

**ON THE GENERATORS OF THE FIRST HOMOLOGY  
 WITH COMPACT SUPPORTS OF THE WEIERSTRASS FAMILY  
 IN CHARACTERISTIC ZERO**

BY  
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**ABSTRACT.** Let  $\mathbf{W}_{\mathbf{Q}} = \text{Proj}(\mathbf{Q}[g_2, g_3, X, Y, Z]/(\text{homogeneous ideal generated by } -Y^2Z + 4X^3 - g_2XZ^2 - g_3Z^3))$ . This is said to be the Weierstrass Family over the field  $\mathbf{Q}$ . Then the first homology with compact supports of the Weierstrass Family is computed explicitly, i.e., it is generated by  $\{C^{-k}dX \wedge dY\}_{k \geq 1}$  and  $\{XC^{-k}dX \wedge dY\}_{k \geq 1}$  over the ring  $\mathbf{Q}[g_2, g_3]$ , where  $C$  is a polynomial  $Y^2 - 4X^3 + g_2X + g_3$ . When one tensors the homology of the Weierstrass Family with  $\Delta^{-1}\mathbf{Q}[g_2, g_3]$ , being localized at the discriminant  $\Delta = g_2^3 - 27g_3^2$ , over  $\mathbf{Q}[g_2, g_3]$ , the first homology is generated by  $C^{-1}dX \wedge dY$  and  $XC^{-1}dX \wedge dY$ . One also obtains the first homologies with compact supports of singular fibres over  $\wp = (g_2 = g_3 = 0)$  and  $\wp = (g_2 = 3, g_3 = 1)$  as corollaries.

**Introduction.** We wish to compute the  $\mathbf{Q}[g_2, g_3]$ -adic homology with compact supports of the Weierstrass Family  $\mathbf{W}_{\mathbf{Q}}$ , where

$$\mathbf{W}_{\mathbf{Q}} = \text{Proj} \left( \frac{\mathbf{Q}[g_2, g_3, X, Y, Z]}{\text{homogeneous ideal generated by } -Y^2Z + 4X^3 - g_2XZ^2 - g_3Z^3} \right).$$

We regard the graded ring  $\mathbf{Q}[g_2, g_3, X, Y, Z]$  as the graded  $\mathbf{Q}[g_2, g_3]$ -algebra such that  $X, Y$  and  $Z$  each has degree  $+1$  and all the elements of  $\mathbf{Q}[g_2, g_3]$  have degree zero. Let  $U$  be the open subset of  $\mathbf{W}_{\mathbf{Q}}$ , "the finite points":  $U = \mathbf{W}_{\mathbf{Q}} \cap \mathbf{A}^2(\text{Spec}(\mathbf{Q}[g_2, g_3]))$ . This is the closed subscheme of  $\mathbf{A}^2(\text{Spec}(\mathbf{Q}[g_2, g_3]))$  given by  $Y^2 = 4X^3 - g_2X - g_3$ . Then we have the long exact sequence of the homology with compact supports,  $\cdots \rightarrow H_{h-2}^c(\{\text{points at } \infty\}, \mathbf{Q}[g_2, g_3]) \rightarrow H_h^c(\mathbf{W}_{\mathbf{Q}}, \mathbf{Q}[g_2, g_3]) \rightarrow H_h^c(U, \mathbf{Q}[g_2, g_3]) \rightarrow \cdots$ . Since  $H_h^c(\{\text{points at } \infty\}, \mathbf{Q}[g_2, g_3])$  vanishes except at  $h = 0$ , we have

$$H_h^c(U, \mathbf{Q}[g_2, g_3]) = \begin{cases} H_h^c(\mathbf{W}_{\mathbf{Q}}, \mathbf{Q}[g_2, g_3]), & h \neq 2, \\ \mathbf{Q}[g_2, g_3], & h = 2. \end{cases}$$

Therefore the knowledge of  $H_h^c(U, \mathbf{Q}[g_2, g_3])$ ,  $h \geq 0$ , determines the homology groups of all the fibres in the family over the various points  $\wp \in \text{Spec}(\mathbf{Q}[g_2, g_3])$ , i.e.,

$$E_{p,q}^2 = \text{Tor}_p^{\mathbf{Q}[g_2, g_3]}(H_q^c(U, \mathbf{Q}[g_2, g_3]), \mathbf{K}(\wp))$$

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with the abutment  $H_n^c(U_\wp, \mathbf{K}(\wp))$ , where  $\mathbf{K}(\wp)$  is the characteristic zero residue field at  $\wp \in \text{Spec}(\mathbf{Q}[g_2, g_3])$ .

