j$ for all $i \neq j$. A trace \hat{w} is said to be *θ -trivial* if any one of its representatives is a basic monomial. If one lets $\mathcal{T}^+(X, \theta)$ be the set of θ -trivial traces, the inversion formula of Cartier and Foata reads as follows.

THEOREM 5 (Cartier and Foata). *For θ an irreflexive symmetric binary relation on X , the traces in $M(X, \theta)$ are generated by*

$$\sum_{\hat{w} \in M(X, \theta)} \hat{w} = \frac{1}{1 + \sum_{\hat{t} \in \mathcal{T}^+(X, \theta)} (-1)^{l(\hat{t})} \hat{t}},$$

where $l(\hat{t})$ denotes the length of any representative of \hat{t} .

A natural question to ask is whether \hat{w} and \hat{t} can be replaced by some canonical representatives so that Theorem 5 remains true as a formula in the free monoid X^* . As resolved by Diekert [5], such canonical representatives exist if and only if θ admits a transitive orientation.

To be precise, a subset $\vec{\theta}$ of θ is said to be an *orientation* of θ if θ is a disjoint union of $\vec{\theta}$ and $\{(x, y) : (y, x) \in \vec{\theta}\}$. The set of $t = t_1 t_2 \cdots t_n \in X^*$ satisfying $t_i \vec{\theta} t_{i+1}$ is denoted by $T^+(X, \vec{\theta})$. Note that $T^+(X, \vec{\theta})$ is a set of representatives for the θ -trivial traces $\mathcal{T}^+(X, \theta)$ whenever $\vec{\theta}$ is transitive. A word $w = x_1 x_2 \cdots x_n \in X^*$ is said to have a $\vec{\theta}$ -adjacency in position k if $x_k \vec{\theta} x_{k+1}$. We denote the number of $\vec{\theta}$ -adjacencies of w by $\vec{\theta}\text{adj } w$. Although Diekert did not explicitly introduce the notion of a $\vec{\theta}$ -adjacency, his lifting theorem may be paraphrased as follows.

THEOREM 6 (Diekert). *Let θ be an irreflexive symmetric binary relation on X and let $\vec{\theta}$ be an orientation of θ . Then, θ is transitive if and only if there exists a complete set W of representatives for the traces of $M(X, \theta)$ such that*

$$\sum_{w \in W} w = \frac{1}{1 + \sum_{t \in T^+(X, \vec{\theta})} (-1)^{l(t)} t}.$$

Moreover, $W = \{w \in X^* : \vec{\theta}\text{adj } w = 0\}$.

To see how Theorem 1 intervenes in the matter, suppose that $\vec{\theta}$ is an orientation of θ (not necessarily transitive for now). If for $x, y \in X$ we set $a_{xy} = a$ when $x \vec{\theta} y$ and $a_{xy} = 1$ otherwise, then Theorem 1 reduces to

$$\sum_{w \in X^*} a^{\vec{\theta}\text{adj } w} w = \frac{1}{1 + \sum_{t \in T^+(X, \vec{\theta})} (-1)^n (1-a)^{l(t)-1} t}. \quad (9)$$

When $\vec{\theta}$ is transitive, setting $a = 0$ in (9) gives the lifting of Theorem 5 to the free monoid as stated in Diekert's theorem. We close this section with two examples.

TRANSITIVE EXAMPLE. Let $X = \{x, y, z\}$ with $\theta = \{(x, y), (y, x), (x, z), (z, x)\}$. Among other possibilities, $\vec{\theta} = \{(y, x), (z, x)\}$ is a transitive orientation of θ . The $\vec{\theta}$ -adjacencies of a word correspond to factors yx and zx . Note that $T^+(X, \vec{\theta}) = \{x, y, z, yx, zx\}$ is a complete set of representatives for the θ -trivial traces $\mathcal{T}^+(X, \theta)$. Also, the only word in

$$\widehat{xzyxy} = \left\{ \begin{array}{ccccc} xzxyy, & xzyxy, & xzyyx, & xxzyy, & zxyyy, \\ zxyxy, & zxyyx, & zyxyx, & zyxyx, & zyxzx \end{array} \right\}$$

having no $\vec{\theta}$ -adjacencies is $xxzyy$. From (9), we have

$$\sum_{w \in \{x, y, z\}^*} a^{\vec{\theta} \text{adj} w} w = \frac{1}{1 - (x + y + z) + (1 - a)(yx + zx)}.$$

Setting $a = 0$ gives an identity that can be viewed as having been lifted from the trace monoid as in Theorem 6.

