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ON THE NOTION OF PRECOHOMOLOGY

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Dedicated to Professor SAUL LUBKIN.

ABSTRACT. For a cochain complex one can have the cohomology functor. In this paper we introduce the notion of precohomology for a cochain that is not a complex, i. e., $d^{q+1} \circ d^q$ *may not be zero. Such a cochain, with objects and morphisms of an abelian category A, is called a cochain precomplex whose category is denoted by Pco (A). If a cochain precomplex is actually a cochain complex, then the notion of precohomology coincides with that of cohomology, i. e., precohomology is a generalization of cohomology. For a left exact functor F from an abelian category A to an abelian category B, the hyperprecohomology of F is defined, and some properties are given. In the last section, a generalization of an inverse limit, called a preinverse limit, is introduced. We discuss some of the links between precohomology and preinverse limit.*

Introduction

Let **Z** be the ring of integers and let A be an abelian category. Suppose a sequence of objects and morphisms in A is given

$$\dashrightarrow C^{q-1} \xrightarrow{d^{q-1}} C^q \xrightarrow{d^q} C^{q+1} \xrightarrow{d^{q+1}} \cdots,$$

which may not satisfy $d^q \circ d^{q-1} = 0$ for certain $q \in \mathbf{Z}$. Then one may not be able to take the cohomology at C^q. We will introduce a functor

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for such a cochain by initially complexifying the cochain to a cochain complex, then taking the cohomology of the complex. For diagram (or element) chasing, we use an exact imbedding of A into the category of abelian groups. It should be noted that precohomology is a self--dual construction and that it is not an exact connected sequence of functors. Furthermore, for each $n \in \mathbf{Z}$, Ph^n is half exact. Hence, they are not derived functors, see § 1.

1. Precohomology

Let A be an abelian category, and let Co (A) and Co^+ (A) be the categories of cochain complexes and positive cochain complexes of objects in A, respectively.

DEFINITION 1.1. A sequence of objects and morphisms of A,

$$\cdots \longrightarrow C^{q-1} \xrightarrow{d^{q-1}} C^q \xrightarrow{d^q} C^{q+1} \xrightarrow{d^q} \cdots$$

is said to be a cochain precomplex, whose category is denoted by Pco (A), and Pco^+ (A) denotes the category of positive cochain precomplexes. A morphism $(f_q)_{q \in \mathbf{Z}}: (C^q, d^q)_{q \in \mathbf{Z}} \to (D^q, e^q)_{q \in \mathbf{Z}}$ in Pco (A) is a sequence of morphisms $f_q : C_q \to D_q$ such that the diagram

$$\begin{array}{ccccccc}
\cdots \longrightarrow & C^q & \xrightarrow{d^q} & C^{q+1} & \to \cdots \\
& \downarrow f_q & & \downarrow f_{q+1} & \\
\cdots \longrightarrow & D^q & \xrightarrow{e^d} & D^{q+1} & \to \cdots
\end{array}$$

commutes, i e., $f_{q+1} \circ d^q = e^q \circ f_q$ for $q \in \mathbf{Z}$.

NOTE. A cochain precomplex $(C^q, d^q)_{q \in \mathbf{Z}}$ is a cochain complex if $d^{q+1} \circ d^q = 0$ for $q \in \mathbf{Z}$.

LEMMA 1.2. Let $(C^q, d^q)_{q \in \mathbf{Z}}$ be an object in Pco (A). Then $(C^q/\mathrm{Im}\, d^{q-1} \circ d^{q-2}, "d^q")_{q \in \mathbf{Z}}$, abbreviated as $("C^q")_{q \in \mathbf{Z}}$, is an object in Co (A), where $"d^q"$ is the morphism induced by d^q as will be described below in the proof.

