MANOVA: Type I Error Rate Analysis

A Senior Project

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In Partial Fulfillment

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Bachelor of Science

By

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Introduction

When testing for differences between the population means of two or more groups with two or more dependent variables, one has two main testing options. The first is to run multiple Analysis of Variance Tests (henceforth ANOVA): one univariate test for each dependent variable. However, performing multiple tests will increase the global Type I error rate. This, if one were more interested in performing a test that simultaneously accounts for differences in each dependent variable, while maintaining an equivalent error rate, we need a different testing method. This second method is known as Multivariate Analysis of Variance. This analysis method was developed to test if the vectors of means of two or more dependent variables significantly differ across three or more populations.

Like with all parametric testing methods, there are mathematical conditions that must be met for the test results to be valid. A test’s ability to correctly find significant results, despite violations to its mathematical conditions for validity is known as robustness. This project was conceived to evaluate this important characteristic of MANOVA testing, particularly concerning violations in MANOVA’s assumption of multivariate normality. The majority of this project was coding and performing simulations to try and evaluate the robustness of MANOVA, i.e. under what circumstances is the test still reliable and when is it not. Using the software package R 2.13.0, I investigated different characteristics of data sets and how to create randomly generate data sets that have these set characteristics. Some of these characteristics would be marginal variances, distribution, correlation structure, number of groups, and sample size.

In this report you will find the methodology, coding philosophy, and findings of my simulations. I will also highlight the most interesting results and several forms to present these results. All other simulation results are included in the appendix.
Literature Review

If MANOVA is properly considered as a viable statistical method, it is easy to run across the issue of robustness. For example, one might come across data that’s more multivariate uniform than multivariate normal. If that’s the case, it would be useful to know if MANOVA is still a viable test and to what degree. Since this is the case there are certainly going to be several people studying and researching this issue.

In the article by Stefan van Aelst and Gert Willems published in the Journal of the American Statistical Association, they propose robust tests as alternatives to the classical Wilk’s Lambda test for MANOVA. This suggests that Wilk’s Lambda is not a statistic that is sufficiently robust. This is further agreed upon by the academic article by Valentin Todorov and Peter Filzmoser, published in Computational Statistics and Data Analysis (Todorov and Filzmoser. 2010, 37-48). They write that Wilk’s Lambda, being based on multivariate normal theory, is generally highly sensitive to outliers. This would suggest that distributions that have many extreme values, such as skewed distributions or even distributions with heavy tails. The Exponential distribution is one such distribution that could adversely affect the MANOVA results.

Todorov has also done previous research into the robustness of MANOVA mainly dealing with the Wilk’s Lambda statistic. In his article in 2007 published in Statistical Methods and Applications (Todorov 2007 395-407) he also evaluates the robustness of the Wilks MANOVA in terms of linear discriminant analysis, in which he concludes that Wilks is not a robust way of testing. It should be noted that Stefan van Aelst and Gert Willems also came to a similar solution, but with a particular focus on the effects of outliers (van Aelst and Willems, 106,494).

Both of these articles use Monte Carlo distributions in some degree, in which they identify a domain of parameters or possible inputs, generate the inputs randomly from a probability distribution and then perform computation. Others in the statistical community also use Monte Carlo Simulations to evaluate the robustness of MANOVA. This makes sense as mathematically computing the power of a MANOVA test in any given situation would be much more tedious and difficult. Taking this into consideration, the simulations done in this report will be of a Monte Carlo nature.
Multivariate Analysis of Variance

A Brief Introduction to MANOVA

Multivariate Analysis of Variance, also known as MANOVA, is an extension of the univariate analysis of variance, also known as ANOVA. MANOVA is a procedure to analyze data where there are two or more dependent variables. In general, where ANOVA compares means and evaluates if at least one difference between groups with respect to a single dependent variable, MANOVA compares vectors of means, where each component of the vector is the mean of a different dependent variable.

Suppose that we have a two sided hypothesis test with p independent variables. Then the hypothesis for the two group test would be as below.

