ON THE GENERATORS OF THE FIRST HOMOLOGY
WITH COMPACT SUPPORTS OF THE WEIERSTRASS FAMILY
IN CHARACTERISTIC ZERO

BY
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ABSTRACT. Let \( W_Q = \text{Proj}(Q[g_2, g_3, X, Y, Z])/(\text{homogeneous ideal generated by } -Y^2Z + 4X^3 - g_2XZ^2 - g_3Z^3) \). This is said to be the Weierstrass Family over the field \( Q \). Then the first homology with compact supports of the Weierstrass Family is computed explicitly, i.e., it is generated by \{C-1dX, dY\} and \{XC-1dX, dY\} over the ring \( Q[g_2, g_3] \), where \( C \) is a polynomial \( Y^2 - 4X^3 + g_2X + g_3 \). When one tensors the homology of the Weierstrass Family with \( Q[g_2, g_3] \), being localized at the discriminant \( \Delta = g_2^3 - 27g_3^2 \), over \( Q[g_2, g_3] \), the first homology is generated by \( C-1dX \) and \( XC-1dX \). One also obtains the first homologies with compact supports of singular fibres over \( \varphi = (g_2 = g_3 = 0) \) and \( \varphi = (g_2 = 3, g_3 = 1) \) as corollaries.

Introduction. We wish to compute the \( Q[g_2, g_3] \)-adic homology with compact supports of the Weierstrass Family \( W_Q \), where

\[
W_Q = \text{Proj}
\left( \frac{Q[g_2, g_3, X, Y, Z]}{\text{homogeneous ideal generated by } -Y^2Z + 4X^3 - g_2XZ^2 - g_3Z^3} \right).
\]

We regard the graded ring \( Q[g_2, g_3, X, Y, Z] \) as the graded \( Q[g_2, g_3] \)-algebra such that \( X, Y \) and \( Z \) each has degree +1 and all the elements of \( Q[g_2, g_3] \) have degree zero. Let \( U \) be the open subset of \( W_Q \), "the finite points": \( U = W_Q \cap A^2(\text{Spec}(Q[g_2, g_3])) \). This is the closed subscheme of \( A^2(\text{Spec}(Q[g_2, g_3])) \) given by \( Y^2 = 4X^3 - g_2X - g_3 \). Then we have the long exact sequence of the homology with compact supports, \( \cdots \to H^h_{\text{c}}(\{\text{points at } \infty\}, Q[g_2, g_3]) \to H^h(W_Q, Q[g_2, g_3]) \to H^h(U, Q[g_2, g_3]) \to \cdots \). Since \( H^h(\{\text{points at } \infty\}, Q[g_2, g_3]) \) vanishes except at \( h = 0 \), we have

\[
H^h(U, Q[g_2, g_3]) = \begin{cases} H^h(W_Q, Q[g_2, g_3]), & h \neq 2, \\ Q[g_2, g_3], & h = 2. \end{cases}
\]

Therefore the knowledge of \( H^h(U, Q[g_2, g_3]) \), \( h \geq 0 \), determines the homology groups of all the fibres in the family over the various points \( \varphi \in \text{Spec}(Q[g_2, g_3]) \), i.e.,

\[
E^2_{p, q} = \text{Tor}^Q_{p}(S^2, S^1)(H^q_{\text{c}}(U, Q[g_2, g_3], K(\varphi))).
\]

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with the abutment $H'_n(U, \mathbf{K}(\varphi))$, where $\mathbf{K}(\varphi)$ is the characteristic zero residue field at $\varphi \in \text{Spec}(\mathbb{Q}[g_2, g_3])$.

