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MINIMAX-OPTIMAL STOP RULES AND DISTRIBUTIONS IN SECRETARY PROBLEMS

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For the secretary (or best-choice) problem with an unknown number N of objects, minimax-optimal stop rules and (worst-case) distributions are derived, under the assumption that N is a random variable with unknown distribution, but known upper bound n. Asymptotically, the probability of selecting the best object in this situation is of order of $(\log n)^{-1}$. For example, even if the only information available is that there are somewhere between 1 and 100 objects, there is still a strategy which will select the best item about one time in five.

1. Introduction. In the classical secretary problem, a known number of rankable objects is presented one by one in random order (all $n!$ possible orderings being equally likely). As each object is presented, the observer must either select it and stop observing or reject it and continue observing. He may never return to a previously rejected object, and his decision to stop must be based solely on the relative ranks of the objects he has observed so far. The goal is to maximize the probability that the best object is selected. This problem, also known as the marriage problem or best-choice problem, is well known, and the reader is referred to Freeman (1983) and Ferguson (1989) for a history and review of the literature.

Suppose now that the total number of objects is not known, but is a random variable N taking values in $\{1, 2, \dots, n\}$, where n is a known fixed positive integer. How should the observer play in order to guarantee the highest probability of selecting the best object, what is this probability and what is the worst distribution for N? The main goal of this paper is to determine these minimax-optimal stop rules, values and distributions as a function of n. For example, if $n = 5$, the strategy "stop with the first object with probability $26/75$; otherwise continue and stop with the second object with probability $26/49$ provided it is better than the first object; and otherwise stop the first time an object is observed which is better than any previously observed object" is minimax-optimal. This strategy will select the best object with probability at least $26/75$ for all distributions of N (≤ 5), and that probability is best possible. Conversely, if N has the distribution $P(N = 1) = 13/75$, $P(N = 2) = 2/75$, $P(N = 5) = 60/75$, then no strategy will select the best object with probability greater than $26/75$, so this distribution is also mini-

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max. (It is assumed that, given N, all $N!$ orderings are equally likely, and that if an object is rejected and no more objects remain, the game is over and the best object has not been selected.)

A number of results are known for the general situation where the number of objects N is a random variable. Presman and Sonin (1972) derive optimal stop rules when N has a known prior distribution and mention the necessarily complex form ("islands") of optimal stop rules for certain prior distributions. Irle (1980) gives a concrete example of such a prior for which the optimal stop rule has these islands and sufficient conditions for existence of simple "non-island" stop rules. Abdel-Hamid, Bather and Trustrum (1982) derive necessary and sufficient conditions for admissibility of randomized stop rules.

Extensions to the situation where the interarrival times of the objects are continuous random variables with known distributions have been studied by Presman and Sonin (1972), Gianini and Samuels (1976) and Stewart (1981). More recently, Bruss (1984) and Bruss and Samuels (1987) derive surprising and very general minimax-optimal strategies in this same context and even for more general loss functions. In contrast to the minimax-optimal stop rules derived in this paper, which are based on knowledge of a bound for N, those of Bruss and Samuels are based on knowledge of the distributions of the continuous i.i.d. interarrival times; in this sense our results complement theirs.

This paper is organized as follows: Section 2 contains notation, results for the classical secretary problem and basic results concerning randomized stop rules; Section 3 contains the statements of the main results and examples; Sections 4 and 5 contain the proofs of the minimax-optimal stop rules and distributions, respectively; and Section 6 contains remarks and asymptotics.

2. Preliminaries. A well-known equivalent formulation of the classical secretary problem is the following. $R_1, R_2, \dots, R_n$ are independent random variables on a probability space $(\Omega, \mathscr{F}, P)$, where n is a fixed positive integer and $P(R_j = i) = j^{-1}$ for all $i \in \{1, 2, \dots, j\}$ and all $j \in \{1, 2, \dots, n\}$. If $\hat{\mathscr{T}}_n$ denotes the stop rules for $R_1, R_2, \dots, R_n$, then the value of a stop rule $t \in \hat{\mathscr{T}}_n$ (given that there are n objects) is

$$V(t|N = n) = P(R_t = 1 \text{ and } R_j > 1 \text{ for all } j > t);$$

that is, $V(t|N = n)$ is the probability of selecting the best object using the stop-rule strategy t, given that there are n objects. The goal is to find a t making $V(t|N = n)$ as large as possible, and the solution to this problem is well known [cf. Ferguson (1989) and Freeman (1983)] and is recorded here for ease of reference. Throughout this paper, $s_0 = 0$, and for $j \geq 1$, $s_j = \sum_{i=1}^{j} i^{-1}$.