Proof. Let.

$$\dashrightarrow C^{q-2} \xrightarrow{d^{q-2}} C^{q-1} \xrightarrow{d^{q-1}} C^q \xrightarrow{d^q} \cdots$$

be a cochain precomplex in Pco (A). Then the morphism $''d^q''$ is defined as the morphism

$$C^q/\mathrm{Im}\ d^{q-1} \circ d^{q-2} \xrightarrow{''d^q''} C^{q+1}/\mathrm{Im}\ d^q \circ d^{q-1}$$

such that $''d^q''([c^q]) = [d^q c^q]$ in $C^{q+1}/\mathrm{Im}\ d^q \circ d^{q-1}$ for $[c^q] \in C^q/\mathrm{Im}\ d^{q-1} \circ d^{q-2}$. Note $''d^q''$ is well-defined. It remains to demonstrate that $''d^{q+1}'' \circ ''d^q''([c^q]) = 0$. By the above definition, $''d^{q+1}'' \circ ''d^q''([c^q]) = [d^{q+1} \circ d^q (c^q)] = 0$ holds in $C^{q+2}/\mathrm{Im}\ d^{q+1} \circ d^q$.

REMARK. The assigment of an object $(C^q, d^q)_{q \in \mathbf{Z}}$ in Pco (A) to the object $(C^q/\mathrm{Im}\ d^{q-1} \circ d^{q-2}, ''d^q'')_{q \in \mathbf{Z}}$ is a right exact functor.

NOTE. We call this process (functor) $(C^q, d^q) \underset{q \in \mathbf{Z}}{\leadsto} (''C^q'', ''d^q'')_{q \in \mathbf{Z}}$ the complexifying functor of the precomplex $(C^q, d^q)_{q \in \mathbf{Z}}$.

DEFINITION 1.3. For an object $(C^q, d^q)_{q \in \mathbf{Z}}$ in Pco (A), define the q-th precohomology of $(C^q, d^q)_{q \in \mathbf{Z}}$, denoted as $Ph^q(C^*)$, by

$$Ph^q(C^*) = H^q(\dashrightarrow C^q/\mathrm{Im}\ d^{q-1} \circ d^{q-2} \xrightarrow{''d^q''} \cdots)$$

$$= \mathrm{Ker}\ ''d^q''/\mathrm{Im}\ ''d^{q-1}'',$$

i. e., by the q-th cohomology of the cochain complex derived from the cochain precomplex $(C^q, d^q)_{q \in \mathbf{Z}}$.

NOTE. We have $\mathrm{Ker}\ ''d^q'' = \{[c^q] \in C^q/\mathrm{Im}\ d^{q-1} \circ d^{q-2} \mid d^q(c^q - d^{q-1} c^{q-1}) = 0$ for some $c^{q-1} \in C^{q-1}\}$ and $\mathrm{Im}\ ''d^{q-1}'' = \{[c^q] \in C^q/\mathrm{Im}\ d^{q-1} \circ d^{q-2} \mid c^q = d^{q-1}(c^{q-1})$ for some $c^{q-1} \in C^{q-1}\}$.

From this note, we plainly have the following proposition.

PROPOSITION 1.4. Precohomology is a generalization of cohomology in the sense that precohomology coincides with cohomology in the case when a cochain precomplex is a cochain complex.

DEFINITION 1.5. Let $(C^q, d^q)_{q\in\mathbf{Z}}$ be a cochain precomplex in Pco (A), then the dual-complexifying functor of the precomplex $(C^q, d^q)_{q\in\mathbf{Z}}$ is defined as $(\mathrm{Ker}\, d^{q+1}\circ d^q, 'd^{q}{}')_{q\in\mathbf{Z}}$, where $'d^q{}'$ is the restriction of d^q on the subobject $\mathrm{Ker}\, d^{q+1}\circ d^q$ of C^q. The object which was obtained above is a cochain complex, denoted by $('C^q{}', 'd^q{}')_{q\in\mathbf{Z}}$ or simply by $('C^q{}')_{q\in\mathbf{Z}}$. Define the q-th dual-precohomology $'Ph^q(C^*)$ of a precomplex C^* as

$$'Ph^q(C^*) = \mathrm{Ker}\ 'd^q{}' / \mathrm{Im}\ 'd^{q-1}{}'.$$

THEOREM 1.6. (Self-Duality of Precohomology). The canonical map from $'C^q{}'$ to $''C^q{}''$ induces an isomorphism from $'Ph^q(C^*)$ to $Ph^q(C^*)$ for each $q\in\mathbf{Z}$.