Table 1: MANOVA Hypothesis

<table>
<thead>
<tr>
<th>Term</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mu_{ij} )</td>
<td>Mean for group ( i ) for variable ( j ).</td>
</tr>
</tbody>
</table>

**Null Hypothesis**

\[
\begin{bmatrix}
\mu_{11} \\
\mu_{12} \\
\vdots \\
\mu_{1p}
\end{bmatrix} =
\begin{bmatrix}
\mu_{21} \\
\mu_{22} \\
\vdots \\
\mu_{2p}
\end{bmatrix}
\]

**Alternative Hypothesis**

\[
\begin{bmatrix}
\mu_{11} \\
\mu_{12} \\
\vdots \\
\mu_{1p}
\end{bmatrix} \neq
\begin{bmatrix}
\mu_{21} \\
\mu_{22} \\
\vdots \\
\mu_{2p}
\end{bmatrix}
\]

Another way to think about it would be graphically. For example, suppose that we have data sampled from two bivariate populations and we would like to assess whether the bivariate sample means of these observations are significantly different. For simplicity, let us assume that the samples are drawn from two populations whose means are expressed as the following:

\[
\mu_1 = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \quad \mu_2 = \begin{bmatrix} 4 \\ 4 \end{bmatrix}
\]

Further suppose that both variables in both populations have a variance of 1.

Through random samples from each of these populations, MANOVA allows us to assess if the population means are jointly different across all dependent variables, without having prior knowledge of the means. Two groups are depicted below in Figure 1. The red dots represents the sample mean vector for each group, the blue dot indicates the true mean vector of each population. The surrounding points represent a sample of approximately 1000 observations from each population.
In this case, we know the population means and since the sample mean vectors are close to the true mean vectors, we can expect a formal MANOVA test to lead to the conclusion that the two groups are significantly different.

**Figure 1: Graphical Representation of Multivariate Data (Population)**

Unlİke ANOVA, MANOVA also has a variety of global test statistics that can be used to test for significance. The most commonly used statistic is known as the Wilk’s test statistic, which is analogous to ANOVA’s F-statistic. Three other global test statistics are Pillai, Hotelling-Lawley and Roy’s statistics. Each of these has different formulae to achieve a similar goal, but the differences will not be covered in this report.

**Conditions of MANOVA**

With any statistical test, there are conditions that need to be satisfied in order for the test results to be valid. Like with ANOVA, this also holds true with MANOVA. The three main conditions for MANOVA are:

- Independence – The observations need to be independent from one another.
- Multivariate Normality – The multivariate data are drawn from a Multivariate Normal distribution.
Homogeneity of Variance – Each group has the same covariance structure as the other group(s) being tested against.

**Why MANOVA?**

MANOVA allows us to test for significant differences between two or more groups, jointly accounting for multiple variables of interest. This essentially controls our Type I error rate without the need for any additional adjustment. MANOVA also accounts for inter-dependencies among the response variables enhancing our power to detect significant differences between groups. Such differences may be missed when only testing one variable at a time with a technique such as MANOVA.

An example of a situation where MANOVA could be used is the following:

Suppose we have a hypothesis that SAT scores vary from one sex to the other. We may want to formally test to see if there is an association between SAT Math and SAT Reading scores and sex. The MANOVA procedure allows us to test our hypothesis that the variables are jointly associated with sex.

**Type I Error**

Recall that a Type I error occurs when the null hypothesis is rejected when in reality, the null hypothesis is true. So for example, if a test procedure finds significant evidence for a difference between the proportion of males and females in the Democratic Party versus the Republican Party, and in reality, these proportions are the same, we have committed a Type I error. We normally control for this by assigning a value $\alpha$ to something like, say, 0.05. That means we design the test such that we have a 5% probability of making a Type I error. (This quantity is also known as a Significance Level.)
Simulations

This investigation deals mainly in an effort to assess the robustness of MANOVA. To do this we will start by purposely violating the Multivariate Normality assumption, to see if that will affect our Type I error rate. In order to evaluate the robustness of MANOVA, we will generate data sets that we know have the same mean. Then with that knowledge kept in mind, we will run 5000 simulations of the test and count the proportion of times we reject the null hypothesis at some nominal \( \alpha \) (in this case 0.05). So if we reject the null hypothesis substantially more or less than the 0.05 mark, we know there is something wrong with the test under the conditions we set. But in order to thoroughly evaluate MANOVA’s robustness against distribution, we also need to identify other parameters to use in the test to generate these conditions.

Multivariate Distribution

One of the assumed conditions for a MANOVA statistical analysis is the assumption that the data being tested is sampled from a Multivariate Normal Distribution. So what would happen to the test if say, we sample from a Multivariate Uniform? A Multivariate Exponential? In this report we will cover both Multivariate Uniform and Multivariate Exponential. We are looking to evaluate if this assumption is truly important to

Variance-Covariance Structure

Although we will not be violating the assumption of homogeneity of variance-covariance, we will be varying the structure, perhaps to see if the robustness of MANOVA is affected by large variance, or certain types of inter-dependencies. This can be described by the Variance-Covariance Matrix.