DEFINITION 2.1. For each positive integer n, k_n is the nonnegative integer satisfying

$$s_{n-1} - s_{k_n - 1} \geq 1 > s_{n-1} - s_{k_n}.$$

PROPOSITION 2.2. *The stop rule* $\hat{t}_n \in \hat{\mathscr{T}}_n$ *defined by* $\hat{t}_n = \min\{\min\{j > k_n: R_j = 1\}, n\}$ *is optimal, that is,*

$$V(\hat{t}_n | N = n) = \sup_{t \in \hat{\mathscr{T}}_n} V(t | N = n).$$

In other words, given that there are n objects, the optimal strategy is to observe the first k_n objects without stopping and then to stop with the first object, if any, that is better than any object previously seen. It is well known that $n/k_n \to e$ as $n \to \infty$, and the next example records a few typical values of n.

EXAMPLE 2.3. $k_1 = k_2 = 0$, $k_3 = k_4 = 1$, $k_5 = k_6 = k_7 = 2$ and $k_8 = k_9 = k_{10} = 3$.

Next, the above notations will be generalized to the setting where the number of objects N is a random variable and randomized stop rules are allowed. (In the classical setting of a fixed known number of objects, it is clear that randomization does not help, that is, $\hat{t}_n$ is also optimal among the larger class of randomized stop rules.)

For each positive integer n, Π_n denotes the set of probabilities on $\{1, 2, \dots, n\}$, so $\mathbf{p} \in \Pi_n$ is of the form $\mathbf{p} = (p_1, p_2, \dots, p_n)$, where $p_i \geq 0$ for all i and $\sum_{i=1}^n p_i = 1$.

N is a random variable with distribution $\mathscr{L}(N) \in \Pi_n$, $R_1, \dots, R_n$ are as above and independent of N and $\mathscr{T}_n$ denotes the set of *randomized* stop rules for $R_1, \dots, R_n$, that is, $t \in \mathscr{T}_n$ means that $\{t = i\}$ is in the σ-algebra generated by $R_1, U_1, \dots, R_i, U_i$, where $U_1, U_2, \dots$ are i.i.d. $U[0,1]$ random variables which are independent of the $\{R_i\}$ process and of N. In other words, the observer is allowed to base his selection rule not only on the observed relative ranks, but also on an independent event, say flipping a coin or using a random number generator. Clearly the only stop rules which are of interest (for the goal of selecting the best object) are those which never stop with an object which is not the best seen so far, so every "reasonable" $t \in \mathscr{T}_n$ may be described by $t = (q_1, q_2, \dots, q_n) \in [0,1]^n$, where q_i is the probability that $t = i$, given that $R_i = 1$ and $t > i - 1$. Accordingly, it will be assumed throughout that only such stop rules are used, so $\mathscr{T}_n$ is essentially $[0,1]^n$. The stop rule $t = (q_1, q_2, \dots, q_n)$ describes the selection strategy "stop with the first object with probability q_1 (i.e., if $U_1 \leq q_1$); otherwise continue observing and if the second object is better than the first, stop with probability q_2 (i.e., $U_2 \leq q_2$); otherwise continue, ..." [see Abdel-Hamid, Bather and Trustrum (1982)]. To relate this to the classical problem, Proposition 2.2 says that if $N \equiv n$, then an optimal stop rule is $(0, \dots, 0, 1, \dots, 1)$, where k_n zeros precede $n - k_n$ ones. [Formally speaking, the above stop rules $(q_1, \dots, q_n)$ are not forced to stop by time n, but since stopping with a relative rank less than 1 is worth nothing, it is easily seen that forcing a stop by time n changes nothing.]