Proof. We will give a proof using [4]. Let us denote the canonical map $'Ph^q(C^*)\to Ph^q(C^*)$ by Φ, i. e., for the cohomologous class $\bar{x}$ of $x\in\mathrm{Ker}\ 'd^q{}'$ $\Phi(\bar{x}) = \overline{\pi_q(i_q x)}$, where i is the monomorphism $\mathrm{Ker}\ d^{q+1}\circ d^q\to C^q$ and π_q denotes the projection $C^q\to C^q/\mathrm{Im}\ d^{q-1}\circ d^{q-2}$. Notice $\overline{\pi_q(i_q x)} = \overline{[x]}$, where $[x]\in {}''C^q{}'' = C^q/\mathrm{Im}\ d^{q-1}\circ d^{q-2}$. This map is well-defined since $''d^q{}''([x]) = 0$ holds in $''C^{q+1}{}''$. This is because $x\in\mathrm{Ker}\ 'd^q{}'$, i. e., $'d^q{}'(x) = d^q(x) = 0$ in $'C^{q+1}{}'$. First we will show that Φ is monomorphic. Suppose $\overline{[x]} = 0$, then $[x]\in\mathrm{Im}\ ''d^{q-1}{}''$. Hence $x = d^{q-1}(x^{q-1})$ as in the note after Def 1.3. We need to check $x^{q-1}\in\mathrm{Ker}\ d^q\circ d^{q-1} = {}'C^{q-1}{}'$. $d^q d^{q-1}(x^{q-1}) = d^q x = 0$ holds from the above. Secondly, we will prove Φ is epimorphic. Let $\overline{[x]}\in Ph^q(C^*)$. Then, since $[x]\in\mathrm{Ker}\ ''d^q{}''$, $d^q(x - d^{q-1}x') = 0$ holds for some $x'\in C^{q-1}$. Then $\Phi(\overline{x - d^{q-1}x'}) = \overline{[x - d^{q-1}x']} = \overline{[x]}$ holds since $-d^{q-1}x' = d^{q-1}(-x')$. Notice also $x - d^{q-1}x'\in\mathrm{Ker}\ d^{q+1}\circ d^q = {}'C^q{}'$.

PROPOSITION 1.7. (Half-Exacteness). Let $0\to C_1^*\xrightarrow{\alpha^*} C_2^*\xrightarrow{\beta^*} C_3^*\to 0$ be a short exact sequence in Pco (A). Then, for each $q\in\mathbf{Z}$, the sequence

$$Ph^q(C_1^*)\xrightarrow{\bar{\alpha}^q} Ph^q(C_2^*)\xrightarrow{\bar{\beta}^q} Ph^q(C_3^*)$$

is exact at $Ph^q(C_2^*)$.

Proof. Suppose $\bar{\beta}^q(\overline{[x]}) = \overline{[\beta^q(x)]} = 0$ holds in $Ph^q(C_3^*)$. That is, $[\beta^q(x)]\in\mathrm{Im}\ ''d_3^{q-1}{}''$ holds, which implies $\beta^q(x) = d_3^{q-1}(y)$ for some $y\in C_3^{q-1}$. Since β_3^{q-1} is an epimorphism, there exists $x'\in C_2^{q-1}$ such that $\beta^{q-1}(x') = y$. Let $x'' = d_2^{q-1} x'$. We obtain $\beta^q(x'' - x) = 0$ since $\beta^q(x'' - x) = \beta^q d_2^{q-1} x' - \beta^q(x) = d_3^{q-1} \beta^{q-1}(x') - \beta^q(x) = d_3^{q-1}(y) - \beta^q(x) = 0$. Therefore one can find $z \in C_1^q$ such that $\alpha^q(z) = x'' - x$ by the exactness. We need to prove $''d_1^q{}''[z] = 0$, i. e.,