Let us consider the unequal characteristic case. Suppose that $\mathfrak{O}$ is a complete discrete valuation ring with the quotient field $K$ and residue class field $k$ and suppose that $A$ is an $\mathfrak{O}$-algebra. Let $X$ be a scheme over $A = (A \otimes_{\mathfrak{O}} k)_{\text{red}}$. Suppose that $\mathbf{K}(\varphi)$ is a finite field at $\varphi \in \text{Spec}(A)$ and let $W(\mathbf{K}(\varphi))$ be the complete discrete valuation ring and denote the quotient field of $W(\mathbf{K}(\varphi))$ by $K_{\varphi} = W(\mathbf{K}(\varphi)) \otimes_{\mathbb{Z}} \mathbb{Q}$. Then the zeta function of the fibre $X_{\varphi}$ at $\varphi$ is given by

\begin{equation}
Z_{X_{\varphi}}(T) = \frac{\prod_{p+q=\text{odd}} P_{p,q}(T)}{\prod_{p+q=\text{even}} P_{p,q}(T)},
\end{equation}

where $P_{p,q}(T)$ is the reverse characteristic polynomial of the endomorphism of

\begin{equation}
E^2_{p,q} = \text{Tor}_p^{d \otimes_{\mathbb{Z}} \mathbb{Q}}(H^*_q(X, A \otimes_{\mathbb{Z}} \mathbb{Q}), K_{\varphi})
\end{equation}

induced by the $p$'th power map, $p' = \text{card}(\mathbf{K}(\varphi))$ (see pp. 448–450, [6]). This homological spectral sequence abuts upon $H^*_q(X, K_{\varphi})$. Therefore if one knows the lifted $p$-adic homology with compact supports of $X$ over $A$, $H^*_q(X, A \otimes_{\mathbb{Z}} \mathbb{Q})$, $h > 0$, and the zeta endomorphisms of these groups, (1) determines the zeta function of every fibre over a finite field in the algebraic family $X$ over the ring $A$. These are the subjects in the forthcoming paper [2].

The main result of the paper is the explicitness of the generation of the first homology with compact supports of the entire Weierstrass Family $W_{\mathbb{Q}}$ in the characteristic zero (Theorem 1) and its consequences.

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1. In this section (notations being the same as in the Introduction) we describe explicitly the basis elements over the ring $\mathbb{Q}[g_2, g_3]$ which generate the first homology with compact supports of the Weierstrass Family over the field of rational numbers $\mathbb{Q}$. $H^i(U, \mathbb{Q}[g_2, g_3])$. By the definition of the lifted $p$-adic homology with compact supports [6, p. 415], applied to the characteristic zero case, we have

\begin{equation}
H^i(U, \mathbb{Q}[g_2, g_3]) = H^3(A^2(\text{Spec}(\mathbb{Q}[g_2, g_3])), A^2(\text{Spec}(\mathbb{Q}[g_2, g_3]))) - U, \Gamma_{Q_{g_2, g_3}}(\text{Spec}(\mathbb{Q}[g_2, g_3]))).
\end{equation}

If one tensors $H^i(U, \mathbb{Q}[g_2, g_3])$ with $\Delta^{-1}\mathbb{Q}[g_2, g_3]$ over $\mathbb{Q}[g_2, g_3]$, one has the free $\Delta^{-1}\mathbb{Q}[g_2, g_3]$-module of rank two, where $\Delta = g_2^3 - 27g_3^2$. This is so because we have the universal coefficients spectral sequence

\begin{equation}
E^2_{0,1} = H^i(U, \mathbb{Q}[g_2, g_3]) \otimes_{\mathbb{Q}[g_2, g_3]} \Delta^{-1}\mathbb{Q}[g_2, g_3] \Rightarrow H^i(U, \Delta^{-1}\mathbb{Q}[g_2, g_3]),
\end{equation}

and $\Delta^{-1}\mathbb{Q}[g_2, g_3]$ means that the ring $\mathbb{Q}[g_2, g_3]$ is localized at the discriminant $\Delta$. The computation has been made even in the $p$-adic case in [1] for this open subfamily of the Weierstrass Family.
THEOREM 1. Consider \( U = W_Q \cap A^2(\text{Spec}(\mathbb{Q}[g_2, g_3])) \), which is the closed affine subscheme of \( A^2(\text{Spec}(\mathbb{Q}[g_2, g_3])) \). Then the first homology with compact supports \( H'_i(U, \mathbb{Q}[g_2, g_3]) \) is generated by \( \{C^{-i}dx \wedge dy\}_{i \geq 1} \) and \( \{X^{-i}dx \wedge dy\}_{i \geq 1} \) as a \( \mathbb{Q}[g_2, g_3] \)-module.