DEFINITION 2.4. For $t = (q_1, \dots, q_n) \in \mathscr{T}_n$ and $\mathbf{p} = (p_1, \dots, p_n) \in \Pi_n$, the value of using t given that the distribution of N is $\mathbf{p}$, $V(t|\mathbf{p})$, is given by

$$V(t|\mathbf{p}) = P(t \le N \text{ and } R_t = 1 \text{ and } R_i > 1$$
$$\forall\, i \in \{t+1, t+2, \dots, N\} | \mathscr{L}(N) = \mathbf{p}).$$

(Recall the assumption that if the observer rejects the jth object and $N = j$, then he loses.)

The next lemma is found in Abdel-Hamid, Bather and Trustrum (1982) and is recorded here for completeness. (For notational convenience, the product over an empty set is taken to be 1.)

LEMMA 2.5. *For $t = (q_1, q_2, \dots, q_n) \in \mathscr{T}_n$ and $\mathbf{p} = (p_1, \dots, p_n) \in \Pi_n$,*

$$V(t|\mathbf{p}) = \sum_{j=1}^{n} p_j j^{-1} \sum_{i=1}^{j} q_i \prod_{m=1}^{i-1} (1 - m^{-1} q_m).$$

PROOF. Using t, the probability that all of the first m objects will be rejected, $r(t, m)$, is

$$r(t, m) = (1 - q_1)(1 - q_2/2) \cdots (1 - q_m/m)$$

and if $N = j$, the probability of winning with this rule t is $V(t|N = j) = j^{-1}\sum_{i=1}^{j} q_i r(t, i-1)$. Since $V(t|\mathbf{p}) = \sum_{j=1}^{n} p_j V(t|N = j)$, this yields the desired equality. □

3. Main theorems and examples. Recall that $s_j = \sum_{i=1}^{j} i^{-1}$ and k_n is the "cutoff" for the optimal rule in the classical secretary problem with n objects (Definition 2.1 and Proposition 2.2).

DEFINITION 3.1. Let $\alpha_1 = 1$, $\alpha_2 = 1/2$ and, for $n > 2$,

$$\alpha_n = \frac{s_{n-1} - s_{k_n - 1}}{(n - k_n)/k_n + (s_{n-1} - s_{k_n - 1}) s_{k_n}}.$$

(See Table 1 for α_n, $n = 3, 4, 5, 10$.)

Recall also that $\mathscr{T}_n$ is the set of randomized stop rules for n objects, Π_n is the set of probabilities on $\{1, \dots, n\}$ and $V(t|\mathbf{p})$ is the probability of selecting the best object using t, given that the distribution of the number of objects is $\mathbf{p}$. The following three theorems are the main results of this paper.

THEOREM A. $\sup_{t \in \mathscr{T}_n} \inf_{\mathbf{p} \in \Pi_n} V(t|\mathbf{p}) = \alpha_n = \inf_{\mathbf{p} \in \Pi_n} \sup_{t \in \mathscr{T}_n} V(t|\mathbf{p})$.

REMARK. Although each of the terms in the definition of α_n has a natural probabilistic interpretation (e.g., $s_i - s_j$ is the expected number of relative rank 1 candidates occurring between the ith and jth candidates), the authors know of no intuitive explanation why α_n should be the minimax constant appearing in Theorem A.

THEOREM B (Minimax-optimal stop rule). *If* $t_n^* = (q_1^*, \dots, q_n^*) \in \mathscr{T}_n$ *is defined by*

$$q_j^* = \begin{cases} \left(\alpha_n^{-1} - s_{j-1}\right)^{-1} & \textit{for } j = 1, \dots, k_n, \\ 1 & \textit{for } k_n < j \le n, \end{cases}$$

then $V(t_n^* | \mathbf{p}) \ge \alpha_n$ *for all* $\mathbf{p} \in \Pi_n$.

