

# Expected Utility Inequalities\*

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## Abstract

Suppose we know the utility function of a risk averse decision maker who values a risky prospect  $\mathbf{X}$  at a price  $CE$ . Based on this information alone I develop upper bounds for the tails of the probabilistic belief about  $\mathbf{X}$  of the decision maker. I also illustrate how to use these *expected utility bounds* in a variety of applications, which include the estimation of risk measures from observed data, option valuation, credit risk and the equity premium puzzle.

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# 1 Introduction

Suppose we know the utility function of a risk averse decision maker who values a risky prospect  $\mathbf{X}$  at a price  $CE$ . Based on this very limited information, can we know anything whatsoever about the beliefs held by the decision maker about  $\mathbf{X}$ ?

The surprising answer is that we can, and the reason is that  $CE$  cannot be too high if the decision maker believes with high probability that the risky prospect will perform extremely poorly. Similarly, the  $CE$  cannot be too low if the decision maker believes with high probability that the risky prospect will perform extremely well. The main result of this paper is an upper bound for the tails of the probabilistic belief of an expected utility decision maker; bounds that depend on the utility function and on  $CE$ , but that do not depend on assumptions made on the shape of the probability distribution that describes the beliefs of the decision maker.

In the paper I also explore applications of these *expected utility bounds* to a variety of economic problems, which include the estimation of risk measures from observed data, the valuation of options, the analysis of credit risk and the equity premium puzzle.

## 2 The Setup

Consider a risky prospect  $\mathbf{X}$  that takes values in a set of possible outcomes  $\mathcal{C} \subseteq \mathbb{R}$ . The decision maker is risk averse and has preferences over risky prospects given by an increasing and continuous (Bernoulli) utility function  $u : \mathcal{C} \rightarrow \mathbb{R}$ , known to the analyst, and by some probability distribution function  $P$  over the (measurable) subsets of  $\mathcal{C}$ . This probability distribution is unknown to the analyst.

Suppose that, in addition, the analyst observes that the decision maker's certainty equivalent for  $\mathbf{X}$  is equal to  $CE$ . The main goal of this paper is to see if we can recover some useful information about  $P$  from the fact that  $\mathbf{X}$  is defined over  $\mathcal{C}$ , the decision maker has utility function  $u$  and values the risky prospect at  $CE$ .

Without loss of generality, I normalize the origin and units of  $u$  in such a way that  $\bar{u} := \sup_{x \in \mathcal{C}} u(x) = 0$ , when an upper bound for  $u$  exists on  $\mathcal{C}$ , and  $\underline{u} := \inf_{x \in \mathcal{C}} u(x) = -1$ , when a lower bound for  $u$  exists on  $\mathcal{C}$ .<sup>1</sup>

Throughout the paper all random variables will be denoted by bold letters.

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<sup>1</sup>To be sure, given any increasing and continuous utility function  $v$  with upper and lower bound on  $\mathcal{C}$  given respectively by  $\bar{v}$  and  $\underline{v}$ , define  $u$  by  $u(x) = \frac{v(x) - \bar{v}}{(\bar{v} - \underline{v})}$ . It can be checked that  $u(x) \leq 0$  for all  $x$  and  $\underline{u} = -1$  as desired.

### 3 The Result

The main results of the paper is the following.

**Theorem 1** *Suppose  $\mathbf{X}$  is a risky prospect valued at  $CE$  by a decision maker with utility function  $u$ . Then, for  $k > 0$ ,*

(i) *If  $\bar{u} = 0$ ,*

$$P(\mathbf{X} \leq CE - k) \leq \frac{u(CE)}{u(CE - k)}, \text{ and}$$

(ii) *If  $\underline{u} = -1$ ,*

$$P(\mathbf{X} \geq CE + k) \leq \frac{u(CE) + 1}{u(CE + k) + 1}.$$

**Proof.** Let  $s_k = \sup\{u(x) : x \leq CE - k\}$ . Since  $u$  is increasing,  $s_k = u(CE - k)$ . The definition of  $s_k$  and the fact that  $u(x) \leq 0$  for all  $x \in \mathcal{C}$  imply that

$$u(CE - k) 1_{\{\mathbf{X} \leq CE - k\}} \geq u(\mathbf{X}) 1_{\{\mathbf{X} \leq CE - k\}} \geq u(\mathbf{X}).$$

Now take expected values to obtain

$$u(CE - k) P(\mathbf{X} \leq CE - k) \geq Eu(\mathbf{X}).$$

By the definition of the certainty equivalent  $CE$  of a risky prospect,  $Eu(\mathbf{X}) = u(CE)$ . Using this fact and rearranging yields (i).

To prove (ii) let  $i_k = \inf \{u(x) : x \geq CE + k\}$ . Since  $u$  is increasing,  $i_k = u(CE + k)$ . The definition of  $i_k$  and the fact that  $u(x) \geq -1$  for all  $x \in \mathcal{C}$  imply that

$$u(\mathbf{X}) \geq u(CE + k) 1_{\{\mathbf{X} \geq CE + k\}} + (-1) 1_{\{\mathbf{X} \leq CE + k\}}.$$

Now take expected values to obtain

$$Eu(\mathbf{X}) = u(CE) \geq u(CE + k) P(\mathbf{X} \geq CE + k) - [1 - P(\mathbf{X} \geq CE + k)].$$

Rearranging this expression yields (ii). ■

**Remark 1** *An upper bound for  $u$  arises naturally, no matter the shape of  $\mathcal{C}$ , if  $u$  is bounded from above, as with constant absolute risk aversion preferences. Alternatively, an upper (resp. lower) bound for  $u$  arises, no matter the*

shape of  $u$ , if  $C$  is bounded above (resp. below), as in Rothschild and Stiglitz (1970, 1971). I use both kinds of assumptions in the applications developed below.