Let us consider the unequal characteristic case. Suppose that  $\mathcal{O}$  is a complete discrete valuation ring with the quotient field  $K$  and residue class field  $k$  and suppose that  $A$  is an  $\mathcal{O}$ -algebra. Let  $X$  be a scheme over  $A = (A \otimes_{\mathcal{O}} k)_{\text{red}}$ . Suppose that  $\mathbf{K}(\wp)$  is a finite field at  $\wp \in \text{Spec}(A)$  and let  $W(\mathbf{K}(\wp))$  be the complete discrete valuation ring and denote the quotient field of  $W(\mathbf{K}(\wp))$  by  $K_\wp = W(\mathbf{K}(\wp)) \otimes_{\mathbf{Z}} \mathbf{Q}$ . Then the zeta function of the fibre  $X_\wp$  at  $\wp$  is given by

$$(0.1) \quad Z_{X_\wp}(T) = \frac{\prod_{p+q=\text{odd}} P_{p,q}(T)}{\prod_{p+q=\text{even}} P_{p,q}(T)}$$

where  $P_{p,q}(T)$  is the reverse characteristic polynomial of the endomorphism of

$$(0.2) \quad E_{p,q}^2 = \text{Tor}_p^{A^\dagger \otimes_{\mathbf{Z}} \mathbf{Q}}(H_q^c(X, A^\dagger \otimes_{\mathbf{Z}} \mathbf{Q}), K_\wp)$$

induced by the  $p^r$ th power map,  $p^r = \text{card}(\mathbf{K}(\wp))$  (see pp. 448–450, [6]). This homological spectral sequence abuts upon  $H_n^c(X_\wp, K_\wp)$ . Therefore if one knows the lifted  $p$ -adic homology with compact supports of  $X$  over  $A$ ,  $H_h^c(X, A^\dagger \otimes_{\mathbf{Z}} \mathbf{Q})$ ,  $h \geq 0$ , and the zeta endomorphisms of these groups, (1) determines the zeta function of every fibre over a finite field in the algebraic family  $X$  over the ring  $A$ . These are the subjects in the forthcoming paper [2].

The main result of the paper is the explicitness of the generation of the first homology with compact supports of the entire Weierstrass Family  $\mathbf{W}_{\mathbf{Q}}$  in the characteristic zero (Theorem 1) and its consequences.

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1. In this section (notations being the same as in the Introduction) we describe explicitly the basis elements over the ring  $\mathbf{Q}[g_2, g_3]$  which generate the first homology with compact supports of the Weierstrass Family over the field of rational numbers  $\mathbf{Q}$ ,  $H_1^c(U, \mathbf{Q}[g_2, g_3])$ . By the definition of the lifted  $p$ -adic homology with compact supports [6, p. 415], applied to the characteristic zero case, we have

$$H_1^c(U, \mathbf{Q}[g_2, g_3]) = H^3(A^2(\text{Spec}(\mathbf{Q}[g_2, g_3])), A^2(\text{Spec}(\mathbf{Q}[g_2, g_3])) - U, \Gamma_{\mathbf{Q}[g_2, g_3]}^*(\text{Spec}(\mathbf{Q}[g_2, g_3])))$$

If one tensors  $H_1^c(U, \mathbf{Q}[g_2, g_3])$  with  $\Delta^{-1}\mathbf{Q}[g_2, g_3]$  over  $\mathbf{Q}[g_2, g_3]$ , one has the free  $\Delta^{-1}\mathbf{Q}[g_2, g_3]$ -module of rank two, where  $\Delta = g_2^3 - 27g_3^2$ . This is so because we have the universal coefficients spectral sequence

$$E_{0,1}^2 = H_1^c(U, \mathbf{Q}[g_2, g_3]) \otimes_{\mathbf{Q}[g_2, g_3]} \Delta^{-1}\mathbf{Q}[g_2, g_3] \xrightarrow{\cong} H_1^c(U, \Delta^{-1}\mathbf{Q}[g_2, g_3]),$$

and  $\Delta^{-1}\mathbf{Q}[g_2, g_3]$  means that the ring  $\mathbf{Q}[g_2, g_3]$  is localized at the discriminant  $\Delta$ . The computation has been made even in the  $p$ -adic case in [1] for this open subfamily of the Weierstrass Family.