Simply put, a Variance-Covariance Matrix is a matrix that is used to incorporate the interdependencies of the dependent variables of a data set, along with their marginal variance. The diagonals of the matrix are the variances of each of the dependent variables and the off diagonal components are the covariances between each of the dependent variables. So if there are \( p \) dependent variables, there will be a \( p \times p \) variance covariance matrix.

\[
\Sigma = \begin{bmatrix}
\sigma_1^2 & \sigma_{12} & \cdots & \sigma_{p1} \\
\sigma_{21} & \sigma_2^2 & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots \\
\sigma_{1p} & \cdots & \cdots & \sigma_p^2
\end{bmatrix}
\]

One way to vary the structure of the Variance-Covariance matrix is to incorporate the correlation (the inter-dependencies) of the dependent variables. To be complete, it also stands to reason that we also need to incorporate the variances of these dependent variables. By including both of these two components we can essentially obtain any variance covariance structure we would like.
To begin, we can use a correlation matrix. A correlation matrix has diagonal components of 1, where the correlation between each dependent variable is in the off-diagonals. We will vary the correlation between each of the variables from 0.1 to 0.9, incrementing by 0.1, essentially determining the degree of association between each of the dependent variables in the data. For simplicity and without loss of generality, I set the variance for one dependent variable and allow the variance of the other dependent variables to be a multiple of that variance, creating a variance ratio. So for a two dimensional case, for example, the ratio will be 1:2, 1:3, or 1:4. It should be noted for the purposes of these simulations the variance-covariance structure across groups will remain homogeneous.

To combine the properties of the correlation matrix with the variances of each of the variables to create a variance covariance matrix we can use some matrix algebra in conjunction with the correlation matrix to create a Variance Covariance Matrix. The technique we will use is known as Outer Multiplication coupled with a simple matrix multiplication. Let S denote a vector of standard deviations obtained by taking the square root of the variances. Also, let R denote a correlation matrix. We can then do the following to obtain $\Sigma$:

$$\Sigma = (SS^T)R$$

**Sample Size**

Sample size is also an important factor, as we know for the single dimension case, the larger the sample, the more robust ANOVA becomes against violations in the distribution condition for ANOVA. Since that is the case, we will try sample sizes of 10, 20, and 30.

**Number of Groups**

We will also vary the number of groups begin tested. So, for example, we will test for significant differences between the mean vectors between 2 and 5 groups.

We will obtain the multivariate distribution by simply sampling from multivariate distributions which have independent components. This allows us to begin with a dataset that theoretically has no covariance between the dependent variables. So from a two dimensional multivariate uniform case, we will sample each dimension from independant univariate uniform distributions with a mean of 0 and a variance of 1. An example of this is shown below. The top, from left to right, is an example of a multivariate normal with mean 0 and variance 1 in two dimensions and three dimensions respectively. The bottom left is an example of a two dimensional multivariate uniform with the same mean and variance. The bottom right is an example of a three dimensional multivariate uniform distribution with the same mean and variance.
Once this data has been generated, we can transform it to have the desired covariance structure. We can use an Outer Matrix Multiplication using that vector of variances on the correlation matrix to generate a Variance-Covariance Matrix, as described above. We can then transform the previously independent multivariate data with this Variance-Covariance Matrix to give the data the appropriate properties.

To transform the data, we can use a method known as Singular Value Decomposition to decompose the Variance-Covariance matrix \( \Sigma \) into two matrices that are transposes of each other - \( V \) and \( V^T \) - and a diagonal matrix \( D \).

\[
\Sigma = VDV^T
\]

The two matrices \( V \) and \( V^T \) represent the linear transformation we need to introduce the variance and correlation properties set from before. So if \( X \) is our desired matrix and \( A \) is a matrix of untransformed, independently simulated data, then we can obtain \( X \) via the following equation.

\[
X = AV
\]

We can manipulate the simulated data matrix \( X \) to have different dimensions to change the number of groups and the sample size.
The simulations were all done in the statistical package R. The functions used for all the computation can be found the Appendix Section 1. An example of the exact way I used these functions can be found in the Appendix Section 2. Table 2 shows the various settings I used.