**Remark 2** *The reader may notice that the proof of this result works in the same way of that for Chebyshev's inequality, an inequality that provides estimates of the tail of a probability distribution based solely on estimates of its mean and standard deviation. There is a sense in which the theorem above provides an estimate of the lower tail of the distribution in the same way, but by using information about economically meaningful variables, such as  $CE$  and  $u$ , rather than mainly statistical measures such as the mean and the standard deviation of  $\mathbf{X}$ .*

The application in the next section makes this interpretive point more precise.

## 4 An unexpected use for the coefficient of risk tolerance

Consider a decision maker with preferences given by a Bernoulli utility function that satisfies constant absolute risk aversion, that is,

$$u(x) = -e^{-\frac{x}{r}},$$

for  $r > 0$ . The parameter  $r$  in this formulation measures the decision maker's tolerance for risk. The following facts are standard: (i) As  $r$  decreases, the individual becomes more more risk averse, (ii) as  $r$  approaches infinity, the individual tends towards risk neutral behavior, (iii) the coefficient of absolute risk aversion for this decision maker is given by  $1/r$ .

It turns out that the following is true: An individual with a constant risk tolerance of  $r$  that values a risky prospect  $\mathbf{X}$  at a price  $CE$  cannot assign probability greater than 14% to  $\mathbf{X}$  taking values less than  $2r$  to the left of  $CE$ . That is, for this decision maker,  $P(\mathbf{X} \leq CE - 2r) \leq 14\%$ , an estimate made without making any assumptions about the shape of the distribution of  $\mathbf{X}$ .

To see how this is true simply notice that the utility function being considered is bounded above by zero and hence, by Theorem 1

$$P(\mathbf{X} \leq CE - k) \leq e^{\frac{CE - k - CE}{r}} = e^{-\frac{k}{r}}.$$

Replacing  $k$  with  $zr$  yields  $P(\mathbf{X} \leq CE - zr) \leq e^{-z}$ , and setting  $z = 2$  gives the desired result, as  $e^{-2} \approx 14\%$ . This simple fact is summarized in the theorem below.

**Theorem 2** *Suppose  $\mathbf{X}$  is a risky prospect valued at  $CE$  by a decision maker with constant risk tolerance parameter given by  $r$ . Then*

$$P(\mathbf{X} \leq CE - zr) \leq e^{-z} \tag{1}$$

**Remark 3** *It is instructive to compare this estimate with the one given by the one-sided Chebyshev's inequality*

$$P(\mathbf{X} \leq \mu - z\sigma) \leq \frac{1}{1 + z^2},$$

where  $\mu$  and  $\sigma$  are, respectively, the mean and the standard deviation of  $\mathbf{X}$  for this decision maker. Notice that  $z = 2$  this gives an upper bound for

$P(\mathbf{X} \leq \mu - 2\sigma)$  of 20%, an estimate made without making any assumptions about the shape of the distribution of  $\mathbf{X}$ .

One can say that, based on the above, there is a precise sense in which, for the purpose of coming up with upper bound estimates of the lower tail of the distribution of beliefs of the decision maker, *the certainty equivalent is to the mean of the belief distribution as the risk tolerance coefficient is to the standard deviation of the belief distribution.*

## 5 Value at Risk

Value at risk is a measure used to estimate how the value of an asset or of a portfolio of assets could decrease over a certain time period under usual conditions. Usual conditions in this context is defined to mean “under most circumstances,” which in turn is described with respect to a probabilistic confidence level, usually 95% or 99%.

Formally, the *Value at Risk at a confidence level of  $p$* , of a risky prospect  $\mathbf{X}$  worth  $CE$  to a decision maker is the maximum loss  $L$  that the decision maker expects to incur with probability  $p$ . It can be calculated by the fol-

lowing formula:

$$CE - (1 - p)^{th} \text{ percentile of } \mathbf{X}.$$

It turns out that the following is true:

**Theorem 3** *Suppose  $\mathbf{X}$  is a risky prospect valued at  $CE$  by a decision maker with invertible utility function  $u$  with  $\bar{u} = 0$ . Then the Value at Risk,  $L$ , at a confidence level of  $p$  for this decision maker must satisfy*

$$L \leq CE - u^{-1} \left[ \frac{u(CE)}{(1 - p)} \right]. \quad (2)$$

**Proof.** From Theorem 1 and from the fact that  $u(CE - L) < 0$  it follows that

$$u(CE - L) \geq \frac{u(CE)}{P(\mathbf{X} \leq CE - L)}.$$

Since  $u$  is increasing and invertible, and setting  $P(\mathbf{X} \leq CE - L) = 1 - p$ , then

$$CE - L \geq u^{-1} \left[ \frac{u(CE)}{1 - p} \right],$$

which gives a lower bound on the  $(1 - p)^{th}$  percentile of the distribution of  $\mathbf{X}$  according to the decision maker. Rearranging gives the desired result. ■

Notice that for the case of decision makers with constant risk tolerance  $r$  expression (2) reduces to

$$L \leq -r \ln(1 - p)$$

**Example 1** *Suppose  $\mathbf{X}$  is a risky prospect valued at 100 by a decision maker with constant risk tolerance parameter given by  $r = 10$ . Then an analyst that knows this about the decision maker will know that the Value at Risk at a confidence level of 95% for the decision maker will be below 30. That is, the analyst will know that the decision maker considers his losses to be below a certain threshold with probability 95% ; this threshold the analyst knows will be below 30.*

## 6 Variance and Half-Variance Bounds

It is standard in many applications to use the variance  $\sigma^2$  of a risky project  $\mathbf{X}$  as a way to measure its riskiness. One potential drawback to this method is that it is not always easy for an analyst to adequately estimate the variance

of  $\mathbf{X}$  that is implicit in the probabilistic beliefs of the decision maker.<sup>2</sup> In this Section I develop upper and lower bounds for  $\sigma^2$  that depend on the decision maker's attitudes towards risk, on the certainty equivalent of the risky prospect, and on its risk premium.

