Nash Bargaining without Convexity*

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Abstract

In this note I study Nash bargaining when the utility possibility set of the bargaining problem is non-convex. A simple variation of Nash's symmetry axiom is all that is necessary to establish a set valued version of Nash's solution in non-convex settings.

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1. Introduction

The Nash bargaining solution needs no introduction. Since its development by Nash in 1950 it has been a central tool in the analysis of bargaining problems and its layout became the departure point for all the subsequent work in the area. It is also extensively used in applications.

Unfortunately, for Nash's solution to be applied, the underlying utility possibility on top of which the bargaining problem is laid out must be a convex set. This convexity is not often found in applications and is often remedied by convexification via lotteries. While this is a satisfactory way to proceed in certain situations, it is not always so (See e.g., Conley and Wilkie, [2]). It would therefore be nice if we could apply Nash's solution to any problem, even non-convex ones.

To deal with non-convexities I relax the requirement of finding a point-valued solution to a bargaining problem. One then has to accept set-valued solutions, as exemplified in Anant, Mukherji and Basu [1]. What distinguishes this work from the related work on Nash bargaining done by Serrano and Shimomura [5], Herrero [3] or Kaneko [4] in non-convex settings is that a simple variation of Nash's symmetry axiom is all that is necessary in my paper to recover Nash's (set valued) solution as the solution to the bargaining problem.

2. Notation

Let $f(U,d) \subseteq U$ be a candidate solution to bargaining problem (U,d). U is compact but need not be convex. It can even contain only finitely many points.

Consider the following axioms:

(IAT) If $U' = \{y : y_i = \alpha_i + \beta_i x_i \text{ for some } x \in U\}$ and $d'_i = \alpha_i + \beta_i d_i$ (where $\alpha_i \in \Re, \beta_i > 0$ for all i) then $f(U', d') = \{y : y_i = \alpha_i + \beta_i x_i \text{ for some } x \in f(U,d)\}$.

(P) If $x, y \in U$ such that $y_i \leq x_i$ for all i with strict inequality for some i then $y \notin f(U,d)$.

(IIA) If $y \in f(U,d)$ and $y \in \overline{U} \subseteq U$ then $y \in f(\overline{U}, d)$.

For the next axiom we need a definition. Point y is *comparably more symmetric than* x if $y = \lambda x + (1-\lambda) s(x)$ for $\lambda \in (0,1)$ and s a symmetry operator.

(Sy) If U is symmetric and $x, y \in U$ with y comparably more symmetric than x then $x \notin f(U,d)$.

If $\{f^k(U,d)\}$ is the family of candidate solutions that satisfy IAT, P, IIA and Sy then the solution to bargaining problem (U,d) is $f(U,d) = \cup_k f^k(U,d)$. It is easy to see that f itself satisfies IAT, P, IIA and Sy.

3. A Solution by Nash

Let $f^*(U,d) = \arg\max_{v \in U} (v_1 - d_1) \times \cdots \times (v_I - d_I)$.

Theorem 3.1. *f^* satisfies IAT, P, IIA and Sy.*

Proof. The reader will note that the proof for the IAT, P and IIA axioms are identical to the case with U convex.

(IAT) First let $U' = \{y : y_i = \alpha_i + x_i \text{ for some } x \in U\}$ and $d'_i = \alpha_i + d_i$ (where $\alpha_i \in \Re$ for all i). Then we have

$$\begin{aligned} f^*(U',d') &= \arg\max_{v \in U'} (v_1 - d'_1) \times \cdots \times (v_I - d'_I) \\ &= \arg\max_{v \in U + \{\alpha\}} (v_1 - d_1) \times \cdots \times (v_I - d_I) \\ &= \{\alpha\} + f^*(U,d) = \{y : y_i = \alpha_i + x_i \text{ for some } x \in f^*(U,d)\} \end{aligned}$$

From this it is always the case that $f^*(U,d) = f^*(U - \{d\}, 0) + \{d\}$, so from now on we normalize so that $d = 0$, and write $f^*(U)$ for $f^*(U,d)$.

Now let $U' = \{y : y_i = \beta_i x_i \text{ for some } x \in U\}$. Then if $y = \beta x \in f^*(U')$ it is the case that $y_1 \times \cdots \times y_I \geq y'_1 \times \cdots \times y'_I$ for all $y' \in U'$, so $\beta_1 x_1 \times \cdots \times \beta_I x_I \geq \beta_1 x'_1 \times \cdots \times \beta_I x'_I$ for all $x' \in U$, that is, $x_1 \times \cdots \times x_I \geq x'_1 \times \cdots \times x'_I$ for all $x' \in U$. Therefore, $x \in f^*(U)$ and $f^*(U') = \{y : y_i = \beta_i x_i \text{ for some } x \in f^*(U)\}$.

(P) If $x, y \in U$ such that $y_i \leq x_i$ for all i with strict inequality for some i then $y_1 \times \cdots \times y_I < x_1 \times \cdots \times x_I$. Therefore, $y \notin f^* (U)$.

(IIA) If $y \in f^* (U)$ then $y_1 \times \cdots \times y_I \geq x_1 \times \cdots \times x_I$ for all $x \in U$, and indeed for every subset $\overline{U}$ of U. Therefore, if $y \in \overline{U}$ then $y \in f^* \left(\overline{U}\right)$.