**Table 2: The Simulation Parameter Settings**

<table>
<thead>
<tr>
<th>Setting</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sample Size</td>
<td>10, 20, 40</td>
</tr>
<tr>
<td>Number of Groups</td>
<td>2, 3, 5</td>
</tr>
<tr>
<td>Distribution</td>
<td>Uniform and Exponential</td>
</tr>
<tr>
<td>Correlation</td>
<td>0.1 to 0.9 in steps of 0.1</td>
</tr>
<tr>
<td>Variance Ratio</td>
<td>1:2, 1:3, 1:4</td>
</tr>
<tr>
<td>Test Statistic</td>
<td>Wilks, Pillai, Hotelling-Lawley</td>
</tr>
</tbody>
</table>

There are many additional parameter settings that can be changed. I have limited myself to what is shown in the table above. For each combination of parameter settings, I simulated 5000 data sets, and then I ran a MANOVA with a specific test statistic such as Wilks on each simulated data set. I then calculated the proportion of times the p-value was less than 0.05, the nominal alpha level. If this proportion was less than 0.05, we have a conservative test. If it is greater, we have a liberal test. Both of these conditions are not desirable when evaluating the robustness of MANOVA.
Results

Wilk’s Test Statistic
In the first batch of simulations, I had used mainly Wilks test statistic. This test statistic is the most commonly used in MANOVA procedures, and thus would be the most practical. As described above, I simulated 5000 p-values for each combination of parameter settings from both Multivariate Uniform and Multivariate Exponential distributions, each time with manipulated covariance structures and simulation settings.

What was uniformly true across all settings was that the test did not seem to be dependent on the actual correlation between each variable. MANOVA, when using the Wilks test statistic, seemed to be robust against the multivariate uniform distribution at any sample size or variance-covariance structure. The simulation results collected for this case can be found in the appendix (Tables A.1 to A.3). A graphical representation of the simulated empirical alpha levels is shown here.

We can see that despite any of the parameter settings, the empirical alpha rate has settled around the nominal 0.05 level of the test. This is good, as it suggests that MANOVA is robust against the uniform distribution.

However, from this we cannot reasonably believe that MANOVA is robust against all non-normal distributions. Since the Uniform is symmetric, we next examine a skewed distribution. I have chosen the exponential distribution for that purpose. This distribution is only one other case, so even if we determine MANOVA is robust, we would still need to check other distributions.

I have found that the Wilks test tends to fail to detect a difference between the population mean vectors more often for the simulations coming from the multivariate exponential distribution. This is, as before, regardless of correlation. This is more apparent when considering the two group case. We see in Figure 4 that for sample sizes of 10 and 20 that the empirical alpha level is lower than that of the nominal 0.05 alpha level we assigned. This suggests that in the exponential case, the Wilks test statistic
is largely conservative. That is to say that the method rejects the null hypothesis fewer times than it should.

For a sample size of 40, we see that just like in the uniform case the empirical alpha level appears to approach 0.05. As the number of groups increase we also can observe that the empirical alpha level approaches about 0.05, the nominal level. I suspect this is due to an increase in overall sample size. The following graphs will demonstrate this phenomenon (Figure 4).

**Figure 4: MANOVA Empirical Alpha Level for the Exponential Distribution**

![Graphs showing the empirical alpha level for different group settings and sample sizes.](image)

The simulation results are also available in the appendix (Tables A.4-A.6).

**Pillai’s Test Statistic**

Another test statistic that can be used for MANOVA procedures is Pillai’s test statistic. Again using the statistical package R, I simulated a specific number of observations for each parameter combination from both Multivariate Uniform and Multivariate Exponential distributions.

As was true with the Wilk’s test statistic, the correlation didn’t seem to affect the results. And again, as was true with the results from the Wilk’s statistic, the empirical alpha rate has settled around the nominal 0.05 alpha level.

However, we find that when we look at the empirical alpha rate for the Multivariate Exponential Distribution, we observe that many are lower than 0.05. Almost every simulation with two groups is
under the 0.05 nominal alpha level. This only begins to change for sample sizes over 10 with more than two groups, such as is demonstrated in the table below (Table 3).