Consider the case of a decision maker who values some asset at  $CE$ . The following is standard: if we know the expected value of the asset,  $\mu$ , and  $\mu - CE > 0$ , then (a) the asset is risky (otherwise  $\mu - CE = 0$ ), and (b) the decision maker is risk averse (otherwise  $\mu - CE \leq 0$ ). Because of this, label the asset as  $\mathbf{X}$  (as a risky prospect) and call  $\mu - CE$  the *risk premium* of  $\mathbf{X}$ .

In this Section I establish, in addition, the following: if we know the value of the asset for the decision maker, the risk premium and the decision maker's attitudes towards risk, then it follows that the variance of  $\mathbf{X}$  cannot be too small (resp. too big) or we would observe a smaller (resp. bigger) risk premium. The details follow.

**Theorem 4** *Suppose  $\mathbf{X}$  is a risky prospect valued at  $CE$  and with risk premium given by  $RP$  for a decision maker with utility function  $u$ , with  $\underline{u} = -1$ .*

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<sup>2</sup>See, e.g., the discussion in Morgan and Henrion (1990). See also Myerson (2005).

Then the variance  $\sigma^2$  of  $\mathbf{X}$  for this decision maker satisfies

$$\sigma^2 \geq \max_{k \in [0, RP]} (RP - k)^2 \left[ 1 - \frac{u(CE) + 1}{u(CE + k) + 1} \right] > 0 \quad (3)$$

**Proof.** With  $\mu = RP + CE$  notice that, for  $k < RP$ ,

$$\inf \{ (\mathbf{X} - \mu)^2 : \mathbf{X} \leq CE + k \} = (RP - k)^2,$$

and hence

$$\begin{aligned} (\mathbf{X} - \mu)^2 &= (\mathbf{X} - \mu)^2 1_{\{\mathbf{X} \leq CE + k\}} + (\mathbf{X} - \mu)^2 1_{\{\mathbf{X} \geq CE + k\}} \\ &\geq (RP - k)^2 1_{\{\mathbf{X} \leq CE + k\}}. \end{aligned}$$

Taking expectations yields

$$\sigma^2 \geq (RP - k)^2 (1 - P(\mathbf{X} \geq CE + k)).$$

From Theorem 1,

$$1 - P(\mathbf{X} \geq CE + k) \geq \left( 1 - \frac{u(CE) + 1}{u(CE + k) + 1} \right)$$

and therefore

$$\sigma^2 \geq (RP - k)^2 \left( 1 - \frac{u(CE) + 1}{u(CE + k) + 1} \right)$$

for all  $k \in [0, RP]$ . Then the best lower bound for the variance is obtained

at

$$\max_{k \in [0, RP]} (RP - k)^2 \left( 1 - \frac{u(CE) + 1}{u(CE + k) + 1} \right),$$

which is greater than zero since the function to be maximized is continuous, non-negative, and takes strictly positive values in the interior of  $[0, RP]$ . ■

**Remark 4** *Another commonly used measure of risk is the half variance, defined as  $HV = E(\min\{\mathbf{X} - \mu, 0\}^2)$ . The half variance takes into account only the so-called “downside risk.” It is not hard to see that the bound in expression (3) is also a lower bound for the half variance of  $\mathbf{X}$ . I omit the details here.*

**Example 2** *Consider a decision maker with constant relative risk aversion preferences, with coefficient of relative risk aversion equal to 1/2. Assume the decision maker is evaluating a risky prospect  $\mathbf{X}$  that can take only non-negative values. If this decision maker values  $\mathbf{X}$  at 100 and believes the risk premium of this prospect to be 20 then the standard deviation of  $\mathbf{X}$  according to this decision maker is at least 2.38.*

For the next result in this Section it is necessary for  $\mathcal{C}$  to be bounded. Let  $a$  and  $b$  be the greatest lower bound and least upper bound of  $\mathcal{C}$ , respectively, with associated lower and upper bounds of  $u$  on  $\mathcal{C}$  given by  $\underline{u} = -1$  and  $\bar{u} = 0$ . Call a risky prospect  $\mathbf{X}$  defined over such set  $\mathcal{C}$  a *bounded project*. If  $\mathbf{X}$  is a bounded project, let  $b - \mu$  be the *upside potential* of  $\mathbf{X}$  and  $d = \min\{b - \mu, \mu - a\}$  be the *narrow range* of  $\mathbf{X}$ .

**Theorem 5** *Suppose  $\mathbf{X}$  is a bounded project valued at  $CE$  and with risk premium given by  $RP$  for a decision maker with utility function  $u$ . Assume further that the risk premium does not exceed the upside potential of  $\mathbf{X}$ . Then the variance  $\sigma^2$  of  $\mathbf{X}$  for this decision maker satisfies*

$$\sigma^2 \leq [(a - \mu)^2 - d^2] \frac{u(CE)}{u(\mu - d)} + [(b - \mu)^2 - d^2] \frac{u(CE) + 1}{u(\mu + d) + 1} + d^2$$

**Proof.** With  $\mu = RP + CE$  notice that, for  $k < d$ ,

$$\begin{aligned} (\mathbf{X} - \mu)^2 &= (\mathbf{X} - \mu)^2 1_{\{\mathbf{X} \leq \mu - k\}} + (\mathbf{X} - \mu)^2 1_{\{\mu - k \leq \mathbf{X} \leq \mu + k\}} + (\mathbf{X} - \mu)^2 1_{\{\mathbf{X} \geq \mu + k\}} \\ &\leq (a - \mu)^2 1_{\{\mathbf{X} \leq \mu - k\}} + k^2 1_{\{\mu - k \leq \mathbf{X} \leq \mu + k\}} + (b - \mu)^2 1_{\{\mathbf{X} \geq \mu + k\}} \end{aligned}$$