REMARK 1. For the pair of affine schemes 
\( A^2(\text{Spec}(\mathbb{Q}[g_2, g_3])) \) and \( A^2(\text{Spec}(\mathbb{Q}[g_2, g_3])) - U \), where \( U \) is the closed subscheme corresponding to the polynomial \( C = Y^2 - 4X^3 + g_2X + g_3 \) in \( \mathbb{Q}[g_2, g_3, X, Y, Z] \), there is induced a long exact sequence of hypercohomology groups,
\[
\cdots \rightarrow H^n(A^2(A) - U, \Gamma^*_A(A^2(A))) \rightarrow H^n(A^2(A), \Gamma^*_A(A^2(A))) \rightarrow \cdots
\]
where \( A = \text{Spec}(\mathbb{Q}[g_2, g_3]) \).

There are three first-quadrant spectral sequences induced which have the above three hypercohomology groups as their abutments:
\[
\begin{align*}
\cdots \rightarrow & \quad H^q(A^2(A) - U, \Gamma^*_A(A^2(A))) \\
& \rightarrow H^q(A^2(A) - U, \Gamma^*_A(A^2(A))) \\
\end{align*}
\]

**LEMMA 1.** We have the following isomorphisms: the abutment
\[
H^3(A^2(A), A^2(A) - U, \Gamma^*_A(A^2(A))) \cong \text{coker}(E_1^{2,0} \rightarrow E_1^{1,0}),
\]
and
\[
H^3(A^2(A) - U, \Gamma^*_A(A^2(A))) \cong \text{coker}(E_2^{2,0} \rightarrow E_2^{1,0}).
\]

**PROOF OF LEMMA 1.** Consider the following diagram (Diagram A) with exact rows. We denote the structure sheaf of the affine scheme \( A^2(A) = A^2(\text{Spec}(\mathbb{Q}[g_2, g_3])) \) by \( \mathcal{O}_{A^2(A)} \). Therefore, we have \( E_1^{p,q} = 0 \) unless \( q = 1 \), which is abutting \( E_2^{3} = H^3(A^2(A), A^2(A) - U, \Gamma^*_A(A^2(A))) \). Then the isomorphism \( E_2^{3,1} \rightarrow E_2^{3} \) in Lemma 1 follows. Furthermore, this diagram can be rewritten as Diagram B. The remaining two isomorphisms in Lemma 1 are obtained from the well-known lemma in homological algebra, i.e., from Diagram B with the exact rows we have the induced exact sequence
\[
0 \rightarrow \ker d_1^{0,0} \rightarrow \ker d_1^{1,0} \rightarrow \ker d_1^{0,0} \rightarrow \text{coker } d_1^{1,0} \rightarrow \text{coker } d_1^{0,0} \rightarrow \text{coker } d_1^{1,1} \rightarrow 0
\]
\[
E_2 \rightarrow E_2^1 \rightarrow E_3 \rightarrow 0
\]
and since the \( \mathbb{Q}[g_2, g_3] \)-homomorphism

\[
d_1^{0}: E_1^{0} = \Gamma_{\mathbb{Q}[g_2, g_3]}(\mathbb{Q}[g_2, g_3, X, Y]) \rightarrow E_2^{0} = \Gamma_{\mathbb{Q}[g_2, g_3]}(\mathbb{Q}[g_2, g_3, X, Y])
\]

is an epimorphism, we have \( E_2 \approx E_2^{0} \approx 0 \). Therefore

\[
E_2 \cong \operatorname{coker} d_1^{1,0} \cong \operatorname{coker} d_1^{1,1} \cong E_3
\]

as stated in Lemma 1. Q.E.D.
Hence our computation of the abutment 
\[
H^3 = H^1(A^2(A), A^1(A) - U, \Gamma^2(A^2(A)))
\]
is reduced to compute
\[
coker\left( \Gamma^2_{Q[g_2, g_3]}(Q[g_2, g_3, X, Y, C^{-1}]) \right) \to \Gamma^2_{Q[g_2, g_3]}(Q[g_2, g_3, X, Y, C^{-1}])
\]

