LARGE SETS OF ZERO ANALYTIC CAPACITY

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ABSTRACT. We prove that certain Cantor sets with non-sigma-finite one-dimensional Hausdorff measure have zero analytic capacity.

1. INTRODUCTION

In this paper we consider a Cantor set $K$ similar to the $\frac{1}{4}$-Cantor set of [G70] and [I84]. Fix $p > 2$ and for $n > 0$ define

$$
\sigma_n = 4^{-n}a_n = 4^{-n}[\log(n + 1)]^{1/p}.
$$

Set $K_0 = [0, 1] \times [0, 1]$ and $K_1 = \bigcup_{j=1}^4 K_{1,j}$, where $K_{1,j} \subset K_0$ is a square of sidelength $\sigma_1$ having sides parallel to the axis and containing one of the four corners of $K_0$. Next take $4^2$ squares $K_{2,j}$ of sidelength $\sigma_2$, one in each corner of each square $K_{1,j}$, and define $K_2 = \bigcup_{j=1}^{4^2} K_{2,j}$. Continuing we obtain $K_n = \bigcup_{j=1}^{4^n} K_{n,j}$, where $K_{n,j}$ is a square of sidelength $\sigma_n$. The Cantor set we study is

$$
K = K(p) = \bigcap_{n=1}^{\infty} K_n.
$$

If $E$ is a compact plane set define

$$
A(E, 1) = \{f : f \text{ analytic on } E^c, \ f(\infty) = 0, \ \|f\|_{L^\infty(E^c)} \leq 1\}
$$

and define the analytic capacity of $E$ by

$$
\gamma(E) = \sup\{\|f'(\infty)\| : f \in A(E, 1)\},
$$

where

$$
f'(\infty) = \lim_{z \to \infty} zf(z).
$$

If $\gamma(E) = 0$, then the only function in $A(E, 1)$ is the constant $f \equiv 0$ and in this case $E$ is removable for bounded analytic functions. For more details about analytic capacity see [G72].

**Theorem 1.** Let $p > 2$, and let $K$ be the four-corner Cantor set $K(p)$. Then $\gamma(K) = 0$ but $K$ does not have $\sigma$-finite one-dimensional measure.
The proof of Theorem 1 depends on a lemma of Jones [J89] used for a proof different from [G70] that the $\frac{1}{4}$-Cantor set has zero analytic capacity.

Let $h(t)$ be an increasing continuous function on $t \geq 0$ with $h(0) = 0$, and write $\Lambda_{h(t)}(E)$ for the Hausdorff $h$-measure of $E$. Now define an increasing function $h(t)$ so that $h(0) = 0$ and $h(\sigma_n) = 4^{-n}$ for all $n$. We say $h(t)$ is a measure function corresponding to the Cantor set $K$. For every $n$ define a measure $\mu_n$ on $K_n$ by $\mu_n(K_{n,j}) = 4^{-n}$ for all $j$. Then $\{\mu_n\}$ converges weak-star to a measure $\mu$ supported on $K$ and satisfying $\mu(K_{n,j}) = 4^{-n}$. Suppose $\frac{\sqrt{2}}{2}\sigma_n \leq r < \frac{\sqrt{2}}{2}\sigma_{n-1}$ and let $D(z, r)$ be a disk of radius $r$ and center $z \in K$. Then $D(z, r)$ can meet at most 4 squares of sidelength $\sigma_n$. Hence

$$
\mu(D(z, r)) \leq 4\mu(K_{n,j}) = 4 \cdot 4^{-n} = 4h(\sigma_n) \leq 4h(r),
$$

so that $\mu(D(z, r)) \leq 16h(r)$ for any disk $D(z, r)$. Therefore $\Lambda_h(K) > 0$ by Frostman’s Theorem [G72]. Since

$$
\lim_{t \to 0} \frac{h(t)}{t} = 0,
$$

if follows that $K$ has non-$\sigma$-finite 1-dimensional measure.

If $h(t)$ is a measure function corresponding to $K$, then

$$
\int_0^1 \frac{h(t)^2}{t^3} \, dt \sim \sum_{n=1}^\infty \frac{1}{(a_n)^2} = \sum_{n=1}^\infty \frac{1}{(\log n)^2} = \infty.
$$

On the other hand, Mattila [M96] proved that $\gamma(K) > 0$ if $K$ is a Cantor set built with squares of side $\sigma_n$ and if

$$
\int_0^1 \frac{h(t)^2}{t^3} \, dt < \infty,
$$

where $h$ is any measure function for corresponding to $K$. Mattila’s proof used Menger curvature (see [Me95] and [MMV96]). However, if the Cantor set $K$ has corresponding measure function $h$ satisfying

$$
\int_0^1 \frac{h(t)^2}{t^3} \, dt = \infty,
$$

then Eiderman [E98] proved that $\gamma^+(K) = 0$, where

$$
\gamma^+(E) = \sup \left\{ \int_E d\mu : \left| \int_E \frac{d\mu(z)}{\zeta - z} \right| < 1, \forall z \in \mathbb{C}\setminus E, \mu > 0, \text{spt}(\mu) \subset E \right\}.
$$

Since $\gamma^+(E) \leq \gamma(E)$, our result is a partial improvement of Eiderman’s theorem. Mattila [M96] has conjectured that for Cantor sets of this type $\gamma(K) = 0$ if and only if

$$
\int_0^1 \frac{h(t)^2}{t^3} \, dt = \infty,
$$

when $h$ corresponds to $K$. This latter condition holds if and only if

$$
\sum_{n=1}^\infty \frac{1}{(a_n)^2} = \infty.
$$

If Mattila’s conjecture is true, then together with Eiderman’s theorem it gives Cantor set evidence supporting the more ambitious conjecture that $\gamma(E) > 0$ implies $\gamma^+(E) > 0$. 
In [G72] it was incorrectly claimed that $\gamma(K) > 0$ if and only if
$$\int_0^1 \frac{h(t)}{t^2} dt < \infty.$$Eiderman, however, found a mistake in the proof. In fact the result in [M96] shows that the claim is false. See L.D. Ivanov [I84] for the first example of a Cantor set of non-$\sigma$-finite linear measure and zero analytic capacity.