Taking expectations and rearranging yields

$$\sigma^2 \leq [(a - \mu)^2 - k^2] P(\mathbf{X} \leq \mu - k) + [(b - \mu)^2 - k^2] P(\mathbf{X} \geq \mu + k) + k^2.$$

To replace the probabilities with their upper estimates from Theorem 1 we require that  $k > RP$ . This is possible, since, by assumption, the risk premium does not exceed the upside potential of  $\mathbf{X}$ , and this guarantees that  $RP < d$ . Then

$$\sigma^2 \leq [(a - \mu)^2 - k^2] \frac{u(CE)}{u(\mu - k)} + [(b - \mu)^2 - k^2] \frac{u(CE) + 1}{u(\mu + k) + 1} + k^2 \quad (4)$$

for all  $k \in [RP, d]$ . It turns out that the right hand side of (4) is decreasing in  $k$  (I show this in Appendix 1). Then the best upper bound for the variance is obtained by replacing  $d$  in place of  $k$  in (4), which gives the desired result.

■

**Example 3** *Consider a decision maker with constant relative risk aversion preferences, with coefficient of relative risk aversion equal to 6. Assume the decision maker is evaluating a risky prospect  $\mathbf{X}$  that can take only non-negative values not greater than 150. If this decision maker values  $\mathbf{X}$  at 100 and believes the risk premium of this prospect is equal to 7 then the standard*

*deviation of  $\mathbf{X}$  according to this decision maker is at most 52.5.*

## **7 Option Valuation**

Having upper estimates of the probabilities of tail events for a decision maker can be of use to see how this decision maker would value financial instruments whose payoffs are tied to the occurrence of those events. In short, they can be of use for the valuation of options and other financial derivatives.

The ability to use the inequalities developed in the present paper to put “expected utility bounds” on option values is potentially very valuable, as it is known that parametric approaches to the estimation of those tail probabilities tend to produce distributions with tails that are either too thin or too unstable. Another potential problem is that sometimes we are dealing with one-of-a-kind projects for which there is not even any previous data to use in the estimation process.

The use of upper bounds to the probability of those tail events sidesteps completely the tail estimation problem, as it avoids making distributional assumptions. This, of course, can come at a cost if one ends up with bounds to the option values that are not very informative. Whether this is so or not

is an empirical question that deserves attention in its own right.

What follows illustrates the kinds of bounds on the value of options that arise from a judicious application of Theorem 1 to the valuation of out-of-the-money put options.<sup>3</sup>

The setup is, again, a risky prospect  $\mathbf{X}$  that takes values in a set of possible outcomes  $\mathcal{C}$ . The decision maker evaluates  $\mathbf{X}$  with an increasing utility function  $u$  and with respect to beliefs given by a distribution  $P$ , unknown to the analyst. All payoffs and prices in this Section are expressed in present value terms.

I am interested in how much this decision maker would value a contract that gives the decision maker the right but not the obligation to sell risky prospect  $\mathbf{X}$  at a predetermined (strike) price  $S$ . I am thus interested in the value for the decision maker of the risky prospect

$$\mathbf{T}_S = \max \{S - \mathbf{X}, 0\},$$

commonly known as a put option on  $\mathbf{X}$  with strike price  $S$ . To find the value of  $\mathbf{T}_S$  for the decision maker one has to find the certainty equivalent of  $\mathbf{T}_S$ ,

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<sup>3</sup>A similar exercise done on the valuation of call options produces bounds on the option values that are worse than well known “arbitrage bounds,” and it is therefore of little interest.

that is, the price  $Q_S$  such that

$$u(Q_S) = Eu(\mathbf{T}_S).$$

The following result will be needed in what follows.

**Lemma 6** *Suppose  $\mathbf{X}$  is a risky prospect to be valued by a decision maker with utility function  $u$ . Let  $\mathbf{T}_S$  be a put option on  $\mathbf{X}$  with strike price  $S$ . Then*

$$Eu(\mathbf{T}_S) \leq u(S)P(\mathbf{X} \leq S) + u(0)[1 - P(\mathbf{X} \leq S)] \quad (5)$$

**Proof.**

$$u(\mathbf{T}_S) = u(\max\{S - \mathbf{X}, 0\}) = u(S - \mathbf{X})1_{\{\mathbf{X} \leq S\}} + u(0)1_{\{\mathbf{X} > S\}}$$

Since  $u$  is increasing,  $u(S - x) \leq u(S)$ , so

$$u(\mathbf{T}_S) \leq u(S)1_{\{\mathbf{X} \leq S\}} + u(0)1_{\{\mathbf{X} > S\}},$$

and taking expectations yields the desired result. ■

Even if an analyst knows the utility function of the decision maker equa-

tion (5) is of little help if one has no information about the beliefs held by the decision maker. This is where an estimate of  $P(\mathbf{X} \leq S)$  can be of help. Such estimate is available for  $S < CE$ , that is, for options that are *out of the money*.

In what follows I assume that  $u$  is bounded from above and normalize the origin of  $u$  in such a way that  $u(x) \leq 0$  for all  $x \in \mathcal{C}$  and the units of the utility function in such a way that  $u(0) = -1$ .<sup>4</sup> Also, let  $\underline{P(\mathbf{X} > S)} = 1 - \frac{u(CE)}{u(S)}$ . From Theorem 1 it follows that one can interpret  $\underline{P(\mathbf{X} > S)}$  to be a lower bound on the probability that the decision maker assigns to  $\mathbf{X}$  taking values greater than  $S$ .