PROOF OF THEOREM 1. From now on we denote, "d", instead of the exterior differential, "d^1", in the spectral sequence. We have that
\[
d(C^{-k}X^iY^j dX) = (-2kC^{-k-1}X^iY^j + jC^{-k}X^iY^{j-1}) dY \wedge dX,
\]
\[
d(C^{-k}X^iY^j dY)
\]
\[
= (12kC^{-k-1}X^i+2Y^j - g_2kC^{-k-1}X^iY^j + iC^{-k}X^i-1Y^j) dX \wedge dY,
\]
in the \( Q[g_2, g_3] \)-module \( \Gamma^2_{Q[g_2, g_3]}(Q[g_2, g_3, X, Y, C^{-1}]) \), where \( C = Y^2 - 4X^3 + g_2X + g_3 \), \( i \), \( j \), and \( k \) are nonnegative integers. The equations (1) and (2) give the cohomological relations, which are denoted by " ~ " as
\[
2kC^{-k-1}X^iY^j+1 dX \wedge dY \sim jC^{-k}X^iY^j dX \wedge dY
\]
and
\[
(12kC^{-k-1}X^i+2Y^j - g_2kC^{-k-1}X^iY^j + iC^{-k}X^i-1Y^j) dX \wedge dY \sim 0.
\]
Notice that, by Lemma 1:
\[
\sim E_2^{1,1} \sim E_1^{1,1} / \text{Im}(\sim E_1^{1,1} \sim E_2^{1,1})
\]
and
\[
\sim E_1^{1,1} \sim E_1^{1,0} / \text{Im}(\sim E_1^{1,0} \sim E_1^{1,0}),
\]
where \( E_1^{1,0} \approx \Gamma^2_{Q[g_2, g_3]}(Q[g_2, g_3, X, Y]) \). Therefore it suffices to consider the integer \( k \geq 1 \) in the equations (1), (2), (3) and (4) above.

If \( j = 0 \) in (3), then \( C^{-k-1}X^iY^j dX \wedge dY \sim 0 \) for all \( i \geq 0 \) and \( k \geq 1 \). But (4) implies that \( C^{-1}X^iY^j dX \wedge dY \sim 0 \) for \( i \geq 0 \) since \( iC^{-1}X^{i-1}Y^j dX \wedge dY \sim g_2C^{-2}X^iY^j dX \wedge dY \). Therefore,
\[
C^{-k}X^iY^j dX \wedge dY \sim 0 \quad \text{for all integers } i, k \geq 0.
\]

