UNSTEADY FLOW IN VERTICAL, CONVERGING TUBES

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ABSTRACT

The discharge of liquids from vertical tubes with various contraction geometries was studied via the unsteady Bernoulli equation. The temporal variations of the exit velocity and fluid level in the tube were found from the numerical integration of nonlinear differential equations. Sudden, quadratic, and exponential contraction geometries were considered. For inlet to exit area ratios greater than two, the flow initially accelerates to a maximum speed and then it decelerates for the geometries studied. The exponential contraction has the shortest discharge time. The solutions also reveal that the largest possible velocity and the shortest discharge time are achieved in a non-converging tube.

NOMENCLATURE

A	area
AR	area contraction ratio
g	gravitational acceleration
h	fluid level in the tube
H	h/L
k	$\sqrt{AR} - 1$
L	tube length
p	pressure
Q	volume flow rate
t	time
V	velocity
z	vertical coordinate
α	$ln(AR)$
ρ	density
Subscripts	
e	exit
fs	free surface
i	inlet
1,2	points on a streamline

INTRODUCTION

Unsteady turbulent flows, the subject of much current research, can be easily generated in the laboratory by employing gravity as the driving mechanism. Discharging of a vertical tube filled with water, produces a jet in which the flow accelerates in time. In order to tailor the variation of acceleration with time, one may utilize various tube geometries. The exit velocity variation with time for a converging tube can be obtained from the integration of the unsteady Bernoulli equation. Since the flow typically starts from rest and lasts only a short time, the boundary layer is confined to the tube walls and has only modest growth. Furthermore, Lefebvre & White (1989) observed that the transition in accelerating pipe flows is delayed to very high Reynolds numbers ($\sim 5 \times 10^5$).

Previous work on transient flow through pipes has typically concentrated on either the waterhammer problem or the velocity profile development in startup conditions. Szymanski (1932) found an exact solution of the Navier-Stokes equations for the laminar, incompressible, startup flow in a circular pipe. Durian (1986) studied the filling and emptying of small diameter tubes filled with viscous fluids. This study was mainly concerned with the time required to establish a fully-developed profile. The present work seeks to determine the velocity variation with time in vertical tubes having various contraction geometries. We are concerned with tubes of considerable diameter and short discharge times where the velocity profile is nearly uniform except near the tube walls.

BASIC EQUATIONS

The unsteady Bernoulli equation for two points on a given streamline in a flow of an incompressible fluid in the presence of gravity is

$$\int_1^2 \frac{\partial V}{\partial t}\, ds + \frac{1}{2}{V_2}^2 + \frac{p_2}{\rho} + g z_2 = \frac{1}{2}{V_1}^2 + \frac{p_1}{\rho} + g z_1. \qquad (1)$$

The integral is taken along the streamline and cannot, in general, be easily evaluated. For a vertical tube with an arbitrary contraction geometry, the integral can be quite closely approximated by an integral along the tube axis. See Fig. 1. In this case, the streamline is taken to extend from the free surface to the tube exit. Neglecting any external pressure gradients (i.e., $p_1 = p_2$) and placing the origin at the tube exit, Eq. 1 simplifies to

$$\int_0^h \frac{\partial V}{\partial t}\, dz + \frac{1}{2} V_{fs}^2 + g h(t) = \frac{1}{2} V_e^2 \quad (2)$$

The volume flow rate, $Q(t)$, relates the free surface and exit velocities via the areas.

$$Q(t) = A(z)V(z) = A_e V_e = A(h) V_{fs} \quad (3)$$

Furthermore, the free surface velocity is the rate at which the surface is falling $V_{fs} = -\,dh/dt$. The integral then becomes

$$\int_0^h \frac{\partial V}{\partial t}\, dz = \int_0^{h(t)} \frac{\partial}{\partial t}\left(\frac{Q(t)}{A(z)}\right) dz = \frac{dQ}{dt} \int_0^h \frac{dz}{A(z)} \quad (4)$$

where

$$\frac{dQ}{dt} = \frac{d}{dt}\left(A(h)\, V_{fs}\right) = \frac{dA}{dh}\left(\frac{dh}{dt}\right)^2 + A(h)\,\frac{d^2h}{dt^2} \quad (5)$$

Rewriting the Bernoulli equation in terms of h, one obtains

$$\left[\frac{dA}{dh}\left(\frac{dh}{dt}\right)^2 + A(h)\,\frac{d^2h}{dt^2}\right] \int_0^h \frac{dz}{A(z)} + g\, h(t) + \frac{1}{2}\left[1 - \left(\frac{A(h)}{A_e}\right)^2\right]\left(\frac{dh}{dt}\right)^2 = 0. \quad (6)$$