**Theorem 7** *Suppose  $\mathbf{X}$  is a risky prospect valued at  $CE$  by a decision maker with utility function  $u$  with  $\bar{u} = 0$  and for which  $u(0) = -1$ . Then, for  $S < CE$ ,*

$$Eu(\mathbf{T}_S) \leq Eu(\mathbf{X}) - \underline{P(\mathbf{X} > S)}. \quad (6)$$

**Proof.** From Lemma 6 we have that

$$Eu(\mathbf{T}_S) \leq u(S)P(\mathbf{X} \leq S) + u(0)[1 - P(\mathbf{X} \leq S)],$$

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<sup>4</sup>Notice that I am not assuming in this case that  $-1$  is the lower bound of  $u$  on  $\mathcal{C}$ .

while Theorem 1 gives us a bound on  $P(\mathbf{X} \leq S)$  :

$$P(\mathbf{X} \leq S) \leq \frac{u(CE)}{u(S)}.$$

Combining these two expressions and rearranging we get

$$Eu(\mathbf{T}_S) \leq \frac{u(CE)}{u(S)}u(S) - \left(1 - \frac{u(CE)}{u(S)}\right),$$

that is,

$$Eu(\mathbf{T}_S) \leq Eu(\mathbf{X}) - \underline{P(\mathbf{X} > S)}.$$

■

Expression (6) then reads as follows: the expected utility of holding the put option cannot exceed the expected utility of holding the original risky prospect, minus the analyst's lower estimate on the probability that the option will expire worthless.<sup>5</sup> This expression can be used to find an upper bound to the value of the option, since  $u(Q_S) = Eu(\mathbf{T}_S)$  and  $u(CE) = Eu(\mathbf{X})$ . I record the formula for obtaining such bound below.

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<sup>5</sup>Of course, discussing expression (6) in this way makes sense specifically for the normalization of the utility function chosen above.

**Corollary 8** *Suppose  $\mathbf{X}$  is a risky prospect valued at  $CE$  by a decision maker with invertible utility function  $u$  with  $\bar{u} = 0$  and for which  $u(0) = -1$ . Then, for  $S < CE$ ,*

$$Q_S \leq u^{-1} \left[ u(CE) - \left( 1 - \frac{u(CE)}{u(S)} \right) \right]. \quad (7)$$

Notice that for the case of decision makers with constant risk tolerance  $r$  expression (7) reduces to

$$Q_S \leq -r \ln \left[ 1 + e^{-\frac{CE}{r}} \left( 1 - e^{\frac{S}{r}} \right) \right]$$

**Example 4** *Suppose  $\mathbf{X}$  is a risky prospect valued at 100 by a decision maker with constant risk tolerance parameter. Table 1 shows upper bounds on the values of put options of different strike prices for varying degrees of tolerance for risk.*

		$S$						
		60	65	70	75	80	85	90
$r$	8	0.05	0.1	0.19	0.36	0.69	1.33	2.70
	10	0.18	0.31	0.51	0.86	1.45	2.52	4.59
	15	1.06	1.51	2.16	3.12	4.56	6.85	10.77

*Table 1. Upper bound on the price of put options given a strike price of  $S$  and constant risk tolerance  $r$  for a risky prospect worth 100 to the decision maker.*

*For example, a decision maker with constant risk tolerance coefficient given by  $r = 10$  that values  $\mathbf{X}$  at 100 will value a put option on  $\mathbf{X}$  with strike price 65 at some level that the analyst knows will never be above 0.31. Another way to put this is as follows: given that the decision maker values  $\mathbf{X}$  at 100 there is no probabilistic belief that the decision maker could have about  $\mathbf{X}$ , no matter how pessimistic it may be, that would justify the analyst believing that the decision maker would pay more than 0.31 for this option. In this sense, the estimates are robust to the tails of the belief distribution being fat or difficult to estimate.*

## 8 Credit Risk

The expected utility bounds developed in this paper can be used for estimating the probability that an entity with limited liability will default on its debt. They can also be used for estimating the value of this debt to some decision maker.

The setup is one in which there is an entity with limited liability that owns the proceeds of risky prospect  $\mathbf{X}$ . The entity has debts by an amount equal to  $B > 0$  that are due at the moment the value of  $\mathbf{X}$  is realized. As before, a decision maker evaluates  $\mathbf{X}$  with an increasing utility function  $u$  and with respect to beliefs given by a distribution  $P$ . The entity and the decision maker in this setup may be the same, although this is not necessary. All payoffs and prices in this Section are expressed in present value terms.

### 8.1 Default probabilities

The first result of this Section is an upper bound on the probability that the entity will default on its debt.

**Theorem 9** *Suppose  $\mathbf{X}$  is a risky prospect valued at CE by a decision maker with utility function  $u$  with  $\bar{u} = 0$ . Suppose the entity that owns  $\mathbf{X}$  has  $B > 0$*

in debts, with  $B < CE$ . Then the probability,  $P^d$ , for this decision maker, that the entity will default on its debt satisfies

$$P^d \leq \frac{u(CE)}{u(B)}$$

**Proof.** The entity enters default when the realized value of  $\mathbf{X}$  is not sufficient to cover the decision maker's obligations, that is, when  $\mathbf{X} < B$ . Hence,  $P^d = P(\mathbf{X} \leq B)$  and an application of Theorem 1 for the case where  $k = CE - B$  yields the desired result. ■

**Example 5** Suppose  $\mathbf{X}$  is a risky prospect valued at 100 by a decision maker that has a constant risk tolerance parameter given by  $r = 20$ . If the entity that owns  $\mathbf{X}$  owes 40 to creditors then this decision maker believes that the entity will default on its debt with probability  $P^d$ , a probability that the analyst knows is below 4.98%.

## 8.2 The Value of Risky Debt

Due to limited liability, owning a risky prospect  $\mathbf{X}$  with  $B$  in debts is just like owning a call option on  $\mathbf{X}$  with strike price  $B$ . This fact, together with the put-call parity, implies that the value of the risky debt equals the value

of riskless debt of identical face value minus the value of a put option on  $\mathbf{X}$  with strike price  $B$ , that is,

$$V_B = B - Q_B, \quad (8)$$

where  $Q_B$  is the certainty equivalent for this decision maker of  $\mathbf{T}_B = \max \{B - \mathbf{X}, 0\}$ .

One can therefore find lower bounds on the value of the risky debt to a particular decision maker using the results from Section 7.

