THE COMBINATORICS OF CERTAIN PRODUCTS

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ABSTRACT. A combinatorial interpretation for the coefficients in the expansion of $\prod (1 + ux^j y^k)(1 - ux^j y^k)^{-1}$ is given.

1. Introduction. The coefficients A(n; x, y), B(n; x, y) and C(n; x, y) defined by

(1.1)
$$\sum_{n\geq 0} \frac{A(n; x, y)u^n}{(x)_n(y)_n} = \frac{1}{(u; x, y)},$$

(1.2)
$$\sum_{n\geq 0} \frac{B(n; x, y)u^n}{(x)_n(y)_n} = (-u; x, y),$$

(1.3)
$$\sum_{n\geq 0} \frac{C(n; x, y)u^n}{(x)_n(y)_n} = \frac{(-u; x, y)}{(u; x, y)},$$

where

$$(1.4) (x)_n = (1-x)(1-x^2)\cdots(1-x^n), (x)_0 = 1,$$

(1.5)
$$(u; x, y) = \prod_{j,k \ge 0} (1 - ux^j y^k),$$

have been considered by a number of mathematicians. Carlitz [2] was the first to demonstrate that these coefficients are polynomials in x and y with positive integral coefficients and provided closed formulas for calculating them. In a subsequent paper Roselle [6] gave combinatorial interpretations for A(n; x, y) and B(n; x, y). More recently, generalizations of (1.1) and (1.2) have appeared in [3,4,5] and [4], respectively.

A natural question, which was suggested to me by Dominique Foata, arises from Roselle's work: Is there a combinatorial interpretation for C(n; x, y)? This note provides one in terms of the already known interpretations of A(n; x, y) and B(n; x, y) and in terms of certain statistics defined on bicolored permutations.

2. Roselle's work. The polynomials A(n; x, y) and B(n; x, y) may be interpreted as the generating functions for various statistics defined on the group G(n) consisting of permutations of $\{1, 2, ..., n\}$. To be precise, if the major index of a permutation $\sigma \in G(n)$, denoted $m(\sigma)$, is defined to be the sum of the elements in the set

(2.1)
$$\{j: \sigma(j) > \sigma(j+1), 1 \le j \le n-1\}$$

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and the i-major and co-i-major indices are defined, respectively, to be

$$i(\sigma) = m(\sigma^{-1}),$$

$$(2.3) c(\sigma) = \binom{n}{2} - i(\sigma),$$

then, as Roselle [6] and Foata [4] have demonstrated,

(2.4)
$$A(n; x, y) = \sum_{\sigma \in G(n)} x^{m(\sigma)} y^{i(\sigma)},$$

(2.5)
$$B(n; x, y) = \sum_{\sigma \in G(n)} x^{m(\sigma)} y^{c(\sigma)}.$$

3. Bicolored permutations. Since Carlitz [2] showed that $C(n; 1, 1) = 2^n \cdot n!$, it is clear that G(n) will not provide a combinatorial structure for interpreting C(n; x, y). For this reason, one is led to consider the set of bicolored permutations BG(n), consisting of words $b = b(1)b(2) \cdots b(n)$ obtained by coloring each letter of some permutation $\sigma = \sigma(1)\sigma(2) \cdots \sigma(n) \in G(n)$ either red (indicated by underlining the letter) or blue (letter not underlined). For instance,

$$(3.1) b = 3274165 \in BG(7).$$

Note that there are 2^n different colorings of each $\sigma \in G(n)$.

Now, possessing a combinatorial set with the correct cardinality, all that remains is to define some appropriate statistics on BG(n) for which C(n; x, y) is the generating function. To this end, let W(n) be the set of words $w = w(1)w(2) \cdots w(n)$ of length n with letters $w(i) \in \{0, 1\}$, and let |w| denote the number of letters equal to 1 in w. The reduction of a permutation γ of $\{a_1 < a_2 < \cdots < a_n\}$ is obtained by replacing a_i in γ by i for $1 \le i \le n$. For instance, the reduction of the permutation $\gamma = 2416$ of $\{1 < 2 < 4 < 6\}$ is 2314.

Each $b \in BG(n)$ is now assigned to a 4-tuple (w, v, θ, α) where $w, v \in W(n)$ with $|w| = |v|, \theta \in G(|w|)$ and $\alpha \in G(n - |w|)$ according to the following rules:

(3.2)
$$w(i) = \begin{cases} 1 & \text{if } b(i) \text{ is red,} \\ 0 & \text{otherwise,} \end{cases}$$

(3.3)
$$v(i) = \begin{cases} 1 & \text{if } i \text{ is red in } b, \\ 0 & \text{otherwise,} \end{cases}$$

(3.4)
$$\theta = \text{reduction of the red subpermutation of } b$$
,

(3.5)
$$\alpha = \text{reduction of the blue subpermutation of } b$$
.

Roughly speaking, w indicates which positions of b are red, v which elements of $\{1, 2, ..., n\}$ are placed in the red positions, θ how the red letters are arranged in the red positions, and α how the blue letters are placed in the remaining positions. For instance, the bicolored permutation b of (3.1) corresponds to the 4-tuple (w, v, θ, α) where w = 0.101110, v = 1.101010, $\theta = 2.314$ and $\alpha = 1.32$. Note that the map $b \to (w, v, \theta, \alpha)$ is a bijection between BG(n) and the set of such 4-tuples.

Finally, for the bicolored permutation $b \to (w, v, \theta, \alpha)$ the appropriate statistics are defined by

$$(3.6) M(b) = m(\theta) + m(\alpha) + m(w),$$

$$(3.7) I(b) = i(\theta) + c(\alpha) + m(v),$$

where m(w) and m(v) are defined in exactly the same way as the major index of a permutation.

4. The interpretation of C(n; x, y). It is now possible to show that

(4.1)
$$C(n; x, y) = \sum_{b \in BG(n)} x^{M(b)} y^{I(b)}.$$

First, as Carlitz [2] demonstrated, identities (1.1)–(1.3) imply

(4.2)
$$C(n; x, y) = \sum_{k \ge 0} {n \choose k}_x {n \choose k}_y A(k; x, y) B(n - k; x, y),$$

where the x-binomial coefficient is defined by

(4.3)
$$\binom{n}{k}_{x} = \frac{(x)_{n}}{(x)_{k}(x)_{n-k}}.$$

Second, the fact that (see [1, p.40])

$$\sum x^{m(w)} = \binom{n}{k}_x,$$

where the summation is over $w \in W(n)$ with |w| = k, along with (2.4), (2.5), (3.6) and (3.7) allow the calculation

(4.5)
$$\sum_{b \in BG(n)} x^{M(b)} y^{I(b)} = \sum_{(w, v, \theta, \alpha)} x^{m(\theta) + m(\alpha) + m(w)} y^{i(\theta) + c(\alpha) + m(v)}$$

$$= \sum_{k \ge 0} \sum_{|w| = k} x^{m(w)} \sum_{|v| = k} y^{m(v)} \sum_{\theta \in G(k)} x^{m(\theta)} y^{i(\theta)} \sum_{\alpha \in G(n-k)} x^{m(\alpha)} y^{c(\alpha)}$$

$$= \sum_{k \ge 0} {n \choose k}_x {n \choose k}_y A(k; x, y) B(n - k; x, y).$$

Identities (4.2) and (4.5) imply (4.1).

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