Elemental principles of t-topos

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Abstract. – In this paper, a sheaf-theoretic approach toward fundamental problems in quantum physics is made. For example, the particle-wave duality depends upon whether or not a presheaf is evaluated at a specified object. The *t-topos* theoretic interpretations of *double-slit interference, uncertainty principle(s)*, and the *EPR-type non-locality* are given. As will be explained, there are more than one type of uncertainty principle: the *absolute* uncertainty principle coming from the direct limit object corresponding to the refinements of coverings, the uncertainty coming from a *micromorphism* of shortest observable states, and the uncertainty of the observation image. A sheaf theoretic approach for quantum gravity has been made by Isham-Butterfield in (*Found. Phys.*, **30** (2000) 1707), and by Raptis based on abstract differential geometry in Mallios A. and Raptis I., *Int. J. Theor. Phys.*, **41** (2002), qr-qc/0110033; Mallios A., *Remarks on "singularities"* (2002) qr-qc/0202028; Mallios A. and Raptis I., *Int. J. Theor. Phys.*, **42** (2003) 1479, qr-qc/0209048. See also the preprint by Requardt M., *The translocal depth-structure of space-time, Connes' "Points, Speaking to Each Other", and the (complex) structure of quantum theory*, for another approach relevant to ours. Special axioms of t-topos formulation are: i) the usual linear-time concept is interpreted as the image of the presheaf (associated with time) evaluated at an object of a *t-site* (*i.e.*, a category with a *Grothendieck topology*). And an object of this t-site, which is said to be a *generalized time period*, may be regarded as a hidden variable and ii) every object (in a *particle ur-state*) of microcosm (or of macrocosm) is regarded as the microcosm (or macrocosm) component of a product category for a presheaf evaluated at an object in the *t-site*. The fundamental category $\hat{S}$ is defined as the category of $\prod_{\alpha \in \Delta} C_\alpha$-valued presheaves on the *t-site* S, where Δ is an index set. The study of topological properties of S with respect to the nature of multi-valued presheaves is left for future study on the *t-topos* version of relativity (see Kato G., *On t.g. Principles of relativistic t-topos*, in preparation; Guts A. K. and Grinkevich E. B., *Toposes in General Theory of Relativity* (1996), arXiv:gr-qc/9610073, 31). We let C_1 and C_2 be microcosm and macrocosm discrete categories, respectively, in what will follow. For further development see also Kato G., *Presheafification of Matter, Space and Time, International Workshop on Topos and Theoretical Physics, July 2003, Imperial College* (invited talk, 2003).

Basic definitions. – For t-topos theory, the notion of a t-site plays the role of hidden variables. More conditions will be added to the site when the further applications in [1] are made. For the concept of a Grothendieck topology, see [2–4] or [5].

Definition 1.1. Let S be a site, namely, a category with a Grothendieck topology and let $\hat{S}$ be the category of presheaves from S to the product category $\prod_{\alpha \in \Delta} C_\alpha$. That is, $\hat{S} = (\prod_{\alpha \in \Delta} C_\alpha)^{S^{\text{opp}}}$, where S^{opp} is the dual category of S. Then site S is said to be a *temporal site* or simply *t-site* when S is used in this context. Category $\hat{S}$ is said to be a *t-topos* or *temporal topos*. We sometimes call an object of $\hat{S}$ an *entity*.

Remarks 1.2. i) See [3] or [5] for Grothendieck topologies which is sufficient for t-topos theory.

ii) For an object F in $\hat{S}$, which we write as $F \in Ob(\hat{S})$ and for an object V in S, *i.e.*, $V \in Ob(S)$, $F(V)$ is an object in $\prod_{\alpha \in \Delta} C_\alpha$. Namely,

$$F(V) = (F(V)_\alpha)_{\alpha \in \Delta}\,,$$

where $F(V)_\alpha$ is the α-th component of $F(V)$. We also say that $F(V)$ is the *manifestation* of F at the generalized time period V.