THEOREM C (Minimax-optimal distribution for N). *If* $P_n^* = (p_1^*, \dots, p_n^*) \in \Pi_n$ *is defined by*

$$p_j^* = \begin{cases} \alpha_n (j+1)^{-1} & \textit{for } j < k_n, \\ \alpha_n\left(1 - (s_{n-1} - s_{k_n - 1})^{-1}\right) & \textit{for } j = k_n, \\ 0 & \textit{for } k_n < j < n \end{cases}$$

(*so for* $n \le 2$, $p_n^* = 1$ *and for* $n > 2$, $p_n^* = n\alpha_n[k_n(s_{n-1} - s_{k_n-1})]^{-1}$), *then*

$$V(t | P_n^*) \le \alpha_n \quad \textit{for all } t \in \mathscr{T}_n.$$

[Verification of the above expression for p_n^* and of the fact that $q_i^* \in [0, 1]$ is left to the reader; this requires only elementary algebra applied to the definitions of α_n, k_n and s_n. For example, to show $q_i^* \le 1$, the monotonicity of the $\{s_j\}$ implies that it is enough to show that $\alpha_n \le (1 + s_{k_n - 1})^{-1}$, and using the definition of α_n and s_j this is equivalent to $(k_n - 1)(s_{n-1} - s_{k_n-1}) \le n - k_n$, which clearly holds.]

Table 1 lists the minimax values $\{\alpha_n\}$, and the minimax-optimal stop rules and distributions for several values of n.

REMARKS. Irle's (1980) example of an "unpleasant" distribution, that is, a distribution for which *no* stop rule of the form $(0, 0, \dots, 0, 1, 1, \dots, 1)$ is optimal, is $\mathbf{p} = (0, 0.895, 0.001, 0.001, \dots, 0.001, 0.1) \in \Pi_8$, for which he calculates the value of the optimal stop rule $(0, 1, 0, 1, 1, 1, 1, 1)$ to be approximately

TABLE 1

n	k_n	α_n	$t_n^* = (q_1^*, \dots, q_n^*)$	$P_n^* = (p_1^*, \dots, p_n^*)$
1	0	1	(1)	(1)
2	0	$\frac{1}{2}$	$(1, 1)$	$(0, 1)$
3	1	$\frac{3}{7}$	$(\frac{3}{7}, 1, 1)$	$(\frac{1}{7}, 0, \frac{6}{7})$
4	1	$\frac{11}{29}$	$(\frac{11}{29}, 1, 1, 1)$	$(\frac{5}{29}, 0, 0, \frac{24}{29})$
5	2	$\frac{26}{75}$	$(\frac{26}{75}, \frac{26}{49}, 1, 1, 1)$	$(\frac{13}{75}, \frac{2}{75}, 0, 0, \frac{60}{75})$
10	3	0.278^+	$(0.278^+, 0.386^+, 0.478^+, 1, 1, \dots, 1)$	$(0.139^+, 0.092^+, 0.068^+, 0, 0, \dots, 0, 0.698^+)$

0.482. Comparison of this value with those in Table 1 suggests that such island distributions are far from being worst-case (i.e., minimax-optimal), although a direct proof of this is not known to the authors.

It should also be observed that the minimax-optimal distribution for N is not one of the other "naive-guess" distributions such as $N \equiv n$ or N uniformly distributed on $\{1, 2, \dots, n\}$ or $N = 1$ with probability p and $= n$ with probability $1 - p$. As far as the authors know, this P_n^* is a new distribution on n points.

Clearly Theorem A follows from Theorems B and C. No direct proof that sup inf = inf sup is known to the authors; although $V(t|\mathbf{p})$ is linear in $\mathbf{p}$ and Π_n is convex and compact, $V(t|\mathbf{p})$ is neither convex nor concave in t, and known generalizations of the classical minimax theorem of game theory do not seem to apply. (The results in this paper may also be interpreted as a zero-sum two-person game as follows. Player I picks the distribution of N, and player II picks the stop-rule or selection-strategy t; if t selects the best of the N objects, then player I pays player II one dollar; and otherwise no money changes hands. The constant α_n then represents the value of this game.)

4. Proof of Theorem B. The conclusions of Theorems B and C are trivial for $n = 1$ and easy for $n = 2$, so for the remainder of this paper, n will be a fixed integer strictly bigger than 2, and to simplify notation, $k = k_n$, $\mathscr{T} = \mathscr{T}_n$ and $\Pi = \Pi_n$. (Observe that $n > 2$ precludes the degenerate cases where $k = 0$; see Example 2.3.)