**THEOREM 1.** Consider  $U = \mathbf{W}_{\mathbf{Q}} \cap \mathbf{A}^2(\text{Spec}(\mathbf{Q}[g_2, g_3]))$ , which is the closed affine subscheme of  $\mathbf{A}^2(\text{Spec}(\mathbf{Q}[g_2, g_3]))$ . Then the first homology with compact supports  $H_1^c(U, \mathbf{Q}[g_2, g_3])$  is generated by  $\{C^{-l}dX \wedge dy\}_{l \geq 1}$  and  $\{XC^{-l}dX \wedge dY\}_{l \geq 1}$  as a  $\mathbf{Q}[g_2, g_3]$ -module.

**REMARK 1.** For the pair of affine schemes

$$\mathbf{A}^2(\text{Spec}(\mathbf{Q}[g_2, g_3])) \quad \text{and} \quad \mathbf{A}^2(\text{Spec}(\mathbf{Q}[g_2, g_3])) - U,$$

where  $U$  is the closed subscheme corresponding to the polynomial  $C = Y^2 - 4X^3 + g_2X + g_3$  in  $\mathbf{Q}[g_2, g_3, X, Y, Z]$ , there is induced a long exact sequence of hypercohomology groups,

$$\begin{aligned} \dots &\xrightarrow{\partial^{n-1}} H^n(\mathbf{A}^2(A), \mathbf{A}^2(A) - U, \Gamma_A^*(\mathbf{A}^2(A))) \rightarrow H^n(\mathbf{A}^2(A), \Gamma_A^*(\mathbf{A}^2(A))) \\ &\rightarrow H^n(\mathbf{A}^2(A) - U, \Gamma_A^*(\mathbf{A}^2(A))) \xrightarrow{\partial^n} \dots \end{aligned}$$

where  $A = \text{Spec}(\mathbf{Q}[g_2, g_3])$ .

There are three first-quadrant spectral sequences induced which have the above three hypercohomology groups as their abutments:

$$\begin{cases} 'E_1^{p,q} = H^q(\mathbf{A}^2(A) - U, \Gamma_A^p(\mathbf{A}^2(A))), \\ E_1^{p,q} = H^q(\mathbf{A}^2(A), \Gamma_A^p(\mathbf{A}^2(A))), \\ ''E_1^{p,q} = H^q(\mathbf{A}^2(A), \mathbf{A}^2(A) - U, \Gamma_A^p(\mathbf{A}^2(A))). \end{cases}$$

**LEMMA 1.** We have the following isomorphisms: the abutment

$$''E^3 = H^3(\mathbf{A}^2(A), \mathbf{A}^2(A) - U, \Gamma_A^*(\mathbf{A}^2(A))) \xrightarrow{\cong} ''E_2^{2,1},$$

and

$$''E^3 \xrightarrow{\cong} 'E^2 = H^2(\mathbf{A}^2(A) - U, \Gamma_A^*(\mathbf{A}^2(A))) \xleftarrow{\cong} \text{coker}('E_1^{2,0} \leftarrow 'E_1^{1,0}).$$

**PROOF OF LEMMA 1.** Consider the following diagram (Diagram A) with exact rows. We denote the structure sheaf of the affine scheme  $\mathbf{A}^2(A) = \mathbf{A}^2(\text{Spec}(\mathbf{Q}[g_2, g_3]))$  by  $\mathcal{O}_{\mathbf{A}^2(A)}$ . Therefore, we have  $''E_1^{p,q} = 0$  unless  $q = 1$ , which is abutting  $''E^3 = H^3(\mathbf{A}^2(A), \mathbf{A}^2(A) - U, \Gamma_A^*(\mathbf{A}^2(A)))$ . Then the isomorphism  $''E_2^{2,1} \rightarrow ''E^3$  in Lemma 1 follows. Furthermore, this diagram can be rewritten as Diagram B. The remaining two isomorphisms in Lemma 1 are obtained from the well-known lemma in homological algebra, i.e., from Diagram B with the exact rows we have the induced exact sequence

$$\begin{array}{ccccccc} 0 & \rightarrow & \ker d_1^{1,0} & \rightarrow & \ker 'd_1^{1,0} & \rightarrow & \ker ''d_1^{1,0} & \rightarrow & \text{coker } d_1^{1,0} & \rightarrow & \text{coker } 'd_1^{1,0} & \rightarrow & \text{coker } ''d_1^{1,0} & \rightarrow & 0 \\ & & & & & & & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & & \\ & & & & & & & & E^2 & \longrightarrow & 'E^2 & \longrightarrow & ''E^3 & \longrightarrow & 0 \end{array}$$