For any odd integer \( j > 0 \) we have \( C^{-k}X^iY^j dX \wedge dY \sim 0 \) by combining (3) and (5) and the repeated use of (4). For example, for \( j = 3 \), we have \( 12kC^{-k-1}X^iY^3 dX \wedge dY \sim 2C^{-k}X^iY^3 dX \wedge dY \), which is cohomologous to zero by (5). Then apply (4) for \( j = 3 \) to get
\[
iC^{-k}X^{i-1}Y^3 dX \wedge dY \sim g_3kC^{-k-1}X^iY^3 dX \wedge dY \sim 12kC^{-k-1}X^{i+1}Y^3 dX \wedge dY.
\]
But the right-hand side is cohomologous to zero from the above result. If \( i = 0 \) in (4), we then have
\[
12kC^{-k-1}X^2Y^j dX \wedge dY \sim g_3kC^{-k-1}Y^j dX \wedge dY
\]
for all integers \( k \geq 1 \) and \( j \geq 0 \). Especially we have, for \( j = 0 \), \( 12kC^{-k-1}X^2 dX \wedge dY \sim g_3kC^{-k-1}X dX \wedge dY \). Then it can be plainly seen that
\[
(C^{-k}dX \wedge dY)_{k \geq 1}, \quad (XC^{-1}dX \wedge dY)_{k \geq 0} \quad \text{and} \quad (X'C^{-1}dX \wedge dY)_{i \geq 2}
\]
generate all the elements of the type \( X^iC^{-l}dX \wedge dY \) for integers \( i \geq 0 \) and \( k \geq 0 \) over the ring \( \mathbb{Q}[g_2, g_3] \) from equations (3) and (4). In particular, \( X^2C^{-1}dX \wedge dY \sim X^2Y^2C^{-2}dX \wedge dY \) by (3) for letting \( i = 2, j = 1 \) and \( k = 1 \), but \( X^2Y^2C^{-2}dX \wedge dY \sim Y^2C^{-3}dX \wedge dY \) by (4) for \( i = 0, j = 2 \) and \( k = 1 \); furthermore, \( Y^2C^{-3}dX \wedge dY \) is cohomologous to \( C^{-1}dX \wedge dY \) from (3) for \( i = 0, j = 1 \) and \( k = 1 \). Hence we have established that \( X^2C^{-1}dX \wedge dY \sim C^{-1}dX \wedge dY \). Next we claim that all the elements of the type \( (X^iC^{-1}dX \wedge dY)_{i \geq 3} \) are generated by the two elements \( C^{-1}dX \wedge dY \) and \( XC^{-1}dX \wedge dY \) over the ring \( \mathbb{Q}[g_2, g_3] \). We have the following recursive formula for integers \( i \geq 3 \) from (3) and (4):

\[
4X^iC^{-1}dX \wedge dY \sim g_2 \left( \frac{1}{12(i-2)} + 1 \right) X^{i-2}C^{-1}dX \wedge dY + \left( g_3 - \frac{1}{i-2} \right) X^{i-3}C^{-1}dX \wedge dY.
\]

Therefore it follows from this recursive formula that \( (X^iC^{-1}dX \wedge dY)_{i \geq 3} \) are generated by \( C^{-1}dX \wedge dY \) and \( XC^{-1}dX \wedge dY \) over \( \mathbb{Q}[g_2, g_3] \). We have established the statement of Theorem 1 for the elements \( X^iY^jC^{-l}dX \wedge dY, i \geq 1, j = 0 \) and \( k \geq 1 \). Now we need consider the elements \( X^iY^jC^{-l}dX \wedge dY, i \geq 1 \) and \( j \geq 1 \), provides the generation of the first homology with compact supports \( H_1(U, \mathbb{Q}[g_2, g_3]) \) of the Weierstrass Family by the elements \( (C^{-1}dX \wedge dY)_k \) and \( (XC^{-1}dX \wedge dY)_k \). Q.E.D.

**PROPOSITION 1.** Assumptions and notations being the same as in Theorem 1, \( H_1(U, \mathbb{Q}[g_2, g_3]) \otimes_{\mathbb{Q}[g_2, g_3]} (\Delta^{-1}\mathbb{Q}[g_2, g_3]) \) is a free \( (\Delta^{-1}\mathbb{Q}[g_2, g_3])-\)module of rank two, i.e., it is generated by \( XC^{-1}dX \wedge dY \) and \( C^{-1}dX \wedge dY \), where \( \Delta \) is the discriminant, \( \Delta = g_2^2 - 27g_3^2 \), and \( \Delta^{-1}\mathbb{Q}[g_2, g_3] \) is localized at the discriminant \( \Delta \).