LEMMA 4.1. *Suppose $\{a_i\}_{i=1}^n$ are real numbers, and j and $\hat{k}$ are positive integers satisfying $n \geq j \geq \hat{k}$. If both*

$$(1) \qquad (a_1 + \cdots + a_{\hat{k}})/\hat{k} \geq (a_1 + \cdots + a_j)/j$$

and

$$(2) \qquad a_m \geq a_{m+1} \quad \text{for all } m \in \{\hat{k}+1, \hat{k}+2, \dots, n\},$$

then $(a_1 + \cdots + a_j)/j \geq (a_1 + \cdots + a_n)/n$.

PROOF. If $j = n$, the conclusion is trivial, so assume $j < n$. Then conditions (1) and (2), respectively, imply

$$\begin{aligned}(a_1 + \cdots + a_j)/j &\geq (a_{\hat{k}+1} + \cdots + a_j)/(j - \hat{k}) \\ &\geq (a_{j+1} + \cdots + a_n)/(n - j),\end{aligned}$$

so $(a_1 + \cdots + a_j)/j \geq (a_1 + \cdots + a_n)/n$. □

PROPOSITION 4.2.

$$\sup_{t \in \mathscr{T}} \inf_{\mathbf{p} \in \Pi} V(t|\mathbf{p}) = \max_{t = (q_1, \dots, q_n) \in \mathscr{T}:\, q_i = 1\, \forall\, i > k} \; \min_{j \in \{1, 2, \dots, k, n\}} V(t|N = j).$$

PROOF. Since $V(t|\mathbf{p})$ is continuous in both t and $\mathbf{p}$, and since $\mathscr{T}$ and Π are compact, the sup and inf are attained. Moreover

$$\inf_{\mathbf{p}\in\Pi} V(t|\mathbf{p}) = \min_{j\le n} V(t|N=j), \tag{3}$$

since $V(t|\mathbf{p})$ is linear in $\mathbf{p}$, and Π is the set of all probabilities on $\{1,2,\dots,n\}$.

The proof of the optimality of the backward induction procedure implies that if t is any stopping time for an adapted sequence of σ-algebras $\mathscr{F}_1 \subset \mathscr{F}_2 \subset \cdots \subset \mathscr{F}_j$ and t^* the optimal stopping time, then $V(t') \ge V(t)$ if t' is obtained from t by stopping at time i on an arbitrary $\mathscr{F}_i$-measurable subset of $\{t > i, t^* = i\}$. Hence, by Proposition 2.2 replacing an arbitrary $t = (q_1,\dots,q_n) \in \mathscr{T}$ by $\hat{t} = (q_1,\dots,q_k,1,1,\dots,1) \in \mathscr{T}$ results in at least as high a probability of selecting the best object for any given (deterministic) number of objects ($\le n$), that is,

$$V(\hat{t}|N=j) \ge V(t|N=j) \quad \text{for all } j \le n. \tag{4}$$

Together, (3), (4) and the compactness of $\mathscr{T}$ imply

$$\sup_{t\in\mathscr{T}} \inf_{\mathbf{p}\in\Pi} V(t|\mathbf{p}) = \max_{t=(q_1,\dots,q_n)\in\mathscr{T}:\, q_i=1\,\forall\, i>k}\ \min_{j\le n} V(t|N=j). \tag{5}$$

To complete the proof of the proposition, it is enough to show that for all $t = (q_1,\dots,q_k,1,\dots,1) \in \mathscr{T}$, and for all $j \in \{k+1, k+2,\dots,n-1\}$,

$$V(t|N=j) \ge \min\{V(t|N=k), V(t|N=n)\}. \tag{6}$$

Fix $t = (q_1,\dots,q_k,1,\dots,1) \in [0,1]^n$ and define real numbers $\{a_i\}_{i=1}^n$ as follows: $a_1 = q_1$ and $a_i = q_i\prod_{m=1}^{i-1}(1 - m^{-1}q_m)$ for $i > 1$.