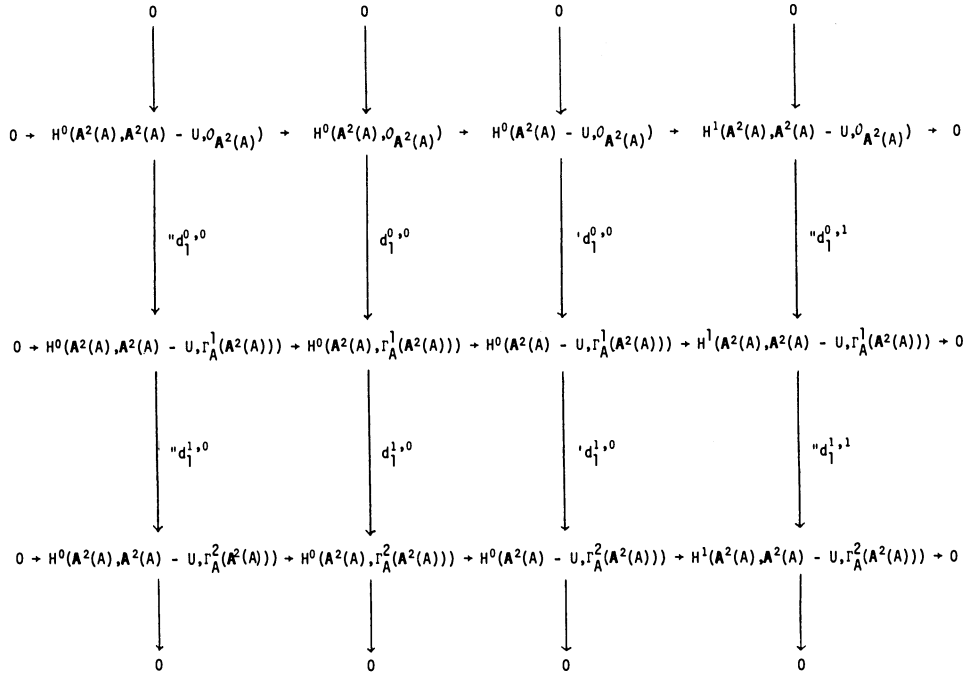


DIAGRAM A

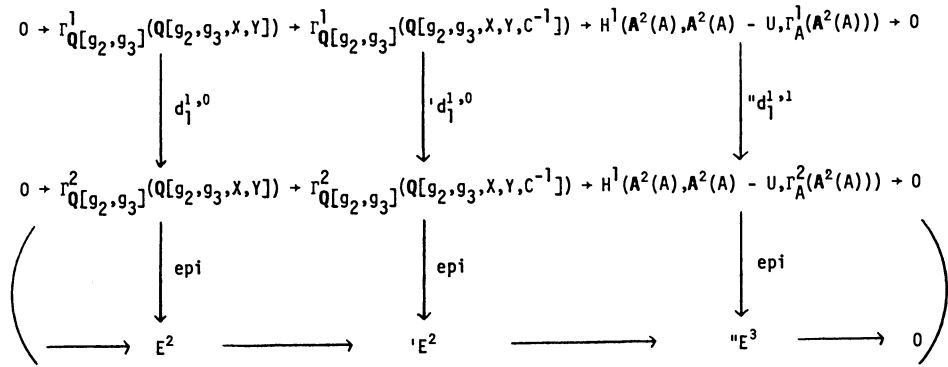


DIAGRAM B

and since the  $\mathbf{Q}[g_2, g_3]$ -homomorphism

$$d_1^{1,0}: E_1^{1,0} = \Gamma_{\mathbf{Q}[g_2, g_3]}^1(\mathbf{Q}[g_2, g_3, X, Y]) \rightarrow E_1^{2,0} = \Gamma_{\mathbf{Q}[g_2, g_3]}^2(\mathbf{Q}[g_2, g_3, X, Y])$$

is an epimorphism, we have  $E^2 \approx E_2^{2,0} \approx 0$ . Therefore

$$'E^2 \xleftarrow{\approx} \text{coker } 'd_1^{1,0} \xrightarrow{\approx} \text{coker } "d_1^{1,1} \approx "E^3$$

as stated in Lemma 1. Q.E.D.