**PROOF OF PROPOSITION 1.** For any integer \( i \geq 2 \) we have

\[
C^{-(i-1)} = C^{-i}(Y^2 - 4X^3 + g_2X + g_3),
\]

where \( dX \wedge dY \) is omitted for simplicity, and from equations (3), (4) and (6) we have the following cohomologous relation for \( i \geq 2 \):

\[
\frac{6i - 11}{6(i-1)} X^{-(i-1)} \sim \frac{2g_2}{3} X^{-(i-1)} + g_3 C^{-i}.
\]

Similarly, one has the corresponding formula for \( XC^{-(i-1)} \) by the equations (3), (4) and (6):

\[
\frac{6i - 13}{6(i-1)} XC^{-(i-1)} \sim \frac{g_2^2}{18} C^{-i} + g_3 X^{-(i-1)}.
\]
We finally have for $i \geq 2$,

\[ C^{-i}dX \wedge dY \sim \frac{12i - 22}{12(i-1)} C^{-(i-1)}dX \wedge dY \]

from equations (1.1) and (1.2).

Equations (1.3) and (1.1) prove that $H_f(U, \mathbb{Q}[g_2, g_3]) \otimes_{\mathbb{Q}[g_2,g_3]} (\Delta^{-1}\mathbb{Q}[g_2,g_3])$ is generated by $XC^{-i}dX \wedge dY$ and $C^{-i}dX \wedge dY$ as a $(\Delta^{-1}\mathbb{Q}[g_2,g_3])$-module. Q.E.D.

**COROLLARY 1.** Let $V^0_Q$ be the closed subfamily defined by “$g_2 = 0$” of the whole Weierstrass Family $W_Q$. Then the first homology with compact supports,

\[ H_f(V^0_Q \cap A^2(\text{Spec } \mathbb{Q}[g_3]), \mathbb{Q}[g_3]) \]

is generated by $XC^{-i}dX \wedge dY$, and $XC^{-i}dX \wedge dY$, as a $\mathbb{Q}[g_3]$-module.

**PROOF.** In (1.1) and (1.2) in the proof of Proposition 1, we have the following corresponding equations for the closed subfamily $V^0_Q$ defined by “$g_2 = 0$”:

\[ \begin{align*}
(1.1)^0 & \quad \frac{12i - 22}{12(i-1)} C^{-(i-1)} \sim g_2 C^{-i}, \\
(1.2)^0 & \quad \frac{6i - 13}{6(i-1)} XC^{-(i-1)} \sim g_3 XC^{-i}.
\end{align*} \]

Then the statement of Corollary 1 follows plainly from (1.1)$^0$ and (1.2)$^0$. Q.E.D.

**Note 1.** The equations (1.1)$^0$ and (1.2)$^0$ also show that Corollaries 2 and 3 are true.

**COROLLARY 2.** The first homology with compact supports of the singular fibre $U_\varphi$ over a point $\varphi = (g_2 = 0, g_3 = 0) \in \text{Spec}(\mathbb{Q}[g_2, g_3])$, a projective line with a cusp (or $\varphi = (g_3 = 0) \in \text{Spec}(\mathbb{Q}[g_3])$, $H_f(U_\varphi, \mathbb{Q})$, is trivial.

**COROLLARY 3.** Notations being the same as in Proposition 1,

\[ H_f(V^0_Q \cap A^2(\text{Spec } \mathbb{Q}[g_3]), \mathbb{Q}[g_3]) \otimes_{\mathbb{Q}[g_3]} (g_3^{-1}\mathbb{Q}[g_3]) \]

is generated by the two elements $C^{-i}dX \wedge dY$ and $XC^{-i}dX \wedge dY$, where $g_3^{-1}\mathbb{Q}[g_3]$ means the localization of the ring $\mathbb{Q}[g_3]$ at $g_3$.

**REMARK 2.** For a point $\varphi \neq (g_3 = 0)$, $H_f(U_\varphi, \mathbb{K}(\varphi))$ is generated by $C^{-i}dX \wedge dY$ and $XC^{-i}dX \wedge dY$ as a $\mathbb{K}(\varphi)$-vector space and where $\mathbb{K}(\varphi)$ is the characteristic zero residue field, i.e., $U_\varphi$ is an elliptic curve. Note that the open subfamily of the Weierstrass Family over $\mathbb{Z}/p\mathbb{Z}$ defined by “$\Delta \neq 0$” has been computed explicitly using the hypercohomology of a flat lifting with coefficients in the $\mathbb{Z}$ of sheaves of differential forms, $H_f(U, (\Delta^{-1}\mathbb{Z}_p[g_2, g_3]) \otimes \mathbb{Q})$, where $\mathbb{Z}_p[g_2, g_3]$ is the $\mathbb{Z}$ of the localization of the ring $\mathbb{Z}_p[g_2, g_3]$ at the discriminant $\Delta = g_2^2 - 27g_3^3$, see [1].