Definition 1.3. Let F be an object of $\hat{S}$. *The state of F during a generalized time period W*, namely, an object of S, is defined by the pair $(F, W) = F(W)$. Then F is said to be *manifested* during W. When a generalized time period is *not given*, F is said to be in a *pre-state* or in an *unmanifested state*. (See *Note 1.4′* below.) For a specified object V, the object $F(V)$ is said to be in the *particle ur-state* of F over the generalized time period V, and when one object in the t-site is not specified for F, then F is said to be in a *wave ur-state* of F and sometimes denoted as $\{F(W)\}_{W \in Ob(S)}$ or $F(-)$.

Definition 1.4. An *observation* of an object m of $\hat{S}$ by another object P of $\hat{S}$ in a non-discrete category C_α, $\alpha \in \Delta$, *over a generalized time period V* is a natural transformation s over V. Namely, the morphism in C_α

$$s_V : m(V) \longrightarrow P(V) \tag{1}$$

is said to be an observation of m by P during the generalized time period V. If such a natural transformation s over a specified object V of t-site exists, then m is said to be *observable or measurable by P during the generalized time period V*. We may also say that m *interacts* with P if there exists such a natural transformation from m to P over some generalized time period. Notice that when m is measured, m needs to be in a particle ur-state since an object in S must be specified for the natural transformation in (1).

Note 1.4′. When an object m of $\hat{S}$ is not observed, not only m is in the wave ur-state, *i.e.*, $\{m(V)\}$ in Definition 1.3, but also (we will be more precise in Definitions 2.1 and 2.2) m may be considered as the totality of decomposed subobjects of m which are to be evaluated at unspecified objects of S. It may be most appropriate to consider an unobserved object m to be simply presheaf "m".

Note 1.5. Let $\{V_i \to V\}$ be a covering of V and let $\{V_{i \leftarrow j} \to V_i\}$ be a covering of V_i as in [2–6] or [7]. Then by composing covering morphisms, $\{V_{i \leftarrow j} \to V\}$ is a covering of V. Similarly, by composing further, one gets a covering $\{V_{k \leftarrow j \leftarrow i} \to V\}$ of V. Then, consider the *inverse limit covering*

$$\{\varprojlim V_{\dots \leftarrow k \leftarrow j \leftarrow i} \longrightarrow V\} \tag{2}$$

of V. In the next section, we will need this notion.

Definition 1.6. Let C_1 be the microcosm discrete category. That is, an object of C_1 is a particle in microcosm, and as a category, C_1 is discrete, namely, no morphisms exist except identity morphisms.

Note 1.7. The topos approach in [8] and [9] by Butterfield-Isham can be interpreted in terms of t-topos as follows. First, we will explain the basic method in [8] and [9]: Let S be the state space and let A be a physical quantity and let $\bar{A}$ be a real-valued function representing A as in [8]. Then the *functional composition principle* (referred to as *FUNC* in [8] and [9]) is the commutative diagram

$$\begin{array}{ccc} H & \xrightarrow{V} & R \\ \downarrow & & \downarrow \\ H & \xrightarrow{V} & R \end{array}$$

where the left-hand side vertical morphism $\tilde{h} : H \to H$ on the Hilbert space H is induced by a function $h : R \to R$ on real numbers. That is, for the value $V(\hat{A})$ of the physical quantity A represented by the operator $\hat{A}$, we have $V(\tilde{h}(\hat{A})) = h(V(\hat{A}))$ which is the commutativity of the above diagram. The Butterfield-Isham topos theory interprets this commutative diagram as follows: Regarding the valuation V as a natural transformation γ from a terminal object 1 to an object X in the topos of presheaves, for $\hat{f} : \hat{B} \to \hat{A}$ in the category of all bounded self-adjoint operators, we first have $X(f) : X(A) \to X(B)$, and γ_A in $X(A)$ and γ_B in $X(B)$, since γ is a natural transformation from 1 to X. Here we make the following interpretation of γ_A as a morphism from A to X using Yoneda Lemma. Then the Kochen-Specker Theorem states that *such a global section γ does not exist to satisfy the commutative diagram*