### Table 3: Exponential Distribution Results using Pillai's Test with Three Groups

<table>
<thead>
<tr>
<th>Variance Ratio of 1:2</th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Sample Size</td>
<td>0.1</td>
<td>0.2</td>
<td>0.3</td>
<td>0.4</td>
<td>0.5</td>
<td>0.6</td>
<td>0.7</td>
<td>0.8</td>
<td>0.9</td>
</tr>
<tr>
<td>10</td>
<td>0.0462</td>
<td>0.0460</td>
<td>0.0412</td>
<td>0.0440</td>
<td>0.0458</td>
<td>0.0462</td>
<td>0.0434</td>
<td>0.0452</td>
<td>0.0438</td>
</tr>
<tr>
<td>20</td>
<td>0.0514</td>
<td>0.0462</td>
<td>0.0452</td>
<td>0.0508</td>
<td>0.0452</td>
<td>0.0498</td>
<td>0.0498</td>
<td>0.0460</td>
<td>0.0402</td>
</tr>
<tr>
<td>40</td>
<td>0.0460</td>
<td>0.0478</td>
<td>0.0526</td>
<td>0.0518</td>
<td>0.0506</td>
<td>0.0488</td>
<td>0.0432</td>
<td>0.0045</td>
<td>0.0486</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Variance Ratio of 1:3</th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Sample Size</td>
<td>0.1</td>
<td>0.2</td>
<td>0.3</td>
<td>0.4</td>
<td>0.5</td>
<td>0.6</td>
<td>0.7</td>
<td>0.8</td>
<td>0.9</td>
</tr>
<tr>
<td>10</td>
<td>0.0506</td>
<td>0.0468</td>
<td>0.0468</td>
<td>0.0486</td>
<td>0.0462</td>
<td>0.0420</td>
<td>0.0444</td>
<td>0.0450</td>
<td>0.0408</td>
</tr>
<tr>
<td>20</td>
<td>0.0424</td>
<td>0.0530</td>
<td>0.0450</td>
<td>0.0458</td>
<td>0.0464</td>
<td>0.0452</td>
<td>0.0410</td>
<td>0.0510</td>
<td>0.0490</td>
</tr>
<tr>
<td>40</td>
<td>0.0450</td>
<td>0.0560</td>
<td>0.0536</td>
<td>0.0462</td>
<td>0.0478</td>
<td>0.0442</td>
<td>0.0556</td>
<td>0.0472</td>
<td>0.0508</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Variance Ratio of 1:4</th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Sample Size</td>
<td>0.1</td>
<td>0.2</td>
<td>0.3</td>
<td>0.4</td>
<td>0.5</td>
<td>0.6</td>
<td>0.7</td>
<td>0.8</td>
<td>0.9</td>
</tr>
<tr>
<td>10</td>
<td>0.0450</td>
<td>0.0464</td>
<td>0.0448</td>
<td>0.0440</td>
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<td>0.0470</td>
<td>0.0446</td>
</tr>
<tr>
<td>20</td>
<td>0.0510</td>
<td>0.0474</td>
<td>0.0436</td>
<td>0.0438</td>
<td>0.0502</td>
<td>0.0416</td>
<td>0.0514</td>
<td>0.0466</td>
<td>0.0438</td>
</tr>
<tr>
<td>40</td>
<td>0.0510</td>
<td>0.0474</td>
<td>0.0436</td>
<td>0.0438</td>
<td>0.0502</td>
<td>0.0416</td>
<td>0.0514</td>
<td>0.0466</td>
<td>0.0438</td>
</tr>
</tbody>
</table>

We can see the alpha values represented visually looking at a graph of the simulation results for Pillai’s test, testing 3 groups, with a variance ratio of 1:2. (Shown in the Figure 5)
Hotelling-Lawley's Test Statistic

The last common MANOVA test statistic is Hotelling-Lawley's test statistic. At this point, I had largely expected that Hotelling-Lawley's test statistic would yield similar results to the Wilk's test statistic, and that was also the case here. The empirical alpha levels of the Hotelling-Lawley test were approximately 0.05 regardless of the setting the simulation was run in, with some degree of error. I suspect that if more simulations were run, this test would be liberal in that it would reject more times than the nominal alpha level would suggest. This is based off the fact that in my simulation that the data yielded using Hotelling-Lawley trace was the only one that saw high empirical alpha levels such as 0.0618.