This is a highly nonlinear, second order differential equation with the following initial conditions

$$h(t=0) = L \quad \text{and} \quad \frac{dh}{dt}(t=0) = 0. \quad (7)$$

For any given area contraction geometry $A(z)$, Eq. 6 can be numerically integrated. To simplify the computations, the fluid height and time were normalized with the tube length L and $\sqrt{L/g}$, respectively. Once $h(t)$ is found, the exit velocity is simply $-\frac{dh}{dt}\frac{A(h)}{A_e}$. A fourth order Runge-Kutta scheme was used in the present study to solve the following geometries of interest.

i) Sudden Contraction: An orifice plate is placed at the exit of a constant area tube. This special case has been worked out by Paterson (1983); however, no solutions were provided. In this geometry $\frac{dA}{dh} = 0$, except at the exit, and the integral in Eq. 6 is simply $h(t)/A_i$. The differential equation in nondimensional form simplifies to

$$\ddot{H}\, H + H + \frac{1}{2}(1 - AR^2)\,\dot{H}^2 = 0, \quad (8)$$

where AR is the overall area contraction ratio (A_i/A_e) and the dots on top of a symbol signify time derivatives.

ii) Quadratic Contraction: The tube area changes quadratically along the tube axis $A(z) = A_e\,(1 + k\,z/L)^2$, where $k = \sqrt{AR} - 1$. This is a tube of conical geometry (i.e., a funnel) with the diameter $d = d_e\,(1 + k\,z/L)$. The differential equation for this geometry is

$$\ddot{H} + \left[4 - \frac{3}{1 + k\,H} - (1 + k\,H)^3\right]\frac{\dot{H}^2}{2H} + \frac{1}{1 + k\,H} = 0. \quad (9)$$

iii) Exponential Contraction: The tube area changes exponentially along the tube axis $A(z) = A_e\, exp(\alpha\, z/L)$, where $\alpha = ln(AR)$. The differential equation for this geometry is

$$\ddot{H} + \alpha\, H\,(e^{\alpha H} - 1)^{-1} - \frac{\alpha}{2}(e^{\alpha H} - 1)\,\dot{H}^2 = 0. \quad (10)$$

The numerical solution of these equations are discussed below.

RESULTS

The area contraction ratio for each geometry was increased from one to ten in order to observe the geometrical effects on the exit velocity and fluid level in the tube. The variation of fluid height with time is shown in Fig. 2. For an area contraction ratio of four (Fig. 2a), the nondimensional fluid level $H = h/L$ initially falls faster for the sudden contraction geometry as compared to the other cases. After this initial period, the fluid level decreases more rapidly for the quadratic and exponential geometries.

The effect of area contraction ratio on the temporal variation of the fluid level is displayed in Fig. 2b for the quadratic tube. As AR increases, it takes a longer time to discharge the fluid. The same pattern was observed for all three geometries. One may argue that tubes with lower area ratios have a larger amount of potential energy (larger volume) to discharge the fluid.

Fig. 3 presents the temporal dependence of the exit velocity. The tube exit velocity accelerates to a maximum value and then decelerates for all three geometries considered. This is quite surprising since one intuitively expects a continuously increasing velocity profile. The area contraction ratio was fixed at four in Fig. 3a although the same behavior was observed for area ratios greater than two. A monotonic velocity profile results for area ratios less than two.

For a fixed geometry, the effect of area contraction ratio on the exit velocity is evident in Fig. 3b. Note that a unity area ratio refers to discharging of a constant area pipe (no contraction). The fluid in this case is accelerating at a constant rate at all times. The solutions reveal that the largest possible exit velocity is achieved in this non-converging tube. As the area ratio increases, initial velocities are also increased. However, the time at which the flow starts to decelerate, is shorter. Furthermore, the final value of exit velocity decreases as the area contraction ratio is increases for all three geometries. For large area ratios, on the order of eight or larger, the exit velocity deceleration occurs at a constant rate.

The time to discharge the fluid, nondimensionalized by $\sqrt{L/g}$, is plotted against the area contraction ratio in Fig. 4 for the three geometries. The sudden contraction geometry has the longest discharge time, while the exponential tube has the shortest. A tube of unity area ratio has the shortest discharge time as well as the largest exit velocity. Beyond the area ratio of four, discharge times for each geometry increase linearly with AR. The performance of the exponential and quadratic tubes is much more alike when compared to the sudden contraction case. Experiments are currently underway to verify these numerical results.