$$\begin{array}{ccc} X & = & X \\ \uparrow & & \uparrow \\ \hat{B} & \xrightarrow{\hat{f}} & \hat{A} \end{array}$$

where the vertical morphisms are γ_A and γ_B. Namely, $\gamma_B = \gamma_A \circ \hat{f}$, *i.e.*, $X(f)(\gamma_A) = \gamma_B$, the *matching condition* in [8, 9]. In terms of t-topos, the value $V^s(A)$ of A at a state $s \in S$ corresponds to $m(V)$ over $V \in Ob(S)$, where $m \in Ob(\hat{S})$. Suppose that usual linear time $\tau(V)$ precedes $\tau(U)$. And let $g : V \to U$ be the associated morphism in the t-site S. (See [10] for the associated morphism induced by the linear ordering on τ.) Then, the t-topos version of Kochen-Specker Theorem states that *there does not exist a natural transformation s over the t-site S (itself) making the diagram*

$$\begin{array}{ccc} m(V) & \xleftarrow{m(g)} & m(U) \\ \downarrow & & \downarrow \\ P(V) & \xleftarrow{P(g)} & P(U) \end{array}$$

commutative, where the left-hand side vertical morphism is s_V and the right-hand side vertical morphism is s_U as in Definition 1.4. Note that such a globally defined natural transformation s (which is γ in Butterfield-Isham) from m to P is defined for the entire objects of S (a global section from 1 to X). As for t-topos, the definition of an observation is defined for a specified object of S as in Definition 1.4.

Note 1.8. Every object in C_1 is the C_1-component of a presheaf in $\hat{S}$ evaluated at a generalized time period in the t-site S. For example, if $\underline{e}$ is a particle in C_1, then there exists an associated presheaf e in $\hat{S}$ such that for an object V in the t-site S, we have $\underline{e} = e(V)$, which is determining the (particle ur-) state of e for $\underline{e}$. Then the particle $\underline{e}$ is said to be *presheafified* by a presheaf e in $\hat{S}$.

Remark 1.9. Let $\underline{e}$ be any object in C_1. For example, $\underline{e}$ can be an electron. For a particle $\underline{e}$ in C_1, the position and time x and t are associated locally. As is mentioned in Note 1.8, we presheafify $\underline{e}$ as $\underline{e} = e(V)$ in C_1, where $e \in Ob(\hat{S})$ and $V \in Ob(S)$. We will presheafify the position by presheaf κ and time by presheaf τ defined over the same object V in the third section, so that $(\kappa(V), \tau(V))$ plays a local coordinate system of $e(V)$.

Definition 1.10. Let $m_1, m_2, \ldots, m_r$ be objects of $\hat{S}$. If the r-tuple $(m_1, m_2, \ldots, m_r)$ can be considered as one object of $\hat{S}$ over a subsite, then objects $m_1, m_2, \ldots, m_r$ are said to be *ur-entangled* (or *ur-correlated*). See [11] for the application to the EPR-type non-locality.

Sheaf-theoretic methods for non-locality and sub-Planck region. – Let m and m' be the presheaves associated with $\underline{m}$ and $\underline{m'}$ and let (κ, τ) be the associated sheaves to space and time to m. Suppose that m and m' are ur-entangled as defined in Definition 1.10. Furthermore, assume that $m(V)$ and $m'(V)$ are physically a distance apart (for the same object V), *e.g.*, 11 kilometers apart. Then the space-time $(\kappa(V), \tau(V))$ of $m(V)$ does not contain $m'(V)$, if necessary by taking V "small" enough in the sense of a covering. That is, space-time presheaves are associated with m in $\hat{S}$. Namely, (κ, τ) should be denoted as (κ_m, τ_m), see also [11].

Definition 2.1. Let M be a matter in the macrocosm discrete category C_2. Let m be the associated presheaf to M. Then a finite direct sum of presheaves $\sum_{\lambda \in \Lambda} m_\lambda$ of m is said to be a *uniform quantum decomposition* of m with respect to a covering $\{V_\lambda \to V\}$ of a generalized time period V if each m_λ is an object of $\hat{S}$ so that $m_\lambda(V_\lambda)$ may be an object of C_1, and $\sum_{\lambda \in \Lambda} m_\lambda(V_\lambda) = m(V)$. See [10] for the notion of a sub-Planck decomposition.

