FILTERING PROPERTY OF A LIGHTLY DAMPED SYSTEM, SUBJECTED TO TRANSIENT EXCITATION

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Consider a SDOF system, excited by a transient force $f(t)$, which is a pulse of finite duration t_*. The equation of motion may be written as

$$m\ddot{x} + 2\alpha m\dot{x} + kx = f(t), \tag{1}$$

where $f(t) \neq 0$ for $0 \leqslant t \leqslant t_*$ and is zero otherwise. Response $x(t)$ can be simply calculated for any given $f(t)$ using convolution integral—see, e.g., textbook [1], where several examples for various specific forms of $f(t)$ are presented.

In this note an explicit expression for $x(t)$, $t \geqslant t_*$ is derived for an *arbitrary* $f(t)$, in terms of its Fourier transform. This expression has important implications, particularly in that it illustrates an important filtering effect. This makes the example instructive for teaching vibrations and dynamics. A special case of the expression for the undamped system ($\alpha = 0$) is presented in book [2].

The convolution integral solution for the response $x(t)$ may be written as

$$x(t) = (m\omega_d)^{-1} \int_0^t f(\tau)\, e^{-\alpha(t-0_\tau)} \sin \omega_d(t-\tau)\, d\tau = (e^{-\alpha t}/m\omega_d)\left[A \sin \omega_d t - B \cos \omega_d t\right], \tag{2}$$

where $\omega_d = \sqrt{\Omega^2 - \alpha^2}$ and $\Omega = \sqrt{k/m}$ are damped and undamped natural frequencies, respectively, and

$$A = \int_0^t f(\tau)\, e^{\alpha\tau} \cos \omega_d \tau\, d\tau, \quad B = \int_0^t f(\tau)\, e^{\alpha\tau} \sin \omega_d \tau\, d\tau, \tag{3}$$

As long as the response is sought for at time instants after the end of the excitation period, i.e., when $f(t) = 0$, the upper integration limits in expressions (3) can be extended to infinity. The lower integration limits can be extended to minus infinity, both in view of the causality property and since $f(t) = 0$ for $t < 0$. Assume also the system to be lightly damped, and the force pulse to be of a moderate duration only, namely $\alpha \ll \Omega$, $\alpha t_* \ll 1$. Then approximations $\omega_d \approx \Omega$ and $\exp(\alpha\tau) \approx 1$ for

$0 \leqslant \tau \leqslant t_*$ may be used in expressions (3). The resulting constants A and B may be reduced as follows:

$$A = \int_{-\infty}^{\infty} f(\tau)\cos\Omega\tau\,d\tau = 2\pi\,\text{Re}\,F(\Omega), \quad B = \int_{-\infty}^{\infty} f(\tau)\sin\Omega\tau\,d\tau = 2\pi\,\text{Im}\,F(\Omega), \quad (4)$$

where $F(\omega) = (2\pi)^{-1}\int_{-\infty}^{\infty} f(\tau)\,e^{-i\omega\tau}\,d\tau$ is clearly seen to be Fourier transform of the force pulse $f(t)$. Using expression (4), the response $x(t)$ may then be represented finally as

$$x(t) = (2\pi/m\Omega)|F(\Omega)|\exp(-\alpha t)\sin(\Omega t + \phi), \tag{5}$$

where $|F(\Omega)| = (2\pi)^{-1}\sqrt{A^2 + B^2}$ is seen to be the value of the Fourier transform of the excitation pulse at the system's natural frequency (and ϕ is some phase, which is not of interest here).

The final formula (5) shows, first of all, that the response is simply a free, damped oscillations; of course, it is this kind of response that should be expected after complete decay of the excitation. The second implication is less obvious. It is seen that the initial amplitude of these free decaying oscillations is proportional to the magnitude of the value of the Fourier transform of the force pulse at the system's natural frequency. Thus, of the whole continuous frequency spectrum of the transient force, the system responds to a single component only—namely, to one with the system's natural frequency; all other frequency components of the excitation are rejected. This filtering property of lightly damped systems, which is of particular importance for Random Vibrations, seems to be illustrated nicely by this example. The simple formula (5) can also be used for quick estimates of the response of structures to pulse-type loadings (say, in Earthquake Engineering).

REFERENCES

1. S. S. Rao 1995 *Mechanical Vibrations*. New York: Addison-Wesley.
2. JA. B. Zel'dovich, and A. D. Myshkis 1965 *Elements of Applied Mathematics* (in Russian). Moscow: Nauka.