$$d_1^q z - d_1^q d_1^{q-1} z' = 0 \text{ holds for some } z' \in C_1^{q-1}.$$

We have that

$$\alpha^{q+1} d_1^q z - \alpha^{q+1} d_1^q d_1^{q-1} z' = \alpha^{q+1} d_1^q z - d_2^q d_2^{q-1} \alpha^{q-1} z' =$$

$$= d_2^q (\alpha^q(z) - d_2^{q-1} \alpha^{q-1} z').$$

Therefore, it is sufficient to prove $[\alpha^q(z)] \in \operatorname{Ker} {}''d_2^q{}''$, i. e., to show $[x'' - x] \in \operatorname{Ker} {}''d_2^q{}''$. Choose $x' - x^0 \in C_2^{q-1}$, where x^0 is chosen such that $d_2^q x - d_2^q d_2^{q-1} x^0 = 0$ for $[x] \in \operatorname{Ker} {}''d_2^q{}''$ above. Then

$$d_2^q (x'' - x - d_2^{q-1}(x' - x^0)) = d_2^q x'' - d^q x - d_2^q d_2^{q-1}(x' - x^0)$$

$$= d_2^q (d_2^{q-1} x' - x - d_2^{q-1}(x' - x^0)) = 0$$

holds. Hence Ph^q is a half-exact functor.

REMARK 1.8. Consider the following short exact sequence of precomplexes, denoted as $0 \to {}^2\mathbf{Z} \to {}^3\mathbf{Z} \to {}^1\mathbf{Z} \to 0$, of $\mathrm{Pco}^+(A)$:

$$\begin{array}{lccccccccc}
 & & & \vdots & & \vdots & & \vdots & & \\
 & & & \uparrow & & \uparrow & & \uparrow & & \\
\vdots & 0 & \to & 0 & \to & 0 & \to & 0 & \to & 0 \\
 & & & \uparrow & & \uparrow & & \uparrow & & \\
2) & 0 & \to & \mathbf{Z} & \xrightarrow{\text{id}} & \mathbf{Z} & \to & 0 & \to & 0 \\
 & & & \uparrow & & \uparrow{\scriptstyle \text{id}} & & \uparrow & & \\
1) & 0 & \to & \mathbf{Z} & \xrightarrow{\text{id}} & \mathbf{Z} & \to & 0 & \to & 0 \\
 & & & \uparrow & & \uparrow{\scriptstyle \text{id}} & & \uparrow & & \\
0) & 0 & \to & 0 & \to & \mathbf{Z} & \xrightarrow{\text{id}} & \mathbf{Z} & \to & 0 \\
 & & & \uparrow & & \uparrow & & \uparrow & & \\
 & 0 & \to & 0 & \to & 0 & \to & 0 & \to & 0 \\
 & & & \uparrow & & \uparrow & & \uparrow & & \\
 & & & \vdots & & \vdots & & \vdots & &
\end{array}$$

Then the complexifying functor " " applied to the above implies the diagram

$$
\begin{array}{lccccccccc}
 & & & \vdots & & \vdots & & \vdots & & \\
 & & & \uparrow & & \uparrow & & \uparrow & & \\
 & 0 & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & 0 \\
 & & & \uparrow & & \uparrow & & \uparrow & & \\
2) & & & \mathbf{Z} & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & 0 \\
 & & & \uparrow{\scriptstyle id} & & \uparrow & & \uparrow & & \\
1) & 0 & \rightarrow & \mathbf{Z} & \xrightarrow{id} & \mathbf{Z} & \rightarrow & 0 & \rightarrow & 0 \\
 & & & \uparrow & & \uparrow{\scriptstyle id} & & \uparrow & & \\
0) & 0 & \rightarrow & 0 & \rightarrow & \mathbf{Z} & \xrightarrow{id} & \mathbf{Z} & \rightarrow & 0 \\
 & & & \uparrow & & \uparrow & & \uparrow & & \\
 & & & \vdots & & \vdots & & \vdots & &
\end{array}
$$