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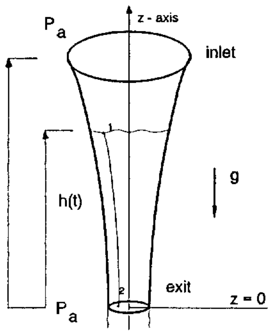

Fig. 1. A schematic of a vertical tube with an arbitrary contraction geometry.

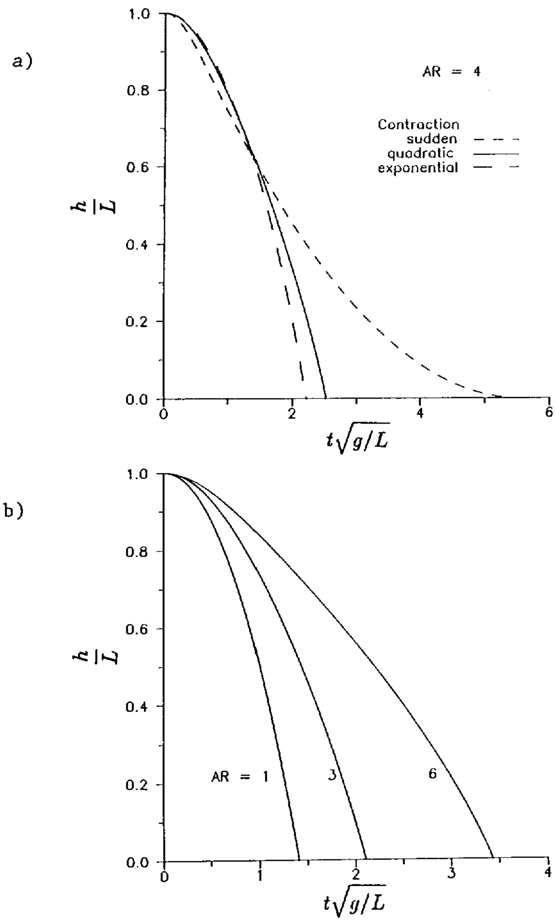

Fig. 2. Variation of fluid level with time. a) Constant area ratio and three geometries. b) Quadratic geometry and three area ratios.

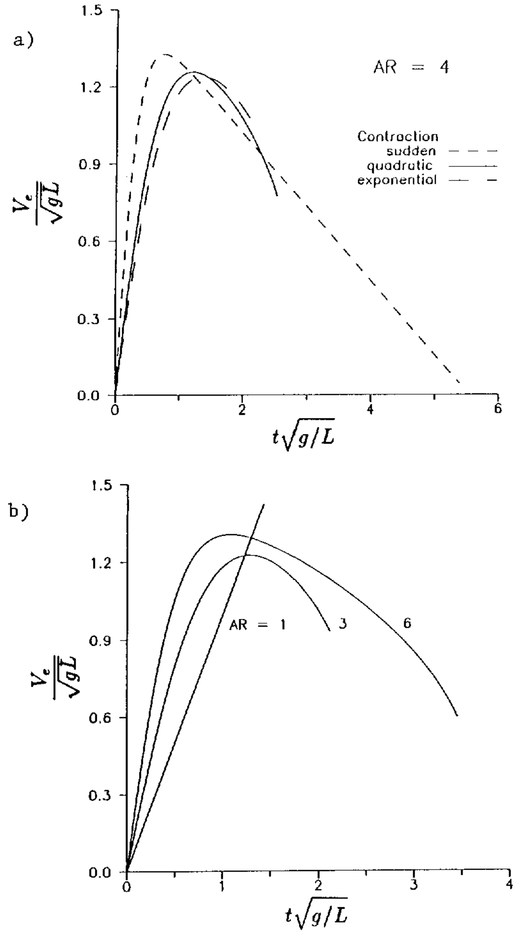

Fig. 3. Variation of exit velocity with time. a) Constant area ratio and three geometries. b) Quadratic geometry and three area ratios.

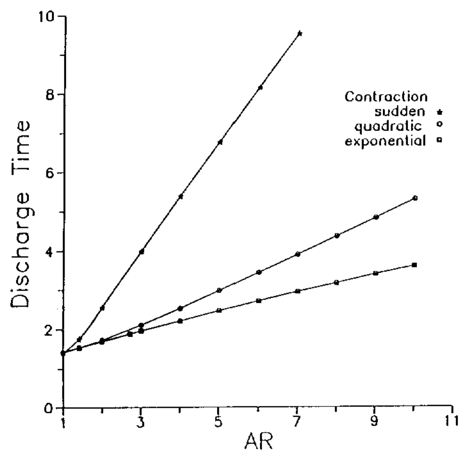

Fig. 4. Effect of area contraction ratio on the nondimensional discharge time for the three geometries.