Since $q_i = 1$ for all $i > k$ and $q_j \in [0,1]$ for all j,

$$a_m > a_{m+1} \quad \text{for all } m > k. \tag{7}$$

By Lemma 2.5, $V(t|N=j) = (a_1 + \cdots + a_j)/j$ for all $j \in \{1,2,\dots,n\}$. To establish (6), suppose $V(t|N=j) \le V(t|N=k)$, that is,

$$(a_1 + \cdots + a_k)/k \ge (a_1 + \cdots + a_j)/j. \tag{8}$$

By (7) and (8) and Lemma 4.1 (with $\hat{k} = k$),

$$V(t|N=j) = (a_1 + \cdots + a_j)/j \ge (a_1 + \cdots + a_n)/n = V(t|N=n),$$

which establishes (6). □

LEMMA 4.3. *For all $t = (q_1,\dots,q_k,1,1,\dots,1) \in \mathscr{T}$,*

$$V(t|N=n) = kn^{-1}\left[(s_{n-1} - s_{k-1})\left(1 - \sum_{j=1}^{k-1}(j+1)^{-1}V(t|N=j)\right) - (s_{n-1} - s_{k-1} - 1)V(t|N=k)\right].$$

PROOF. First it will be shown that

$$\prod_{m=1}^{j}\left(1-m^{-1}q_m\right)=1-\sum_{m=1}^{j-1}(m+1)^{-1}V(t|N=m) - V(t|N=j) \quad \text{for all } j\le k. \tag{9}$$

The proof of (9) is by induction on j. For $j=1$, $(1-q_1)=1-V(t|N=1)$ by Lemma 2.5. Assume that the equality in (9) holds for all $j\le \hat{k}$ and calculate

$$\begin{aligned}\prod_{m=1}^{\hat{k}+1}\left(1-m^{-1}q_m\right) &= \prod_{m=1}^{\hat{k}}\left(1-m^{-1}q_m\right)-(\hat{k}+1)^{-1}q_{\hat{k}+1}\prod_{m=1}^{\hat{k}}\left(1-m^{-1}q_m\right)\\ &=1-\sum_{m=1}^{\hat{k}-1}(m+1)^{-1}V(t|N=m)-V(t|N=\hat{k})\\ &\quad -V(t|N=\hat{k}+1)+\hat{k}(\hat{k}+1)^{-1}V(t|N=\hat{k})\\ &=1-\sum_{m=1}^{\hat{k}}(m+1)^{-1}V(t|N=m)-V(t|N=\hat{k}+1),\end{aligned}\tag{10}$$

where the second equality in (10) follows by the induction hypothesis and the fact (from Lemma 2.5) that

$$V(t|N=\hat{k}+1)=(\hat{k}+1)^{-1}q_{\hat{k}}\prod_{m=1}^{\hat{k}}\left(1-m^{-1}q_m\right)+\hat{k}(\hat{k}+1)^{-1}V(t|N=\hat{k}),$$

which establishes (9).

Since $q_j=1$ for all $j>k$, Lemma 2.5 and the definition of $\{s_j\}$ imply that

$$\begin{aligned}V(t|N=n)=n^{-1}\Bigg[&\sum_{i=1}^{k}q_i\prod_{m=1}^{i-1}\left(1-m^{-1}q_m\right)\\ &+k(s_{n-1}-s_{k-1})\prod_{m=1}^{k}\left(1-m^{-1}q_m\right)\Bigg].\end{aligned}\tag{11}$$

But $\sum_{i=1}^{k}q_i\prod_{m=1}^{i-1}(1-m^{-1}q_m)=kV(t|N=k)$ (Lemma 2.5 again), so (9) (with $j=k$) and (11) yield the desired equality. □

Heuristics. Although a direct calculus-based proof of Theorem B should be possible, the proof given below is greatly facilitated by Proposition 4.2 and Lemma 4.3, which both also serve as heuristics for the structure of the minimax-optimal stop rule. For example, Proposition 4.2 says that any general stop rule can be replaced by a stop rule with $q_i=1$ for all $i>k$, and that with such stop rules, the *critical* values occur when $N=j$ for some j in $\{1,2,\dots,k,n\}$; that is, if $N=j\in\{k+1,\dots,n-1\}$, the observer's probability of selecting the best object is at least as high as the *minimum* of the other possible values for j. (Incidentally this also suggests why the minimax-optimal distribution in Theorem C places no mass on $\{k+1,\dots,n-1\}$. For fixed t of the form known to be optimal (i.e., $q_i=1$ for all $i>k$), Lemma 4.3 implies that $V(t|N = n)$ is a decreasing function of $V(t|N = j)$ for $j \le k$. Together with Proposition 4.2, this suggests via a "Robin Hood principle" (shifting mass to decrease the maximum and increase the minimum) that the extremal case occurs when $V(t|N = 1) = V(t|N = 2) = \cdots = V(t|N = k) = V(t|N = n)$. Solving this set of k equations for the k unknowns $q_1, \ldots, q_k$ leads to the minimax-optimal stop rule in Theorem B. Once the correct extremal stop rule is guessed, of course it is then much easier to prove directly that it is in fact optimal, without justifying the derivation of the guess.