Hence our computation of the abutment " $E^3 = H^3(A^2(A), A^2(A) - U, \Gamma_A^*(A^2(A)))$ " is reduced to compute

$$\text{coker}\left(\Gamma_{\mathbb{Q}[g_2, g_3]}^1(\mathbb{Q}[g_2, g_3, X, Y, C^{-1}]) \xrightarrow{d_1^{1,0}} \Gamma_{\mathbb{Q}[g_2, g_3]}^2(\mathbb{Q}[g_2, g_3, X, Y, C^{-1}])\right).$$

PROOF OF THEOREM 1. From now on we denote, " $d$ ", instead of the exterior differential, " $d_1^{1,0}$ " in the spectral sequence. We have that

$$(1) \quad d(C^{-k}X^iY^j dX) = (-2kC^{-k-1}X^iY^{j+1} + jC^{-k}X^iY^{j-1}) dY \wedge dX,$$

$$(2) \quad d(C^{-k}X^iY^j dY) = (12kC^{-k-1}X^{i+2}Y^j - g_2kC^{-k-1}X^iY^j + iC^{-k}X^{i-1}Y^j) dX \wedge dY,$$

in the  $\mathbb{Q}[g_2, g_3]$ -module  $\Gamma_{\mathbb{Q}[g_2, g_3]}^2(\mathbb{Q}[g_2, g_3, X, Y, C^{-1}])$ , where  $C = Y^2 - 4X^3 + g_2X + g_3$ ,  $i, j$  and  $k$  are nonnegative integers. The equations (1) and (2) give the cohomologous relations, which are denoted by " $\sim$ ", as

$$(3) \quad 2kC^{-k-1}X^iY^{j+1}dX \wedge dY \sim jC^{-k}X^iY^j dX \wedge dY$$

and

$$(4) \quad (12kC^{-k-1}X^{i+2}Y^j - g_2kC^{-k-1}X^iY^j + iC^{-k}X^{i-1}Y^j) dX \wedge dY \sim 0.$$

Notice that, by Lemma 1:

$${}''E_2^{2,1} \leftarrow {}''E_1^{2,1} / \text{Im}({}'E_1^{1,1} \rightarrow {}''E_1^{2,1})$$

and

$${}''E_1^{1,1} \leftarrow {}'E_1^{1,0} / \text{Im}({}'E_1^{1,0} \leftarrow E_1^{1,0}),$$

where  $E_1^{1,0} \approx \Gamma_{\mathbb{Q}[g_2, g_3]}^1(\mathbb{Q}[g_2, g_3, X, Y])$ . Therefore it suffices to consider the integer  $k \geq 1$  in the equations (1), (2), (3) and (4) above.

If  $j = 0$  in (3), then  $C^{-k-1}X^iYdX \wedge dY \sim 0$  for all  $i \geq 0$  and  $k \geq 1$ . But (4) implies that  $C^{-1}X^iYdX \wedge dY \sim 0$  for  $i \geq 0$  since  $iC^{-1}X^{i-1}YdX \wedge dY \sim g_2C^{-2}X^iYdX \wedge dY - 12C^{-2}X^{i+2}YdX \wedge dY$ . Therefore,

$$(5) \quad C^{-k}X^iYdX \wedge dY \sim 0 \quad \text{for all integers } i, k \geq 0.$$

For any odd integer  $j > 1$  we have  $C^{-k}X^iY^jdX \wedge dY \sim 0$  by combining (3) and (5) and the repeated use of (4). For example, for  $j = 3$ , we have  $12kC^{-k-1}X^iY^3dX \wedge dY \sim 2C^{-k}X^iYdX \wedge dY$ , which is cohomologous to zero by (5). Then apply (4) for  $j = 3$  to get

$$iC^{-k}X^{i-1}Y^3dX \wedge dY \sim g_3kC^{-k-1}X^iY^3dX \wedge dY - 12kC^{-k-1}X^{i+3}Y^3dX \wedge dY.$$