**Theorem 10** *Suppose  $\mathbf{X}$  is a risky prospect valued at  $CE$  by a decision maker with invertible utility function  $u$  with  $\bar{u} = 0$  and for which  $u(0) = -1$ . If the entity that owns  $\mathbf{X}$  has debts given by  $B < CE$  then the decision maker values this debt by an amount  $V_B$  that satisfies*

$$V_B \geq B - u^{-1} \left[ u(CE) - \left( 1 - \frac{u(CE)}{u(B)} \right) \right].$$

**Proof.** Combine equation (7), with strike price equal to  $B$ , and equation (8) to get the desired result. ■

**Example 6** *Suppose  $\mathbf{X}$  is a risky prospect valued at 100 by a decision maker that has a constant risk tolerance parameter given by  $r = 30$ . If the entity*

*that owns  $X$  also owes 40 to creditors then this decision maker values this debt at a level  $V_B$ , a level that the analyst knows is above 36.9.*

## **9 The Equity Premium Puzzle**

In an economy populated with risk averse investors the return on stocks must exceed that of riskless bonds to compensate investors for the risk they bear when holding the stocks. The equity premium puzzle, simply stated, is the fact that the excess return of stocks over bonds in the data is much larger than predicted by the standard consumption based asset pricing model under most reasonable assumptions about the investors' attitudes towards risk.

As this is not a paper about asset pricing I will not go into developing the full details of the consumption based model here.<sup>6</sup> I will instead focus on explaining how inequalities like the ones developed above can shed light on whether assuming that the representative investor has probabilistic beliefs over the stock returns with a “fat left tail” could account for at least part of the puzzle.

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<sup>6</sup>For an excellent treatment of the relevant literature see, for example, Cochrane (2005a). I follow his presentation of the equity premium puzzle below.

## 9.1 The Puzzle

The starting point is the basic pricing equation in a consumption based model: that asset prices are generated by expected discounted payoffs,

$$p = E(\mathbf{m}\mathbf{x}), \quad (9)$$

where  $\mathbf{x}$  is the random payoff of some asset,  $\mathbf{m}$  is a stochastic discount factor and  $p$  is the price of the asset. As this equation must hold for all assets, in particular it holds for the gross asset returns  $\mathbf{R} = \mathbf{x}/p$ , and for the gross risk free asset return  $R^f$ , and therefore

$$1 = E(\mathbf{m}\mathbf{R}), \quad (10)$$

$$\text{and } 1 = E(\mathbf{m})R^f. \quad (11)$$

By definition,  $E(\mathbf{m}\mathbf{R}) = E(\mathbf{m})E(\mathbf{R}) + \sigma(\mathbf{m})\sigma(\mathbf{R})\rho_{\mathbf{m},\mathbf{R}}$ , where  $\rho_{\mathbf{m},\mathbf{R}}$  is the correlation coefficient between  $\mathbf{m}$  and  $\mathbf{R}$ . Then equation (10) becomes, after rearranging and recognizing that,  $|\rho_{\mathbf{m},\mathbf{R}}| \leq 1$ ,

$$\frac{|E(\mathbf{R}) - R^f|}{\sigma(\mathbf{R})} \leq \frac{\sigma(\mathbf{m})}{E(\mathbf{m})}. \quad (12)$$

The equity premium puzzle can be expressed using expression (12) as follows:<sup>7</sup> according to postwar US data  $E(\mathbf{R}) \approx 1.09$ ,  $E(\mathbf{m})^{-1} = R^f \approx 1.01$  and  $\sigma(\mathbf{R}) \approx 16\%$ , which, using expression (12), translates into  $\sigma(\mathbf{m}) \geq 50\%$  on an annual basis.

To see the problem with such high discount factor volatility consider that time separable utility and constant relative risk aversion preferences implies that

$$\sigma(\mathbf{m})/E(\mathbf{m}) = \gamma\sigma(\Delta\mathbf{c}), \quad (13)$$

where  $\gamma$  is the coefficient of relative risk aversion of the representative investor and  $\Delta\mathbf{c}$  is aggregate consumption growth. With the consumption growth volatility of about 1.5% per year observed in US postwar data, this implies that the coefficient of relative risk aversion consistent with the data satisfies  $\gamma \geq 50\%/1.5\% = 33$ , which is a level that seems much larger than most economists find acceptable.

A huge literature has developed that attempts to expand and/or explain this puzzle. In the rest of this Section I build an argument based on the inequalities developed above that attempts to shed light on whether belief

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<sup>7</sup>The facts presented below are standard and they can be found, for example, in Cochrane (2005b).

distributions of returns with a fat left tail could account for at least part of the puzzle.

## 9.2 Part of the solution?

One possibility for reconciling expression (12) with the data stems from the recognition that (12) involves expectations of random variables, and those expectations are to be computed with respect to the probabilistic beliefs of the representative investor. Notice, however, that the computations used in the literature to illustrate the puzzle replace those expectations with historical sample moments of the variables of interest. It is therefore plausible that part of the problem is that we are bringing the wrong magnitudes into expression (12).<sup>8</sup>

In this Section I adopt the perspective that, in particular, the problem is most severe with regards to the standard deviation of returns. Put simply, the representative investor may believe that the stock market is much riskier than implied by a standard deviation of 16% but such perceived riskiness need not be reflected in the data, as the tails of the empirical distribution of returns may understate the “thickness” of the left tail of the belief distribution of the

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<sup>8</sup>In this sense the present Section adopts a perspective similar to that of Weitzman (2005), Barro (2005) and Rietz (1988).

representative investor. Put simply, the representative investor may believe that stock market crashes are far more common than the data suggests, and hence require a high premium as compensation for holding stocks. The following question arises: how likely do very negative returns have to be to the representative investor for the observed equity premium to be consistent with expression (12) under “reasonable” levels of risk aversion?