Next the multivariate exponential distribution was explored. What was most interesting about Hotelling-Lawley was that in the two group simulations the empirical alpha values seemed to be mostly below 0.05. However, as the number of groups increased, the empirical alpha level approaches 0.05, where sample sizes were greater than 10. This is the same behavior we saw in the simulations utilizing Pillai's test. Table 4 gives the data for testing with three groups. We see that instead of sample sizes over 10 where test seems viable, it is now slightly higher. We needed 40 observations to see the empirical alpha level converge to 0.05 with more than 2 groups. (Table 4)
Table 4: Exponential Distribution Results using Hotelling-Lawley's Test with Three Groups

<table>
<thead>
<tr>
<th>Sample Size</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
<th>0.7</th>
<th>0.8</th>
<th>0.9</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>0.0446</td>
<td>0.0458</td>
<td>0.0556</td>
<td>0.0405</td>
<td>0.0438</td>
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<td>0.0452</td>
<td>0.0418</td>
</tr>
<tr>
<td>20</td>
<td>0.0490</td>
<td>0.0484</td>
<td>0.0446</td>
<td>0.0510</td>
<td>0.0488</td>
<td>0.0444</td>
<td>0.0438</td>
<td>0.0458</td>
<td>0.0496</td>
</tr>
<tr>
<td>40</td>
<td>0.0508</td>
<td>0.0494</td>
<td>0.0486</td>
<td>0.0458</td>
<td>0.0504</td>
<td>0.0494</td>
<td>0.0448</td>
<td>0.0434</td>
<td>0.0458</td>
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As before, we can also show this graphically. Using the results for Hotelling Lawley’s test, testing 3 groups, with a variance ratio of 1:2 yields the following graph (Figure 6). We also note the large empirical alpha value at sample size 40, highlighted in Table 4 above. Although this report does not cover why or how this value happened, it is worth noting and perhaps further investigating.
Figure 6: MANOVA Empirical Alpha Level for the Exponential Distribution (Hotelling-Lawley)
Conclusion

After all the simulations had been run, there were some observations towards the robustness of MANOVA that could be made. I have observed that with regards to the Uniform distribution that the MANOVA test, regardless of the test statistic, was robust. I suspect this is due to the symmetric nature of a multivariate uniform distribution being close to that of a multivariate normal distribution (See Figure 2). It can also be noted that if the distribution is symmetric, I suspect MANOVA to be robust against it.

In regards to other distributions, it seems that Pillai’s test statistic and Hotelling-Lawley’s test statistic are largely conservative, Hotelling-Lawley slightly more so than Pillai’s. We also considered the possibility of a difference when the true distribution was asymmetric. For that purpose we considered one of the most extreme asymmetric distributions; the exponential distribution. We can also observe that the simulations seem to suggest that the larger the sample size, be it group or overall sample size, affects the results. It seems that larger sample sizes seem to help the robustness of the test, perhaps because of some multivariate version of the central limit theorem. Although it was not apparent where a violation of multivariate normality would be negligent with a large enough sample, it seemed that for the Wilk’s test statistic that number was around 30. For Pillai and Hotelling-Lawley, we see that we need more than 2 groups and a sample size greater than 10. We also note that it seems as the number of variables increase, the empirical alpha rate approaches the nominal one.

The most significant discovery yielded by these simulations was in terms of the two group tests (Table 5). I had suspected that the two group tests would yield similar results regardless of the test statistic. This should be the case, because when all conditions are satisfied, the two group case should yield identical (or extremely similar) conclusions, since MANOVA simply reduces to something equivalent to Hotelling’s $T^2$ Test. But we find that none of the MANOVA test statistics yielded similar results to one another. Regardless of sample size, variance ratio, and correlation, we notice that empirical alpha values constantly hit below the 0.5 nominal mark. This could be due to random variation, although it seems strange Pillai and Hotelling-Lawley would be largely conservative.
Table 5: Exponential Distribution Results using Hotelling-Lawley Test with Two Groups

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</table>

What I have done in this report is only a small part of possible research done to evaluate the robustness of MANOVA. Many more distributions could be studied, such as Chi-Squared or the F distribution. More simulations could be run, to check for varying conclusions due to random variation. It is also possible to combine several assumption violations together and run simulations, such as heterogeneous variance-covariance structures and a multivariate Gamma distribution. There also could be studies on non-parametric multivariate tests that would surely be useful to the statistical community.