But the right-hand side is cohomologous to zero from the above result. If  $i = 0$  in (4), we then have

$$(6) \quad 12kC^{-k-1}X^2Y^jdX \wedge dY \sim g_2kC^{-k-1}Y^jdX \wedge dY$$

for all integers  $k \geq 1$  and  $j \geq 0$ . Especially we have, for  $j = 0$ ,  $12kC^{-k-1}X^2dX \wedge dY \sim g_2kC^{-k-1}dX \wedge dY$ . Then it can be plainly seen that

$$(C^{-k}dX \wedge dY)_{k \geq 1}, \quad (XC^{-k}dX \wedge dY)_{k \geq 1} \quad \text{and} \quad (X^iC^{-1}dX \wedge dY)_{i \geq 2}$$

generate all the elements of the type  $X^i C^{-k} dX \wedge dY$  for integers  $i \geq 0$  and  $k \geq 0$  over the ring  $\mathbf{Q}[g_2, g_3]$  from equations (3) and (4). In particular,  $X^2 C^{-1} dX \wedge dY \sim X^2 Y^2 C^{-2} dX \wedge dY$  by (3) for letting  $i = 2, j = 1$  and  $k = 1$ , but  $X^2 Y^2 C^{-2} dX \wedge dY \sim Y^2 C^{-2} dX \wedge dY$  by (4) for  $i = 0, j = 2$  and  $k = 1$ ; furthermore,  $Y^2 C^{-2} dX \wedge dY$  is cohomologous to  $C^{-1} dX \wedge dY$  from (3) for  $i = 0, j = 1$  and  $k = 1$ . Hence we have established that  $X^2 C^{-1} dX \wedge dY \sim C^{-1} dX \wedge dY$ . Next we claim that all the elements of the type  $(X^i C^{-1} dX \wedge dY)_{i \geq 3}$  are generated by the two elements  $C^{-1} dX \wedge dY$  and  $XC^{-1} dX \wedge dY$  over the ring  $\mathbf{Q}[g_2, g_3]$ . We have the following recursive formula for integers  $i \geq 3$  from (3) and (4):

$$4X^i C^{-1} dX \wedge dY \sim g_2 \left( \frac{1}{12(i-2)} + 1 \right) X^{i-2} C^{-1} dX \wedge dY + \left( g_3 - \frac{1}{i-2} \right) X^{i-3} C^{-1} dX \wedge dY.$$

Therefore it follows from this recursive formula that  $(X^i C^{-1} dX \wedge dY)_{i \geq 3}$  are generated by  $C^{-1} dX \wedge dY$  and  $XC^{-1} dX \wedge dY$  over  $\mathbf{Q}[g_2, g_3]$ . We have established the statement of Theorem 1 for the elements  $X^i Y^j C^{-k} dX \wedge dY$  with  $i \geq 1, j = 0$  and  $k \geq 1$ . Now we need consider the elements  $X^i Y^j C^{-k} dX \wedge dY$  for  $j = 1, 2, 3, \dots$ . As noted before, we know that if  $j$  is an odd integer,  $X^i Y^j C^{-k} dX \wedge dY \sim 0$ . If  $j$  is an even integer, the repeated use of (3) and (4) for the elements  $X^i Y^j C^{-1} dX \wedge dY, i \geq 1$  and  $j \geq 1$ , provides the generation of the first homology with compact supports  $H_1^c(U, \mathbf{Q}[g_2, g_3])$  of the Weierstrass Family by the elements  $(C^{-k} dX \wedge dY)_{k \geq 1}$  and  $(XC^{-k} dX \wedge dY)_{k \geq 1}$ . Q.E.D.

**PROPOSITION 1.** *Assumptions and notations being the same as in Theorem 1,  $H_1^c(U, \mathbf{Q}[g_2, g_3]) \otimes_{\mathbf{Q}[g_2, g_3]} (\Delta^{-1} \mathbf{Q}[g_2, g_3])$  is a free  $(\Delta^{-1} \mathbf{Q}[g_2, g_3])$ -module of rank two, i.e., it is generated by  $XC^{-1} dX \wedge dY$  and  $C^{-1} dX \wedge dY$ , where  $\Delta$  is the discriminant,  $\Delta = g_2^3 - 27g_3^2$ , and  $\Delta^{-1} \mathbf{Q}[g_2, g_3]$  is localized at the discriminant  $\Delta$ .*

**PROOF OF PROPOSITION 1.** For any integer  $i \geq 2$  we have

$$C^{-(i-1)} = C^{-i} (Y^2 - 4X^3 + g_2 X + g_3),$$

where  $dX \wedge dY$  is omitted for simplicity, and from equations (3), (4) and (6) we have the following cohomologous relation for  $i \geq 2$ :

$$(1.1) \quad \frac{6i-11}{6(i-1)} C^{-(i-1)} \sim \frac{2g_2}{3} XC^{-i} + g_3 C^{-i}.$$