From this sequence of complexes, if Ph^* were an exact connected sequence of functors, one would obtain

$$
\begin{array}{ccccccccc}
0 \rightarrow & Ph^0({}^2\mathbf{Z}) & \rightarrow & Ph^0({}^3\mathbf{Z}) & \rightarrow & Ph^0({}^1\mathbf{Z}) & \rightarrow & Ph^1({}^2\mathbf{Z}) & \rightarrow \cdots \\
 & \| & & \| & & \| & & \| & \\
 & 0 & & 0 & & \mathbf{Z} & & 0 &
\end{array}
$$

Hence, Ph^n, $n \in \mathbf{Z}$, is not an exact connected sequence of functors.

REMARK 1.9. The right derived functors of Ph^0 on $Pco^+(A)$ are given by

$$
\begin{cases}
Ph^0 = \mathrm{Ker}\,(d^0), & n = 0 \\
\mathrm{Coker}\,(d^0), & n = 1 \\
0, & n \geqq 2.
\end{cases}
$$

2. Hyperprecohomology of a left exact functor

Let A and B be abelian categories and let $F : A \rightsquigarrow B$ be a left exact additive functor.

DEFINITION 2.1. Let $(C^q, d^q)_{q \in \mathbf{Z}} \in Pco^+(A)$. By the complexifying functor, denoted by " " in the previous section, one has

$(C^q / \mathrm{Im}\, d^{q-1} \circ d^{q-2},\ ''d^q'')_{q \in \mathbf{Z}}$ as an object $Co^+(A)$. We will abbreviate the above associated cochain complex as $''C^*''$. Then $F''C^*''$ is an object of $Co^+(B)$. The q-th hyperprecohomology of F evaluated at C^*, denoted as $Ph^qF(C^*)$, is defined as the q-th hyperderived functor of F evaluated at $''C^*''$.

NOTE 1. We have the following diagram of categories and functors

$$\begin{array}{ccccc} Pco^+(A) & \overset{''\ ''}{\rightsquigarrow} & Co^+(A) & \overset{Co(F)}{\rightsquigarrow} & Co^+(B) \\ & & \Big\downarrow{\scriptstyle H^0} & & \Big\downarrow{\scriptstyle H^0} \\ & & A & \overset{F}{\rightsquigarrow} & B, \end{array}$$

where functors $(C^q, d^q)_{q \in \mathbf{Z}} \overset{''\ ''}{\rightsquigarrow} ''C^*'' \in Co^+(A)$, $''C^*'' \overset{Co(F)}{\rightsquigarrow} F''C^*'' \in$ $\in Co^+(B)$ and $F''C^*'' \overset{H^0}{\rightsquigarrow} \mathrm{Ker}\, F''\, d^0{}''$ are defined as in Definition 2.1, and $H^0 : Co^+(A) \rightsquigarrow A$ is defined by $H^0(''C^*'') = \mathrm{Ker}\, ''d^0'' =$ $= H^0(C^*) = \mathrm{Ker}\, d^0$ and $F : A \rightsquigarrow B$ by $\mathrm{Ker}\, d^0 \rightsquigarrow F(\mathrm{Ker}\, d^0)$. Notice $F(\mathrm{Ker}\, d^0) \xrightarrow{\simeq} \mathrm{Ker}\, F\, d^0$ holds since F is left exact. Then there are induced spectral sequences

$$(2.1.1) \quad E_2^{p,q} = H^p(R^q F(''C^\bullet'')) =$$

$$= H^p(\cdots \to R^q F(''C^p'') \to R^q F(''C^{p+1}'') \to \cdots)$$

$$(2.1.2) \quad {}'E_2^{p,q} = (R^pF)(Ph^q(C^*))$$

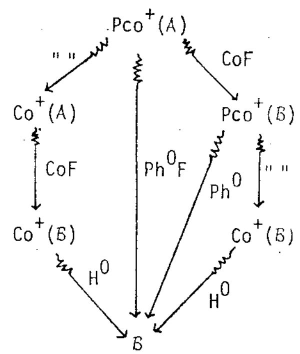
with their abutement the hyperprecohomology $Ph^nF(C^*)$, where R^pF denotes the p-th derived functor of F.