These are just some suggestions to future studies on this topic. What I have done here is simply explore the surface of a very deep and complex problem. I, as well as many other budding statisticians, welcome anyone to investigate further.
Appendix

**Section 1: The Simulation Functions**

```
###############################################################
##### MANOVA
##### Empirical Type I Error
##### Functions to evaluate Robustness of MANOVA
##### Written and Maintained by Chris Ling
##### for Partial Fulfillment of Degree, Statistics B.S.
##### California Polytechnic University, San Luis Obispo
###############################################################

require(corpcor);

dependant.bivar.unif = function(Sig, mu = c(0,0), n = 1000, Check=TRUE){
  library(corp.cor);
  library(MASS);
  p< length(mu);
  if (!all(dim(Sig) == c(p, p))){
    stop("incompatible arguments")
  }
  if (is.positive.definite(Sig) && isSymmetric(Sig)){
    decomp <- svd(Sig);
    A = decomp$u;
    SigX = diag(decomp$d);
    diagX = decomp$d;
  }else{
    warning("Matrix is not a Symmetric Positive Definite Matrix. ");
  }
  Range = NULL;
  Range[1] = (diagX[1]*12)^0.5;
  Range[2] = (diagX[2]*12)^0.5;
  X1 <- runif(n, min= mu-Range[1]/2, max=mu+Range[1]/2);
  X2 <- runif(n, min= mu-Range[2]/2, max=mu+Range[2]/2);
  X <- cbind(X1,X2);
  Xtransform <- X %*% A;
```
if (Check){
    Output = list(Xtransform, var(Xtransform));
    names(Output) = c('Data','Variance-Covariance Matrix');
}else{
    Output = list(Xtransform);
    names(Output) = c('Data');
}
Output;

dependant.gen = function(Sig, mu, n = 1000, dist = 'norm', empirical = FALSE, Check=FALSE){
    ##########################################################################
    ##### Sig is a covariance matrix. It must be symmetric and semi-positive definite.
    ##### mu is a vector of means that allows us to generate a sample.
    ##### n is the number of observations requested.
    ##### dist is an argument that will define the multivariate distribution. It can only take 'norm', 'exp' and 'unif' at this time.
    ##### empirical is a logical argument that will take into account emperical data for the normal argument. (See mvt norm for usage)
    ##### Check will add the covariance matrix of the new data in the output if TRUE.
    ##########################################################################
    library(corpcor);
    p<- length(mu);
    if (!all(dim(Sig) == c(p, p))){
        stop("incompatible arguments");
    }
    if (is.positive.definite(Sig) && isSymmetric(Sig)){
        decomp <- svd(Sig);
        A = decomp$u;
        SigX = diag(decomp$d);
        diagX = decomp$d;
    }else{
        stop("Matrix is not a Symmetric Positive Definite Matrix. ");
    }
    if (dist == 'unif'){
        eS <- eigen(Sig, symmetric = TRUE, EISPACK = TRUE);
        ev <- eS$values;
        X <- matrix(runif(p*n,-sqrt(12)/2,sqrt(12)/2), n);
        if (empirical) {
            X <- scale(X, TRUE, FALSE);
            X <- X %*% svd(X, nu = 0)$v;
            X <- scale(X, FALSE, TRUE);
        }
\begin{align*}
X & \leftarrow \text{drop}(\mu) + \text{eS}$\text{vectors}$ \%\% \text{diag}(\text{sqrt}(\text{pmax}(\text{ev}, 0)), p) \%\% t(X);
\end{align*}

\begin{align*}
\text{nm} & \leftarrow \text{names}(\mu);
\text{if (is.null(\text{nm}) \&\& !is.null(\text{dn} \leftarrow \text{dimnames}(\text{Sig})))}\{
\text{nm} & \leftarrow \text{dn}[1L];
\text{dimnames}(X) & \leftarrow \text{list}(\text{nm}, \text{NULL});
\}
\text{if (n == 1)}\{
\text{drop}(X);
\}\text{else}{
\text{Xtransform} & \leftarrow t(X);
\}
\end{align*}

\begin{align*}
\text{if(\text{dist} == \text{'norm'})}\{
\text{eS} & \leftarrow \text{eigen}(\text{Sig}, \text{symmetric} = \text{TRUE}, \text{EISPACK} = \text{TRUE});
\text{ev} & \leftarrow \text{eS}$\text{values}$;
\text{X} & \leftarrow \text{matrix}(\text{rnorm}(p \times n), n);
\text{if (\text{empirical})}\{
\text{X} & \leftarrow \text{scale}(X, \text{TRUE}, \text{FALSE});
\text{X} & \leftarrow \text{X} \%\% \text{svd}(X, \text{nu} = 0)$\text{v}$;
\text{X} & \leftarrow \text{scale}(X, \text{FALSE}, \text{TRUE});
\}
\text{X} & \leftarrow \text{drop}(\mu) + \text{eS}$\text{vectors}$ \%\% \text{diag}(\text{sqrt}(\text{pmax}(\text{ev}, 0)), p) \%\% t(X);
\end{align*}