PROOF OF THEOREM B. By (6) it suffices to show that

$$V(t_n^*|N = j) = \alpha_n \quad \text{for } j = \{1, 2, \ldots, k, n\}. \tag{12}$$

To establish (12), first check by induction that $q_j^* \prod_{m=1}^{j-1}(1 - m^{-1}q_m^*) = \alpha_n$ for all $j \le k$, so Lemma 2.5 implies that $V(t_n^*|N = j) = \alpha_n$ for all $j \le k$. To check that $V(t_n^*|N = n) = \alpha_n$, use Lemma 4.3 and the fact that $(s_{n-1} - s_{k-1}) = k^{-1}\alpha_n(n - k)(1 - \alpha_n s_k)^{-1}$ to calculate

$$\begin{aligned} V(t_n^*|N = n) &= n^{-1}k\Bigg[(s_{n-1} - s_{k-1})\left(1 - \alpha_n \sum_{j=1}^{k-1} (j+1)^{-1}\right) \\ &\qquad\qquad -(s_{n-1} - s_{k-1} - 1)\alpha_n\Bigg] \\ &= n^{-1}k\left[(s_{n-1} - s_{k-1})\left(1 - \alpha_n \sum_{j=1}^{k} j^{-1}\right) + \alpha_n\right] \\ &= n^{-1}k\left[\alpha_n k^{-1}(n-k)(1 - \alpha_n s_k)^{-1}(1 - \alpha_n s_k) + \alpha_n\right] = \alpha_n. \qquad \square \end{aligned}$$

5. Proof of Theorem C. As mentioned above, Proposition 4.2 suggests that any minimax-optimal (worst-case for the observer) distribution places no mass on $\{k + 1, \ldots, n - 1\}$, and again a Robin Hood principle leads to a guess which has break-even values for each j in $\{1, 2, \ldots, k, n\}$. For example, clearly $p_1^* \le \alpha_n$, since otherwise taking $t = (1, 1, \ldots, 1)$ yields $V(t|P_n^*) > \alpha_n$. As was the case for the optimal stop rule, once a worst-case distribution P_n^* has been guessed, the check that it is in fact minimax is then much easier. Thus most of the work was hidden in the heuristics which generated the guess for P_n^*.

FORMAL ARGUMENT. It is enough to show

$$V(t|P_n^*) \le \alpha_n \quad \text{for all } t = (q_1, \ldots, q_k, 1, 1, \ldots, 1) \in \mathscr{T}, \tag{13}$$

since by (4), $V((q_1, \ldots, q_n)|\mathbf{p}) \le V((q_1, \ldots, q_k, 1, \ldots, 1)|\mathbf{p})$ for all $\{q_i\} \in [0, 1]$ and all $\mathbf{p} \in \Pi$.

[In fact, it will be seen that (13) holds with *equality* throughout, which says intuitively that against P_n^*, *all* "reasonable" stop rules, i.e., all stop rules with $q_i = 1$ for all $i > k$, select the best object with the same probability.]