Remark 2.2. A short remark on Double-Slit Interference may be appropriate, see [12] for details. Suppose that an electron $\underline{e}$ is fired at a certain time. In terms of t-topos, $\underline{e} = e(V)$ is fired at $(\kappa(V), \tau(V))$, where e, κ, and τ are associated presheaves to the electron $\underline{e}$, space and time. Assume also that two slits are appropriately narrow and the spacing between the slits is much larger than the width of the slit. Let $(\kappa(U), \tau(U))$ be the position and the time when the electron hits the screen, inducing a morphism $g : V \to U$. For the two slits, let W and W' be the associated objects of the t-site S for which $(\kappa(W), \tau(W))$ and $(\kappa(W'), \tau(W'))$ would be the corresponding slits that $\underline{e}$ would go through. Without an observation at either one of the slits, there are two objects, *i.e.*, W and W', in S. Hence, by Definition 1.3, e remains to be in a wave ur-state. In the case where there is no mask between the screen and an electron gun, one needs to consider not only via W and W', but also all the factorizations of $g : V \to U$. Then e is in the wave ur-state $e(\{g : V \to U\})$, where $\{g : V \to U\} = \{W \in Ob(S) : g = f \circ h$, where $f : V \to W$ and $h : W \to U\}$, see [12] for details.

Remark 2.3. One can choose a covering $\{V_i \to V\}$ and another covering $\{V_{i \leftarrow j} \to V\}$ as in Note 1.5, so that $m(V_i)$ and $m(V_{i \leftarrow j})$ may belong to C_1 and the Planck scale category C_{Pl}, respectively.

Remark 2.4. First note, for example, when we consider the C_1-components of $m(V)$ and $P(V)$ in Definition 1.4, such a morphism as s_V in (1) belongs to a non-discrete category C_α. However, in the following, we simply say that s_V is an observation of $m(V)$ by $P(V)$ in C_1. An $\hat{S}$-theoretic interpretation of an observation of an electron by an observer is the following. Let e be the presheaf in $\hat{S}$ corresponding to an electron $\underline{e}$. Let P be an observer, *i.e.*, an object of $\hat{S}$ and let V be an generalized time period. As defined in Definition 1.4, an observation of e by P is a natural transformation s_V from e to P over $V \in Ob(S)$.

Remark 2.5 (Uncertainty principles). Suppose time $\tau(V)$ precedes $\tau(U)$ inducing a morphism $V \xrightarrow{\varepsilon} U$ as in Note 1.7. We define the notion of a micromorphism as follows: the morphism $V \xrightarrow{\varepsilon} U$ is said to be a *micromorphism* if the morphism ε can not be factored as $\varepsilon = \beta \circ \alpha$, where $\alpha : V \to W$ and $\beta : W \to U$ and so that $\tau(V)$ may precede $\tau(W)$ which precedes $\tau(U)$. For a general morphism $g : V \to U$, one can consider a micro-decomposition of g as follows: $g = g_n \circ \ldots \circ g_0$ and each $g_j : V_{j-1} \to V_j$ is a micromorphism. A consequence of a micromorphism $V \xrightarrow{\varepsilon} U$ is that it is impossible to observe a particle (presheaf) between $\tau(V)$ and $\tau(U)$ by the definition. Consequently, the position of *particle* m between $\tau(V)$ and $\tau(U)$ cannot be known (observed). Since κ and τ are ur-entangled, we also obtain the uncertainty in position as well. (See the following third section and [10], and [4] for the relativistic version.) The above uncertainty corresponds to the usual Heisenberg uncertainty principle in the following sense. For a micromorphism $V \xrightarrow{\varepsilon} U$, the difference in position $\kappa(U) - \kappa(V)$ *times* the difference in momentum $p(U) - p(V)$ is not less than $\hbar$, where p is the presheaf associated with momentum.