\begin{align*}
\text{nm} & \leftarrow \text{names}(\mu);
\text{if (is.null(\text{nm}) \&\& !is.null(\text{dn} \leftarrow \text{dimnames}(\text{Sig})))}\{
\text{nm} & \leftarrow \text{dn}[1L];
\text{dimnames}(X) & \leftarrow \text{list}(\text{nm}, \text{NULL});
\}
\text{if (n == 1)}\{
\text{drop}(X);
\}\text{else}{
\text{Xtransform} & \leftarrow t(X);
\}
\end{align*}

\begin{align*}
\text{if(\text{dist} == \text{'exp'})}\{
\text{eS} & \leftarrow \text{eigen}(\text{Sig}, \text{symmetric} = \text{TRUE}, \text{EISPACK} = \text{TRUE});
\text{ev} & \leftarrow \text{eS}$\text{values}$;
\text{X} & \leftarrow \text{matrix}((\text{rexp}(p*n,1) - 1), n);
\text{if (\text{empirical})}\{
\text{X} & \leftarrow \text{scale}(X, \text{TRUE}, \text{FALSE});
\text{X} & \leftarrow \text{X} \%\% \text{svd}(X, \text{nu} = 0)$\text{v}$;
\text{X} & \leftarrow \text{scale}(X, \text{FALSE}, \text{TRUE});
\}
\text{X} & \leftarrow \text{drop}(\mu) + \text{eS}$\text{vectors}$ \%\% \text{diag}(\text{sqrt}(\text{pmax}(\text{ev}, 0)), p) \%\% t(X);
\end{align*}

\begin{align*}
\text{nm} & \leftarrow \text{names}(\mu);
\text{if (is.null(\text{nm}) \&\& !is.null(\text{dn} \leftarrow \text{dimnames}(\text{Sig})))}\{
\text{nm} & \leftarrow \text{dn}[1L];
\}
\end{align*}
dimnames(X) <- list(nm, NULL);
}
if (n == 1){
    drop(X);
}else{
    Xtransform = t(X);
}

if (Check){
    Output = list(Xtransform, var(Xtransform));
    names(Output) = c('Data','Variance-Covariance Matrix');
}else{
    Output = list(Xtransform);
    names(Output) = c('Data');
}
Output;

manova.typeI = function(simsize = 5000, dist, mean = c(0,0), cor =
cbind(c(1,0),c(0,1)), var = c(1,2), n=10, groups = 2, test = 'wilks'){
    ####################################################
    #### simsize is the simulation size to obtain empirical alpha from.
    #### dist is the distribution to be sampled from.
    #### cor is a correlation matrix.
    #### var is a vector of the variances(in this case variance ratios).
    #### mean is a vector of means that allows us to generate a sample.
    #### n is the number of observations requested.
    #### groups is the number of groups to be tested in the manova.
    ####
    ####################################################
    sds <- sqrt(var);
    sig <- outer(sds, sds) * cor;
    p = NULL;
    for(i in 1:simsize){
        X = NULL;
        for (j in 1:groups){
            X<- rbind(X,cbind(dependant.gen(sig,mu = mean, n = n, dist=
                         dist)$Data,rep(j,n)));
        }
        fit <- manova(X[,1:2]-X[,3]);
        p <- rbind(p,summary(fit, , test = test)$stats[11]);
    }
    table(p < 0.05)[2]/simsize;
}
Section 2: Example Simulation Code

```r
set.seed(100);

# Wilks test with two groups; Uniform Distribution, n = 10
Wilks2Unif = matrix();
length(Wilks2Unif) = 27;
dim(Wilks2Unif) = c(3,9);
for(varrat in 2:4){
    for(corcount in 1:9){
        Wilks2Unif[varrat-1,corcount] =
            manova.typeI(dist='unif',cor=cbind(c(1,corcount/10),c(corcount/10,1)),var =
            c(1,varrat),n=10,groups=2,test='Wilks');
    }
}
Wilks2Unif;
```
### Section 3: Simulation Results

Table A.1

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Table A.2

Multivariate Uniform Distribution
Wilks Test Empirical α level with 3 Groups

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#### Multivariate Uniform Distribution

**Wilks Test Empirical α level with 5 Groups**

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Multivariate Exponential Distribution
Wilks Test Empirical $\alpha$ level with 2 Groups

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Multivariate Exponential Distribution
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Section 4: References


Todorov, V. “Robust selection of variables in linear discriminant analysis” *Statistical Methods and Applications* 15, 395-407. 2007