NON-TRANSITIVE EXAMPLE. Let X and θ be as in the previous example. The orientation $\vec{\theta} = \{(y, x), (x, z)\}$ is not transitive. Observe that the word yxz in $T^+(X, \vec{\theta}) = \{x, y, z, yx, xz, yxz\}$ is not a θ -trivial trace. Also, $\widehat{yxz} = \{yxz, xyz, yzx\}$ contains two words having no $\vec{\theta}$ -adjacencies. Nevertheless, (9) implies

$$\sum_{w \in \{x, y, z\}^*} a^{\vec{\theta} \text{adj} w} w = \frac{1}{1 - (x + y + z) + (1 - a)(yx + xz) - (1 - a)^2 yxz}.$$

6. PROOFS FOR THEOREMS 1 AND 2

To establish Theorem 1, we begin by noting that (1) is equivalent to

$$\sum_{w \in X^*} a(w)w - \sum_{w \in X^*} \left(\sum_{w=uv, v \neq 1} a(u)\bar{a}(v) \right) w = 1.$$

Thus, by equating coefficients, it suffices to show that

$$a(w) = \sum_{w=uv, v \neq 1} a(u)\bar{a}(v) \quad (10)$$

for all $w \in X^+$. We proceed by induction on the length $l(w)$ of w . For $l(w) = 1$, (10) is trivially true. Suppose $l(w) \geq 2$. Then w factorizes as $w = w_1xy$, where $w_1 \in X^*$ and $x, y \in X$. Assuming (10) holds for words of length smaller than w , it follows that

$$\begin{aligned}
 a(w) &= a_{xy}a(w_1x) = a(w_1x) + (a_{xy} - 1)a(w_1x) \\
 &= a(w_1x)\bar{a}(y) + \bar{a}(xy) \sum_{w_1x=uv_1x} a(u)\bar{a}(v_1x) \\
 &= a(w_1x)\bar{a}(y) + \sum_{w_1xy=uv_1xy} a(u)\bar{a}(v_1xy) \\
 &= \sum_{w=uv, v \neq 1} a(u)\bar{a}(v),
 \end{aligned}$$

and the proof is complete.

We use an alternate approach to prove Theorem 2. Let W denote the left-hand side of (2) and define

$$W_{n+1} = \sum_w a(w)w,$$

where the sum is over words $w = x_1x_2 \cdots x_{n+1} \in U^*V$ of length $(n+1)$. Note that $W = \sum_{n \geq 0} W_{n+1}$. Since $\bar{a}(x_1) = 1$ and $\bar{a}(x_1x_2) = a_{x_1x_2} - 1$, it is a triviality that

$$W_{n+1} = \sum_w \bar{a}(x_1)a(x_2, \dots, x_{n+1})w + \sum_w \bar{a}(x_1x_2)a(x_2 \cdots x_{n+1})w$$

for $n \geq 1$. Similarly, the second sum on the above right may be split as

$$\sum_w \bar{a}(x_1x_2)a(x_3 \cdots x_{n+1})w + \sum_w \bar{a}(x_1x_2x_3)a(x_3 \cdots x_{n+1})w$$

so that

$$\begin{aligned}
 W_{n+1} &= \sum_{k=1}^2 \sum_w \bar{a}(x_1 \cdots x_k)a(x_{k+1} \cdots x_{n+1})w \\
 &\quad + \sum_w \bar{a}(x_1x_2x_3)a(x_3 \cdots x_{n+1})w.
 \end{aligned}$$

Iterating the above argument and then factoring give

$$\begin{aligned} W_{n+1} &= \sum_{k=1}^n \sum_w \bar{a}(x_1, \dots, x_k) a(x_{k+1} \cdots x_{n+1}) w + \sum_w \bar{a}(w) w \\ &= \sum_{k=1}^n \left(\sum_u \bar{a}(u) u \right) W_{n+1-k} + \sum_w \bar{a}(w) w, \end{aligned}$$

where the sum to the immediate left of W_{n+1-k} is over words $u = x_1 \cdots x_k \in U^+$ of length k . As the above recurrence relationship for W_{n+1} is valid for $n \geq 0$, it follows that

$$W = \left(\sum_{w \in U^+} \bar{a}(w) w \right) W + \sum_{w \in U^+ V} \bar{a}(w) w,$$

which implies Theorem 2.

Either of the preceding arguments may be easily modified to give an inversion formula for words in X^* that end in a fixed word v . Without giving the details, we have

THEOREM 7. *According to adjacencies, words ending in a word $v = b_1 b_2 \cdots b_m \in X^*$ of length m are generated by*

$$\sum_{w \in X^*} a(wv) wv = \left(1 - \sum_{w \in X^+} \bar{a}(w) w \right)^{-1} \left(a(v) \sum_{w \in X^*} \bar{a}(wb_1) wv \right).$$

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