There is another uncertainty principle that is absolute in nature. Since we have replaced the notion of a set theoretic point with the notion of an object of a category, we have a finite value of the direct limit over coverings $\varinjlim \tau(V_{\ldots \leftarrow j \leftarrow i})$. (Note that it is not the inverse limit since a presheaf is contravariant.) Similarly, we have the absolute uncertainty for position sheaf. This material is expected to be expanded in a forthcoming paper [1].

Remark 2.6 (Definition 1.10 and the EPR). Let $\underline{e}$ and $\underline{e}'$ be entangled electrons, and let e and e' be the associated presheaves which are ur-entangled. Namely, the pair (e, e') is an object of $\hat{S}$ satisfying $\underline{e} = e(V)$ and $\underline{e}' = e'(V)$ for the common object V in a subsite as in Definition 1.10. Then $e^* = (e, e') \in Ob(\hat{S})$. For a specified generalized time period V, we have objects $e(V)$ and $e'(V)$. That is, the states of $\underline{e}$ and $\underline{e}'$ are determined by the generalized time period V and are independent of the physical distance between $e(V)$ and $e'(V)$ in C_2. When e is observed or measured by P in the sense of Definition 1.4, there is a morphism $s_V : e(V) \to P(V)$ for some V in S. This V determining the state of the object e simultaneously determines the state of e' in the sense of Definition 1.10, see [11] for the full-length description of this topic.

Sheafification of space and time.

Axiom 3.1 (Interpretation of the physical time as a sheaf). As in Remark 1.9, we have already noted that the physical time depends upon generalized time. That is, we hypothesize that τ is an object of $\hat{S}$ so that $\tau(V)$ is the (local) physical time. Then by this definition of the usual physical time, time is of local nature in the sense that for any object V of t-site S, $\tau(V)$ may exist only locally and may not be globally extended. (See the first paragraph of the second section.) For the dependency of τ on a (ur-)particle is a consequence of (ur-)entanglement as we noted earlier.

Axiom 3.2 (Interpretation of the physical space as a sheaf). Let κ be the sheaf associated with the physical space with dimension d. That is, for an object V of S, $\kappa(V)$ is the local physical space in C_1 (or in C_2) of dimension d. Then decompose $\kappa(V)$ as $\kappa(V) = (\kappa(V)^3, \kappa(V)^{d-3})$ so that $\kappa(V)^3$ may be the observable object of C_1, and $\kappa(V)^{d-3}$ may be non-observable in C_1.

Note 3.3. A motivation for Axioms 3.1 and 3.2 is the following. The objects κ and τ in $\hat{S}$ are not only presheaves but also need to be sheaves so that the discrete concept of (pre)sheaves can give the continuum notion of space-time in macrocosm when local data agree on overlaps as in the definition of a sheaf, see [10] a for more thorough treatment of sheaves κ and τ. In [1], as an application to quantum gravity, the following case is considered: let m, m',

P, P' κ and τ be in $\hat{S}$ and let $V \xrightarrow{\varepsilon} U$ be a micromorphism in S, where $m(V)$ and $m'(V)$ have non-zero mass and the intersection of $\kappa(V)$ and $\kappa(U)$ is not empty. Then the commutative diagram is induced from $V \xrightarrow{\varepsilon} U$ and morphisms among m, m', P, and P'. When $m(V)$ is massless, the morphism from $P(V)$ to $P'(V)$ becomes a Lorentz morphism.

Conclusion. – With our model in terms of t-topos $\hat{S}$, particles, space and time are presheafified, and then the interplay among the concepts of observation, wave-particle ur-states, uncertainty principles, non-locality (entanglement), and quantum fluctuation are phrased in terms of objects, morphisms, and sheaves. The sequence of dependency is the following:

$$\hat{S} \xrightarrow{\text{evaluated at } Ob(S)} \prod_{\alpha \in \Delta} C_\alpha \xrightarrow{\text{projection}} C_1 \text{ (or } C_2).$$

* * *

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