The following universal coefficient spectral sequence explains the relationship between Corollary 2 and Theorem 1.

\[ E^2_{p,q} = \text{Tor}_{\mathbb{Q}[g_2,g_3]}^p(H_f(U, \mathbb{Q}[g_2, g_3]), \mathbb{K}(\varphi)), \text{ with the abutment } H_f^n(U_\varphi, \mathbb{K}(\varphi)), \text{ where } \varphi = (g_2 = g_3 = 0) \in \text{Spec}(\mathbb{Q}[g_2, g_3]) \] and $\mathbb{Q} = \mathbb{K}(\varphi)$. 


Corollary 4. Let $\mathcal{V}_g^3$ be the closed subfamily of the Weierstrass Family $W_3$, defined by $g_2 = 3$. Then $H^1(\mathcal{V}_g^3 \cap \mathbb{A}^2(\text{Spec} \mathbb{Q}[g_3]), \mathbb{Q}[g_3])$ is generated by $\{C^{-k}dX \wedge dY\}_{k \geq 1}$ and $(XC^{-k}dX \wedge dY)_{k \geq 1}$ as a $\mathbb{Q}[g_3]$-module. Moreover the first homology with compact supports of the singular fibre over the point $\psi = (g_3 = 1)$ in the base $\text{Spec}(\mathbb{Q}[g_3])$, a projective line with an ordinary double point over $K(\psi)$, is generated by one element as a $K(\psi)$-vector space. One can then take either $C^{-1}dX \wedge dY$ or $XC^{-1}dX \wedge dY$ as the base element for the vector space.

Proof. We only need prove the latter statement. From equations (1.1) and (1.2), we have (1.1)$^\prime$ and (1.2)$^\prime$ as follows:

\begin{align*}
(1.1)^\prime & \quad \frac{6i - 11}{6(i - 1)} C^{-i - 1} \sim 2XC^{-i} + C^{-i}, \\
(1.2)^\prime & \quad \frac{6i - 13}{6(i - 1)} XC^{-i - 1} \sim \frac{1}{2} C^{-i} + XC^{-i}.
\end{align*}

Then we have $2(6i - 13)XC^{-i - 1} \sim (6i - 11)C^{-i - 1}$ for $i \geq 2$. Hence this vector space is one dimensional and the statement of Corollary 4 follows. Q.E.D.

Note 2. For the closed subfamily $\mathcal{V}_g^3$ of the Weierstrass Family we have the following equations (1.1)$^3$, (1.2)$^3$ and (1.3)$^3$:

\begin{align*}
(1.1)^3 & \quad \frac{6i - 11}{6(i - 1)} C^{-i - 1} \sim 2XC^{-i} + g_3C^{-i}, \\
(1.2)^3 & \quad \frac{6(i - 13)}{6(i - 1)} XC^{-i - 1} \sim \frac{1}{2} C^{-i} + g_3XC^{-i}, \\
(1.3)^3 & \quad (g_3^2 - 1)C^{-i} \sim \frac{1}{6(i - 1)} \left\{g_3(6i - 11)C^{-i - 1} - 2(6i - 13)XC^{-i - 1}\right\},
\end{align*}

for integers $i \geq 2$.

Note 3. This paper has been entirely in characteristic zero. The case of nonzero characteristic $p \neq 2, 3$ will appear in a forthcoming paper [2], which is a generalization of the paper [1], where an open subfamily "$\Delta \neq 0$" of the Weierstrass Family was studied.

References


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