(Sy) If U is symmetric and $x, y \in U$ with y comparably more symmetric than x then $y = \lambda x + (1 - \lambda) s (x)$ for $\lambda \in (0, 1)$. By the convexity of the function $v_1 \times \cdots \times v_I$ we have that

$$\begin{aligned} (\lambda x_1 + (1 - \lambda) s_1 (x)) \times \cdots \times (\lambda x_I + (1 - \lambda) s_I (x)) &\geq \\ \lambda (x_1 \times \cdots \times x_I) + (1 - \lambda) (s_1 (x) \times \cdots \times s_I (x)) &= (x_1 \times \cdots \times x_I) \end{aligned}$$

Therefore, $x \notin f^* (U)$. ■

4. The Only Solution

Theorem 4.1. *If $\widetilde{f}$ is a candidate solution to (U, d) that satisfies IAT, P, IIA and Sy then $\widetilde{f} (U, d) \subseteq f^* (U, d)$.*

Proof. Let $\widetilde{f}$ be a candidate solution to (U, d) that satisfies IAT, P, IIA and Sy. Allow U to be an arbitrary bargaining problem with d normalized to zero. By the independence of utility origins implied by IAT, this can be done.

Pick $x^* \in f^*(U)$ and let $\overline{v} = \max_{x\in U} \sum_{i=1}^{I} \frac{x_i}{x_i^*}$ and $\underline{v}= \max_{x\in U} x_1 \times \cdots \times x_I$. Let $U' = \left\{x \in \Re^I_+ : \sum_{i=1}^{I} \frac{x_i}{x_i^*} \leq \overline{v}\right\}$ and $U'' = \left\{x \in \Re^I_+ : x_1 \times \cdots \times x_I \leq \underline{v}\right\}$. Let $U^S = U' \cap U''$. This intersection is always non-empty. Notice that $U \subseteq U^S$, that $\max_{x\in U^S} x_1 \times \cdots \times x_I = \underline{v}$ and that $f^*(U) \subseteq f^*\left(U^S\right)$. Let $U''' = \left\{x \in \Re^I_+ : \sum_{i=1}^{I} x_i \leq \overline{v}\right\}$, $U^{IV} = \left\{x \in \Re^I_+ : x_1 \times \cdots \times x_I \leq 1\right\}$ and $U^T = U''' \cap U^{IV}$. Notice that U^T is always non-empty and symmetric. Let $P\left(U^T\right)$ be the Pareto frontier of U^T. By P, if $y \in \widetilde{f}\left(U^T\right)$ then $y \in P\left(U^T\right)$. By Sy, if $y \in \widetilde{f}\left(U^T\right)$ then there is no $z \in P\left(U^T\right)$ that is comparably more symmetric than y. But this is the case for every point in $P(U''')$ that is not in $P\left(U^{IV}\right)$. Therefore, $y \notin P(U''')$. We have thus shown that if $y \in \widetilde{f}\left(U^T\right)$ then $y \in P\left(U^{IV}\right) \cap U^T$. But notice that $P\left(U^{IV}\right) \cap U^T = \arg\max_{v\in U^T} v_1 \times \cdots \times v_I = f^*\left(U^T\right)$. Hence, $\widetilde{f}\left(U^T\right) \subseteq f^*\left(U^T\right)$. From this it follows that if $x \in \widetilde{f}\left(U^T\right)$ then $x_1 \times \cdots \times x_I = 1$. It is easy to verify that $x \in U^T$ if and only if $(x_1^* x_1, ..., x_I^* x_I) \in U^S$. Therefore, by the independence of utility units implied by IAT, it follows that $\widetilde{f}\left(U^S\right) = \left\{y : y_i = x_i^* x_i \text{ for some } x \in \widetilde{f}\left(U^T\right)\right\}$. As a consequence, if $y \in \widetilde{f}\left(U^S\right)$ then $y_1 \times \cdots \times y_I = x_1^* x_1 \times \cdots \times x_I^* x_I = (x_1^* \times \cdots \times x_I^*)(x_1 \times \cdots \times x_I) = \underline{v}$ and $y \in f^*\left(U^S\right)$. This means that $\widetilde{f}\left(U^S\right) \subseteq f^*\left(U^S\right)$. Pick $y \in \widetilde{f}\left(U^S\right)$ such that $y \in U \subseteq U^S$. Then by IIA $y \in \widetilde{f}(U)$ but since $y_1 \times \cdots \times y_I = \underline{v}$ then $y \in f^*(U)$. Hence, $\widetilde{f}(U) \subseteq f^*(U)$. ■

Theorem 4.2. *f^* is the only solution to (U,d) that satisfies IAT, P, IIA and Sy.*

Proof. Let $\{f^k(U)\}$ be the family of candidate solutions that satisfy IAT, P, IIA and Sy. Then any solution to bargaining problem U that satisfies IAT, P, IIA and Sy is $\widetilde{f}(U) = \cup_k f^k(U)$. Theorem 4.2 indicates that $f^k(U) \subseteq f^*(U)$ for all k, and therefore $\widetilde{f}(U) = \cup_k f^k(U) \subseteq f^*(U)$. On the other hand, Theorem 3.1 says that $f^*(U) = f^k(U)$ for some k, so $\widetilde{f}(U) = \cup_k f^k(U) \supseteq f^*(U)$. As a consequence, $\widetilde{f}(U) = f^*(U)$. ■

References

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