Furthermore, (2.1.1) can be extended to

$$(2.1.1'') \quad E_1^{p,q} = (R^qF)(''C^p''),$$

see [2, pp. 118].

REMARK. We have the commutative diagram of categories and functors:



See Definition 2.1 and the above Note 1 for the description of each functor. The composition of functors leaving $Pco^+(A)$ to B, counter-clockwise, defines the zero-th hyperprecohomology $Ph^0F(C^*)$ of F at C^* in $Pco^+(A)$. The composition of functors leaving $Pco^+(B)$ to B, clockwise, defines the zero-th precohomology of FC^*.

3. Preinverse Limit

Let $(C^q, d^q)_{q \in \mathbf{Z}}$ be a cochain precomplex and be regarded as an inverse system:

$$\cdots \longrightarrow C^{q-1} \xrightarrow{d^{q-1}} C^q \xrightarrow{d^q} C^{q+1} \longrightarrow \cdots .$$

DEFINITION 3.1. Let A be an abelian category such that denumerable direct products of objects exist and such that the denumerable direct product functor is exact. Let $\mathbf{C}^0 = \mathbf{C}^1 = \prod_{q \in \mathbf{Z}} C^q$ and define a morphism

$$\delta^0 : \mathbf{C}^0 \to \mathbf{C}^1$$

by $\pi_{q+1} \circ \delta^0 = d^q \circ \pi_q - d^q d^{q-1} \circ \pi_{q-1}$, where $\pi_q : \prod_{q \in \mathbf{Z}} C^q \to C^q$ is the projection. Let $\mathbf{C}^n = 0$ for $n \neq 0,1$ and $\delta^n = 0$ for $n \neq 0$. Then

$$0 \longrightarrow \mathbf{C}^0 \xrightarrow{\delta^0} \mathbf{C}^1 \xrightarrow{\delta^1} 0 \longrightarrow \cdots$$

is a cochain complex, denoted by $\mathbf{C}^*$. Define the preinverse limit, denoted as $\underleftarrow{\mathrm{Pim}}$,

$$\underset{q\in\mathbf{Z}}{\underleftarrow{\mathrm{Pim}}}\ C^q = H^0(\mathbf{C}^*) = \mathrm{Ker}\ \delta^0$$

and define the 1-st preinverse limit, denoted as $\underleftarrow{\mathrm{Pim}}^1$,

$$\underset{q\in\mathbf{Z}}{\underleftarrow{\mathrm{Pim}}^1}\ C^q = H^1(\mathbf{C}^*) = \mathbf{C}^1 / \mathrm{Im}\ \delta^0.$$