Similarly, one has the corresponding formula for  $XC^{-(i-1)}$  by the equations (3), (4) and (6):

$$(1.2) \quad \frac{6i-13}{6(i-1)} XC^{-(i-1)} \sim \frac{g_2^2}{18} C^{-i} + g_3 XC^{-i}.$$

We finally have for  $i \geq 2$ ,  
(1.3)

$$C^{-i}dX \wedge dY \sim \frac{18}{\Delta} \left\{ \frac{g_2(6i - 13)}{6(i - 1)} XC^{-(i-1)} dX \wedge dY - \frac{g_3(6i - 11)}{4(i - 1)} C^{-(i-1)}dX \wedge dY \right\}$$

from equations (1.1) and (1.2).

Equations (1.3) and (1.1) prove that  $H_1^c(U, \mathbf{Q}[g_2, g_3]) \otimes_{\mathbf{Q}[g_2, g_3]} (\Delta^{-1}\mathbf{Q}[g_2, g_3])$  is generated by  $XC^{-1}dX \wedge dY$  and  $C^{-1}dX \wedge dY$  as a  $(\Delta^{-1}\mathbf{Q}[g_2, g_3])$ -module. Q.E.D.

**COROLLARY 1.** *Let  $\mathbf{V}_Q^0$  be the closed subfamily defined by “ $g_2 = 0$ ” of the whole Weierstrass Family  $W_Q$ . Then the first homology with compact supports,*

$$H_1^c(\mathbf{V}_Q^0 \cap \mathbf{A}^2(\text{Spec } \mathbf{Q}[g_3]), \mathbf{Q}[g_3]),$$

*is generated by  $\{C^{-k}dX \wedge dY\}_{k \geq 1}$  and  $\{XC^{-k}dX \wedge dY\}_{k \geq 1}$  as a  $\mathbf{Q}[g_3]$ -module.*

**PROOF.** In (1.1) and (1.2) in the proof of Proposition 1, we have the following corresponding equations for the closed subfamily  $\mathbf{V}_Q^0$  defined by “ $g_2 = 0$ ”:

$$(1.1)^0 \quad \frac{12i - 22}{12(i - 1)} C^{-(i-1)} \sim g_3 C^{-i},$$

$$(1.2)^0 \quad \frac{6i - 13}{6(i - 1)} XC^{-(i-1)} \sim g_3 XC^{-i}.$$

Then the statement of Corollary 1 follows plainly from (1.1)<sup>0</sup> and (1.2)<sup>0</sup>. Q.E.D.

*Note 1.* The equations (1.1)<sup>0</sup> and (1.2)<sup>0</sup> also show that Corollaries 2 and 3 are true.

**COROLLARY 2.** *The first homology with compact supports of the singular fibre  $U_\varphi$  over a point  $\varphi = (g_2 = 0, g_3 = 0) \in \text{Spec}(\mathbf{Q}[g_2, g_3])$ , a projective line with a cusp (or  $\varphi = (g_3 = 0) \in \text{Spec}(\mathbf{Q}[g_3])$ ),  $H_1^c(U_\varphi, \mathbf{Q})$ , is trivial.*

**COROLLARY 3.** *Notations being the same as in Proposition 1,*

$$H_1^c(\mathbf{V}_Q^0 \cap \mathbf{A}^2(\text{Spec}(\mathbf{Q}[g_3])), \mathbf{Q}[g_3]) \otimes_{\mathbf{Q}[g_3]} (g_3^{-1}\mathbf{Q}[g_3])$$

*is generated by the two elements  $C^{-1}dX \wedge dY$  and  $XC^{-1}dX \wedge dY$ , where  $g_3^{-1}\mathbf{Q}[g_3]$  means the localization of the ring  $\mathbf{Q}[g_3]$  at  $g_3$ .*