Below I show how bounds like the ones developed earlier in this paper may be of help to address this question. I first find an upper bound to the right tail of the belief distribution over the returns of any asset, given that we know the investor’s expected return for the asset. This last bound, jointly with expression (12), and a "reasonable" value for the coefficient of relative risk aversion, are sufficient to produce a lower bound on the probability that the risky assets perform extremely poorly.

### **9.3 The probability of very poor returns**

Consider a decision maker that estimates the expected gross return for a risky asset to be equal to  $E(\mathbf{R})$ . Those returns may or may not be bounded according to the decision maker. Let  $a$  and  $b$  be the greatest lower bound and least upper bound for  $\mathbf{R}$ , according to the decision maker, when they

exist.

**Theorem 11** *Assume the expected gross return for a risky asset is equal to  $E(\mathbf{R})$  for some decision maker. Then, for  $k > 0$ ,*

(i) *If  $\mathbf{R}$  is bounded above by  $b$ ,*

$$P(\mathbf{R} \leq E(\mathbf{R}) - k) \leq \frac{b - E(\mathbf{R})}{b - E(\mathbf{R}) + k}, \text{ and}$$

(ii) *If  $\mathbf{R}$  is bounded below by  $a$ ,*

$$P(\mathbf{R} \geq E(\mathbf{R}) + k) \leq \frac{E(\mathbf{R}) - a}{E(\mathbf{R}) - a + k}.$$

**Proof.** The proof is similar to the proof of Theorem 1. I present the details in Appendix 2. ■

Now suppose for a moment that a lower bound to the standard deviation of returns for the decision maker is known. This, in combination with Theorem 11, is sufficient information for the derivation of a lower bound to the probability of  $\mathbf{R}$  taking values substantially below  $E(\mathbf{R})$ .

**Theorem 12** *Assume the expected gross return for a risky asset is equal to  $E(\mathbf{R})$  according to some decision maker. Assume that to this decision maker*

the returns are bounded below and above by  $a$  and  $b$ , respectively, and that the variance  $\sigma^2$  of  $\mathbf{R}$  for this decision maker is bounded below by  $z$ . Then, for  $k \in (0, E(\mathbf{R}) - a)$

$$P(\mathbf{R} \leq E(\mathbf{R}) - k) \geq \frac{z - [(b - E(\mathbf{R}))^2 - k^2] \frac{E(\mathbf{R}) - a}{E(\mathbf{R}) - a + k} - k^2}{(a - E(\mathbf{R}))^2 - k^2}$$

**Proof.** The proof is similar to the proof of Theorem 5. I present the details in Appendix 3. ■

## 9.4 Example

The example illustrates how to use the bounds developed above to answer the following: How likely do very negative returns have to be to the representative investor for the observed equity premium to be consistent with expression (12) under “reasonable” levels of risk aversion?

Assume that  $\gamma \leq 5$  is the upper bound on the level of relative risk aversion that is considered “reasonable” for the representative investor in a standard consumption-based asset pricing model with constant relative risk aversion preferences, such as the one described at the beginning of this Section. Plugging this number into expression (13) with a consumption growth volatility

of 1.5% and  $R^f \approx 1.01$  implies that  $\sigma(\mathbf{m}) \leq 7.43\%$  on an annual basis.

From this information, expression (12) and an expected gross return for the risky asset given by  $E(\mathbf{R}) \approx 1.09$ , as in the historical data, it follows that the the standard deviation of returns for the representative investor that is consistent with the observed premium satisfies

$$\sigma(\mathbf{R}) \geq \frac{E(\mathbf{m}) |E(\mathbf{R}) - R^f|}{\sigma(\mathbf{m})} \geq \frac{|1.09 - 1.08|}{1.01 \times 7.43\%},$$

in other words, it would have to be *at least* 107% in annual terms. This makes sense only if the representative investor puts sufficiently high probability on returns being either extremely positive or extremely negative. Is this a sensible conclusion to believe?

To shed light on this issue assume that according to the representative investor the gross return for the risky asset is *never* above  $2 * E(\mathbf{R}) = 2.18$  in any given year although it can be as low as zero. These are very, very ample bounds.<sup>9</sup> Then, by Theorem 12, the probability of the risky asset performing

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<sup>9</sup>For comparison, the maximum and minimum gross returns of the S&P composite index over the period 1873-2004 were 1.46 and 0.54 respectively.

very poorly satisfies

$$P(\mathbf{R} \leq 1.09 - k) \geq \frac{1.14 - [1.19 - k^2] \frac{1.09}{1.09+k} - k^2}{1.19 - k^2},$$

which has to be satisfied for all  $k \in (0, 1.09)$ . This inequality is most demanding for  $k \approx .82$ , point at which

$$P(\mathbf{R} \leq 1.09 - .82) = P(\mathbf{R} \leq .27) \geq 33\%.$$

The implication is that the representative investor has to believe that *net* returns of  $-73\%$  per year or worse may occur with probability of at least  $1/3$  for the observed equity premium to be consistent with expression (12) if the representative investor has a coefficient of relative risk aversion that is no greater than 5 and is convinced that the risky asset never yields annual *net* returns above  $118\%$ .

The assumptions made here are not restrictive at all, use inequalities that, by design, provide ample bounds in general, yet they require the representative investor to assign a quite sizable probability to very poor returns. In that sense, belief distributions of returns would have to have an extremely fat left tail for them to be able to explain, by themselves, the equity premium

puzzle.

To be sure, this is not meant to be a definitive answer on the issue. Rather, the point of this Section has simply been to illustrate the applicability of the bounds espoused in this paper to a well-known problem in financial economics; to show that they could be useful tools in the elucidation of an answer to this puzzle.