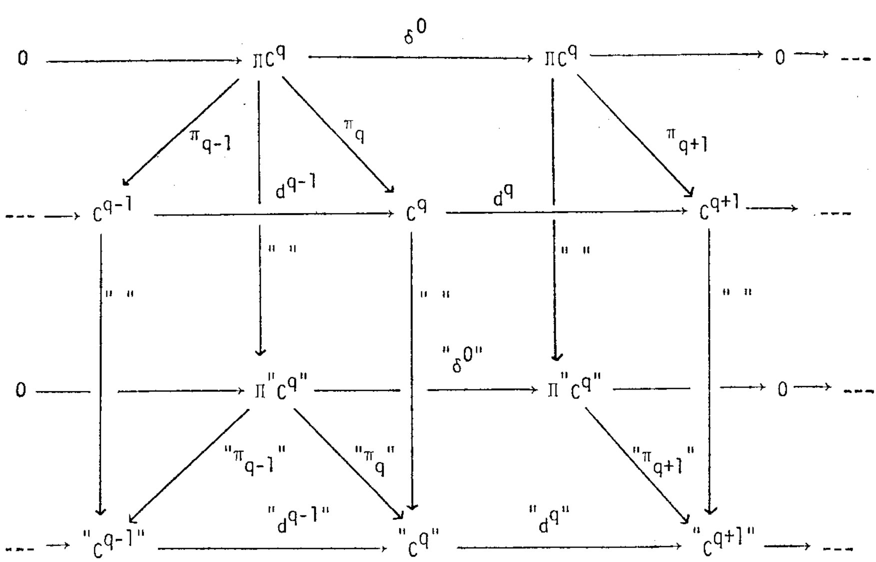
NOTE. $\varprojlim C^q \subset \underleftarrow{\mathrm{Pim}}\ C^q$ and $\varprojlim^1 C^q \xrightarrow{\text{epi}} \underleftarrow{\mathrm{Pim}}^1\ C^q$ hold, where $\varprojlim$ and $\varprojlim^1$ are the usual inverse limits.

THEOREM 3.2. Let $(C^q, d^q)_{q\in\mathbf{Z}}$ be a cochain precomplex, regarded as an inverse system. There exists an isomorphism

$$\prod_{q\in\mathbf{Z}} {''C^q{''}} / \underleftarrow{\mathrm{Pim}}\ {''C^q{''}} \xrightarrow{\cong} \underleftarrow{\mathrm{Pim}}^1\ {''C^q{''}} / \prod_{q\in\mathbf{Z}} Ph^q(C^*)$$

where $''\ ''$ is the canonical epimorphism $\Pi\, C^q \to \Pi\, ''C^q{''}$.

Proof. Consider the following diagram.



From the definition of $''d^q''$, one has Π Ker $''d^q'' =$ Ker $''\delta^0''$ and Π Im $''d^q'' =$ Im $''\delta^0''$. Hence, the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Pi\ \mathrm{Im}\ ''d^{q-1}'' & \longrightarrow & \Pi\ \mathrm{Ker}\ ''d^q'' & \longrightarrow & \Pi\ \mathrm{Ker}\ ''d^q''/\mathrm{Im}\ ''d^{q-1}'' & \longrightarrow & 0 \\
 & & \| & & \| & & \| & & \\
0 & \longrightarrow & \mathrm{Im}\ ''\delta^0'' & \longrightarrow & \mathrm{Ker}\ ''\delta^0'' & \longrightarrow & \Pi\ Ph^q(C^*) & \longrightarrow & 0 \\
 & & \|\iota' & & \downarrow\iota & & \downarrow\iota'' & & \\
0 & \longrightarrow & \mathrm{Im}\ ''\delta^0'' & \longrightarrow & \Pi\ ''C^q'' & \longrightarrow & \underleftarrow{\mathrm{Pim}}^1\ ''C^q'' & \longrightarrow & 0
\end{array}$$

implies, by a well-known lemma applied to the second and third short exact sequences, the following exact sequence,

$$\begin{array}{ccccccccccccc}
0 \rightarrow & \mathrm{Ker}\ l' & \rightarrow & \mathrm{Ker}\ l & \rightarrow & \mathrm{Ker}\ l'' & \rightarrow & \mathrm{Coker}\ l' & \rightarrow & \mathrm{Coker}\ l & \rightarrow & \mathrm{Coker}\ l'' & \rightarrow 0. \\
 & \| & & \| & & & & \| & & & & & \\
 & 0 & & 0 & & & & 0 & & & & &
\end{array}$$

Hence, one obtains the isomorphism

$$\mathrm{Coker}\ l = \prod_{q \in \mathbf{Z}} ''C^q''/\underleftarrow{\mathrm{Pim}}\ ''C^q'' \xrightarrow{\simeq} \mathrm{Coker}\ l'' = \underleftarrow{\mathrm{Pim}}^1\ ''C^q''/\prod_{q \in \mathbf{Z}} Ph^q(C^*).$$

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