Fix $t = (q_1, \dots, q_k, 1, 1, \dots, 1) \in [0, 1]^n$ and calculate

$$
\begin{aligned}
V(t|P_n^*) &= \alpha_n \left(\sum_{j=1}^{k-1} (j(j+1))^{-1} \sum_{i=1}^{j} q_i \prod_{m=1}^{i-1} (1 - m^{-1} q_m) \right. \\
&\qquad \left. + k^{-1} \sum_{i=1}^{k} q_i \prod_{m=1}^{i-1} (1 - m^{-1} q_m) + \sum_{m=1}^{k} (1 - m^{-1} q_m) \right) \\
&= \alpha_n \left(\sum_{i=1}^{k-1} \sum_{j=i}^{k-1} (j^{-1} - (j+1)^{-1}) q_i \prod_{m=1}^{i-1} (1 - m^{-1} q_m) \right. \\
&\qquad \left. + k^{-1} \sum_{i=1}^{k} q_i \prod_{m=1}^{i-1} (1 - m^{-1} q_m) + \prod_{m=1}^{k} (1 - m^{-1} q_m) \right) \\
&= \alpha_n \left(\sum_{i=1}^{k} (i^{-1} - k^{-1}) q_i \prod_{m=1}^{i-1} (1 - m^{-1} q_m) \right. \\
&\qquad \left. + k^{-1} \sum_{i=1}^{k} q_i \prod_{m=1}^{i-1} (1 - m^{-1} q_m) + \prod_{m=1}^{k} (1 - m^{-1} q_m) \right) \\
&= \alpha_n \left[\sum_{i=1}^{k} i^{-1} q_i \prod_{m=1}^{i-1} (1 - m^{-1} q_m) + \prod_{m=1}^{k} (1 - m^{-1} q_m) \right] \\
&= \alpha_n \left[\sum_{i=1}^{k} \left\{ (i^{-1} q_i - 1) \prod_{m=1}^{i-1} (1 - m^{-1} q_m) + \prod_{m=1}^{i-1} (1 - m^{-1} q_m) \right\} \right. \\
&\qquad \left. + \prod_{m=1}^{k} (1 - m^{-1} q_m) \right] \\
&= \alpha_n \left[\sum_{i=1}^{k} \left\{ \prod_{m=1}^{i-1} (1 - m^{-1} q_m) - \prod_{m=1}^{i} (1 - m^{-1} q_m) \right\} \right. \\
&\qquad \left. + \prod_{m=1}^{k} (1 - m^{-1} q_m) \right] \\
&= \alpha_n,
\end{aligned}
$$

where the first equality follows by Lemma 2.5, the second since $\sum_{j=1}^{k-1} \sum_{i=1}^{j} = \sum_{i=1}^{k-1} \sum_{j=i}^{k-1}$ and the third since the first summand disappears for $i = k$. This completes the proof of (13) and the theorem. □

6. Asymptotics. Since Π_n can be viewed as a subset of Π_{n+1}, Theorem A shows, indirectly, that the sequence $\{\alpha_n\}$ is nonincreasing. A direct check using the definition of α_n and general observations about k_n (e.g., k_{n+1} is either k_n or $k_n + 1$) shows that in fact the $\{\alpha_n\}$ are strictly decreasing in n.

Since $s_n \sim \log n$ and $k_n \sim n e^{-1}$ (where $a_n \sim b_n$ means $\lim_{n \to \infty} a_n / b_n = 1$), it follows easily that

$$\alpha_n \sim (\log n)^{-1},$$

$$q_j^* \sim \begin{cases} (\log n - \log j)^{-1} & \text{for } j \le e^{-1} n, \\ 1 & \text{for } j > e^{-1} n \end{cases}$$

and

$$p_j^* \sim \begin{cases} ((j+1)\log n)^{-1} & \text{for } j \le e^{-1} n, \\ 0 & \text{for } e^{-1} n < j < n, \\ 2(\log n)^{-1} & \text{for } j = n. \end{cases}$$

In particular, $\lim_{n \to \infty} \alpha_n = 0$, in contrast to the well-known classical result that for the deterministic case $N \equiv n$, the probability of selecting the best object (using an optimal strategy) decreases monotonically to e^{-1} as $n \to \infty$. The optimal "stopping-probabilities" $\{q_j^*\}$ are nondecreasing, which is also intuitively plausible, since if it is optimal to stop with a certain probability at time i (given $R_i = 1$), then at later times with even more information accrued it should be optimal to stop with at least as high as probability if a rank 1 object is observed.

The following alternative possible derivation of the asymptotic result $\alpha_n \sim (\log n)^{-1}$ has been given by Samuels (1989). Since the *expected* number of relatively best ones ("records") will be about $\log N$, this suggests that the rule *stop with probability* $1/\log n$ *at each of the first* $\log n$ *records* will succeed with probability about $1/\log n$ no matter what the distribution of N is. (A formal derivation using this approach seems to require more information about the actual distribution of the number of records than just its expectation.)

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