**REMARK 2.** For a point  $\varphi \neq (g_3 = 0)$ ,  $H_1^c(U_\varphi, \mathbf{K}(\varphi))$  is generated by  $C^{-1}dX \wedge dY$  and  $XC^{-1}dX \wedge dY$  as a  $\mathbf{K}(\varphi)$ -vector space and where  $\mathbf{K}(\varphi)$  is the characteristic zero residue field, i.e.,  $U_\varphi$  is an elliptic curve. Note that the open subfamily of the Weierstrass Family over  $\mathbf{Z}/P\mathbf{Z}$  defined by “ $\Delta \neq 0$ ” has been computed explicitly using the hypercohomology of a flat lifting with coefficients in the  $\dagger$  of sheaves of differential forms,  $H^1(U, (\Delta^{-1}\hat{\mathbf{Z}}_p[g_2, g_3])\dagger \otimes_{\mathbf{Z}} \mathbf{Q})$ , where  $(\Delta^{-1}\hat{\mathbf{Z}}_p[g_2, g_3])\dagger$  is the  $\dagger$  of the localization of the ring  $\hat{\mathbf{Z}}_p[g_2, g_3]$  at the discriminant  $\Delta = g_2^3 - 27g_3^2$ , see [1]. The following universal coefficient spectral sequence explains the relationship between Corollary 2 and Theorem 1.

$E_{p,q}^2 = \text{Tor}_p^{\mathbf{Q}[g_2, g_3]}(H_q^c(U, \mathbf{Q}[g_2, g_3]), \mathbf{K}(\varphi))$  with the abutment  $H_n^c(U_\varphi, \mathbf{K}(\varphi))$ , where  $\varphi = (g_2 = g_3 = 0) \in \text{Spec}(\mathbf{Q}[g_2, g_3])$  and  $\mathbf{Q} = \mathbf{K}(\varphi)$ .

**COROLLARY 4.** Let  $V_Q^3$  be the closed subfamily of the Weierstrass Family  $W_Q$ , defined by “ $g_2 = 3$ ”. Then  $H_1^c(V_Q^3 \cap A^2(\text{Spec } \mathbf{Q}[g_3]), \mathbf{Q}[g_3])$  is generated by  $\{C^{-k}dX \wedge dY\}_{k \geq 1}$  and  $\{XC^{-k}dX \wedge dY\}_{k \geq 1}$  as a  $\mathbf{Q}[g_3]$ -module. Moreover the first homology with compact supports of the singular fibre over the point  $\wp = (g_3 = 1)$  in the base  $\text{Spec}(\mathbf{Q}[g_3])$ , a projective line with an ordinary double point over  $\mathbf{K}(\wp)$ , is generated by one element as a  $\mathbf{K}(\wp)$ -vector space. One can then take either  $C^{-1}dX \wedge dY$  or  $XC^{-1}dX \wedge dY$  as the base element for the vector space.

**PROOF.** We only need prove the latter statement. From equations (1.1) and (1.2), we have (1.1)<sub>1</sub><sup>3</sup> and (1.2)<sub>1</sub><sup>3</sup> as follows:

$$(1.1)_1^3 \quad \frac{6i-11}{6(i-1)} C^{-(i-1)} \sim 2XC^{-i} + C^{-i},$$

$$(1.2)_1^3 \quad \frac{6i-13}{6(i-1)} XC^{-(i-1)} \sim \frac{1}{2} C^{-i} + XC^{-i}.$$

Then we have  $2(6i-13)XC^{-(i-1)} \sim (6i-11)C^{-(i-1)}$  for  $i \geq 2$ . Hence this vector space is one dimensional and the statement of Corollary 4 follows. Q.E.D.

*Note 2.* For the closed subfamily  $V_Q^3$  of the Weierstrass Family we have the following equations (1.1)<sup>3</sup>, (1.2)<sup>3</sup> and (1.3)<sup>3</sup>:

$$(1.1)^3 \quad \frac{6i-11}{6(i-1)} C^{-(i-1)} \sim 2XC^{-i} + g_3 C^{-i},$$

$$(1.2)^3 \quad \frac{6(i-13)}{6(i-1)} XC^{-(i-1)} \sim \frac{1}{2} C^{-i} + g_3 XC^{-i},$$

$$(1.3)^3 \quad (g_3^2 - 1)C^{-i} \sim \frac{1}{6(i-1)} \{g_3(6i-11)C^{-(i-1)} - 2(6i-13)XC^{-(i-1)}\},$$

for integers  $i \geq 2$ .

*Note 3.* This paper has been entirely in characteristic zero. The case of nonzero characteristic  $p \neq 2, 3$  will appear in a forthcoming paper [2], which is a generalization of the paper [1], where an open subfamily “ $\Delta \neq 0$ ” of the Weierstrass Family was studied.

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