## 10 Conclusions

In this paper I developed expected utility bounds to the tails of the probabilistic beliefs of a risk averse decision maker over a risky prospect based on economic magnitudes such as the certainty equivalent and the risk premium of the risky prospect, and on the decision maker's attitudes towards risk. I also developed applications of these expected utility bounds to several economic problems, such as the estimation of risk measures, option valuation, credit risk and the equity premium puzzle. The bounds are very general in that they require virtually no knowledge about the functional form of the probabilistic beliefs of the decision maker.

As shown in the applications, the bounds can be used in one of two ways:

(i) to generate estimates of certain unobservable variables, based on what is observable, and (ii) as an intermediate step in a more elaborate theoretical argument. Consequently, they should be of interest to both theoretical and empirical researchers alike.

## APPENDIX 1

I wish to show that

$$(a - \mu)^2 \frac{u(CE)}{u(\mu - k)} + (b - \mu)^2 \frac{u(CE) + 1}{u(\mu + k) + 1} + k^2 \left[ 1 - \frac{u(CE)}{u(\mu - k)} - \frac{u(CE) + 1}{u(\mu + k) + 1} \right]$$

is strictly decreasing in  $k$ . Let  $A = (a - \mu)^2$ ,  $B = (b - \mu)^2$ ,  $f(k) = \frac{u(CE)}{u(\mu - k)}$ ,  $g(k) = \frac{u(CE) + 1}{u(\mu + k) + 1}$  and  $h(k) = 1 - f(k) - g(k)$ . The expression above becomes

$$Af(k) + Bg(k) + k^2h(k). \tag{14}$$

It is not hard to see that  $0 < f(k) < 1$ ,  $0 < g(k) < 1$ ,  $h(k) < 0$ , and that  $f'(k) < 0$ ,  $g'(k) < 0$ ,  $h'(k) = -(f'(k) + g'(k)) > 0$ .

The first derivative of (14) with respect to  $k$  is given by

$$Af'(k) + Bg'(k) + 2k h(k) + k^2h'(k),$$

or

$$(A - k^2) f'(k) + (B - k^2) g'(k) + 2k h(k). \quad (15)$$

The fact that every summation term in (15) is negative for  $k < d$  completes the proof.

## APPENDIX 2

Write the gross return as

$$\begin{aligned} \mathbf{R} &= \mathbf{R} 1_{\{\mathbf{R} \leq E(\mathbf{R}) - k\}} + \mathbf{R} 1_{\{\mathbf{R} \geq E(\mathbf{R}) - k\}} \\ &\leq (E(\mathbf{R}) - k) 1_{\{\mathbf{R} \leq E(\mathbf{R}) - k\}} + b 1_{\{\mathbf{R} \geq E(\mathbf{R}) - k\}}. \end{aligned}$$

Taking expectations we get

$$E(\mathbf{R}) \leq (E(\mathbf{R}) - k) P(\mathbf{R} \leq E(\mathbf{R}) - k) + b [1 - P(\mathbf{R} \leq E(\mathbf{R}) - k)].$$

Rearranging this expression produces expression (i) in Theorem 11.

To show part (ii) of the Theorem notice that

$$\begin{aligned} \mathbf{R} &= \mathbf{R} 1_{\{\mathbf{R} \leq E(\mathbf{R}) + k\}} + \mathbf{R} 1_{\{\mathbf{R} \geq E(\mathbf{R}) + k\}} \\ &\geq a 1_{\{\mathbf{R} \leq E(\mathbf{R}) + k\}} + (E(\mathbf{R}) + k) 1_{\{\mathbf{R} \geq E(\mathbf{R}) + k\}}. \end{aligned}$$

Taking expectations we get

$$E(\mathbf{R}) \geq a [1 - P(\mathbf{R} \geq E(\mathbf{R}) + k)] + (E(\mathbf{R}) + k) P(\mathbf{R} \geq E(\mathbf{R}) + k).$$

Rearranging produces the desired result.

### APPENDIX 3

Notice that, for  $k < d := \min\{E(\mathbf{R}) - a, b - E(\mathbf{R})\}$ ,

$$\begin{aligned} (\mathbf{R} - E(\mathbf{R}))^2 &= (\mathbf{R} - E(\mathbf{R}))^2 1_{\{\mathbf{R} \leq E(\mathbf{R}) - k\}} \\ &\quad + (\mathbf{R} - E(\mathbf{R}))^2 1_{\{E(\mathbf{R}) - k \leq \mathbf{R} \leq E(\mathbf{R}) + k\}} \\ &\quad + (\mathbf{R} - E(\mathbf{R}))^2 1_{\{\mathbf{R} \geq E(\mathbf{R}) + k\}} \\ &\leq (a - E(\mathbf{R}))^2 1_{\{\mathbf{R} \leq E(\mathbf{R}) - k\}} + k^2 1_{\{E(\mathbf{R}) - k \leq \mathbf{R} \leq E(\mathbf{R}) + k\}} \\ &\quad + (b - E(\mathbf{R}))^2 1_{\{\mathbf{R} \geq E(\mathbf{R}) + k\}} \end{aligned}$$

Taking expectations and rearranging yields

$$\begin{aligned} \sigma^2 &\leq [(a - E(\mathbf{R}))^2 - k^2] P(\mathbf{R} \leq E(\mathbf{R}) - k) \\ &\quad + [(b - E(\mathbf{R}))^2 - k^2] P(\mathbf{R} \geq E(\mathbf{R}) + k) + k^2. \end{aligned} \quad (16)$$

Replace  $P(\mathbf{R} \geq E(\mathbf{R}) + k)$  in (16) with its upper bound, as in Theorem 11, and  $\sigma^2$  with its lower bound,  $z$ , and get

$$z \leq [(a - E(\mathbf{R}))^2 - k^2] P(\mathbf{R} \leq E(\mathbf{R}) - k) + [(b - E(\mathbf{R}))^2 - k^2] \frac{E(\mathbf{R}) - a}{E(\mathbf{R}) - a + k} + k^2. \quad (17)$$

Solve for  $P(\mathbf{R} \leq E(\mathbf{R}) - k)$  to get the desired result.

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