2. TWO LEMMAS OF PETER JONES

We need the following two lemmas from [J89]. The proofs we give are small variations on [J89] and [C90].

Define $\gamma^m_j = \partial cK_{n,j}$, where $cK_{n,j}$ is the square concentric to $K_{n,j}$ with sidelength $c\sigma_n$ and where $c > 1$ is chosen so that the $\gamma^m_j$ do not overlap. We refer to $\gamma^m_k$ as a square, although it is only the boundary of a square. Notice that
$$A_1(\gamma^m_k) = cA_1(\partial K_{m,k})$$
for the same constant $c$. We associate to each $\gamma^m_k$ a “square annulus”

$$A^m_k = \{w : \text{dist}(w, \gamma^m_k) \leq \varepsilon_0\sigma_m\}$$
and we choose $\varepsilon_0 > 0$ so small that the annuli $A^m_k$ are pairwise disjoint.

Define $\Omega = \mathbb{C}\setminus K$. Since $K$ has positive logarithmic capacity, Green’s function $G(z, \zeta)$ exists for $\zeta, z \notin K$, and harmonic measure $\omega(\zeta, E)$ exists for $\zeta \notin K$ and $E \subset K$. We write $\omega(\zeta, K_{m,k})$ for $\omega(\zeta, K_{m,k} \cap K)$.

We also define the slightly larger “squares”
$$S_{m,k} = \{w : \text{dist}(w, K_{m,k}) \leq \varepsilon_1\sigma_m\}$$
and set
$$S_m = \bigcup_{k=1}^{4^m} S_{m,k},$$
where $\varepsilon_1 > 0$ is so small that $S_{m,k} \cap A^m_k = \emptyset$. Then $K = \bigcap_{m=1}^{\infty} S_m$. Green’s function and harmonic measure also exist for the domain $\Omega_m = \mathbb{C}\setminus S_m$. Denote these by $G_m(z, \zeta)$ and $\omega_m(\zeta, E)$ respectively.

**Lemma 2.** Let $z \in A^m_k$.

(a) There are constants $0 < c_1 < c_2 < 1$, independent of $k$ and $m$, such that
$$c_1 \leq \omega_m(z, \partial S_{m,k}) \leq c_2.$$

(b) If $\zeta \in \Omega$ and $1 \geq \text{dist}(\zeta, K) \geq 2 \text{dist}(z, K)$, then
$$G_m(z, \zeta) \sim \omega_m(\zeta, \partial S_{m,k}).$$

**Proof.** For (a) note that there is $c' > 0$ such that there exists a second square annulus $B_k^m$ so that $A_k^m \subset B_k^m \subset \Omega_m$ and $\text{dist}(z, \partial B_k^m) \geq c'\sigma_m$. The lower bound then follows by a comparison with $B_k^m$. There is $S_{m,j}$ with $j \neq k$ such that $\text{dist}(S_{m,j}, S_{m,k}) \leq c_4\sigma_m$ and the upper bound follows by a comparison with $\mathbb{C}\setminus(S_{m,k} \cup S_{m,j})$, using symmetry and Harnack’s inequality.

To prove (b) note first that as in the proof of (a) there are constants $C_1$ and $C_2$ such that by Harnack’s inequality and a comparison
$$C_1 \leq G_m(z, w) \leq C_2$$
for $w \in \partial B_k^m$. Then using the symmetry of Green’s function and (a) for a larger square we obtain

\[ C_1 \omega_m(\zeta, \partial S_{m,k}) \leq G_m(\zeta, z) \leq C_2 \omega_m(\zeta, \partial S_{m,k}). \]

We write $\gamma_k^m \prec \gamma_j^n$ and say $\gamma_k^m$ is subordinate to $\gamma_j^n$ if $\gamma_j^n$ has winding number one about $\gamma_k^m$. If the winding number is zero, we write $\gamma_k^m \not\prec \gamma_j^n$. For any $f \in A(K, 1)$ and $\gamma_k^m$ define

\[ D(\gamma_k^m) = \sup_{w \in \gamma_k^m} |f'(w)| \sigma_n. \]

We say a square $\gamma_k^m$ has condition J if

\[ D(\gamma_k^m) \leq \delta \]

for some previously defined $f$ and $\delta > 0$.

**Lemma 3.** Let $f \in A(K, 1)$. For every $\delta > 0$ there exists a $C_0 > 0$ such that for every $\gamma_j^n$ there exists $\gamma_k^m \prec \gamma_j^n$ such that $m \leq n + C_0 \delta^{-2}$ and such that $\gamma_k^m$ has condition J.

**Proof.** Observe that by Harnack’s inequality

\[ \sup_{\gamma_{n,j}} |f'(z)|^2 \sim \int \int_{A_k^e} \frac{|f'|^2}{\sigma_n^2}. \]

Suppose the lemma is false. Choose $\zeta$ with $\text{dist}(\zeta, K) = 1$. Then by Green’s theorem and the above observation

\[ 4 \geq \int_{\partial \Omega_n} |f(z) - f(\zeta)|^2 d\omega_n(\zeta, z) \]
\[ = \int_{\Omega_n} |f'(z)|^2 G_n(z, \zeta) dx dy \]
\[ \geq n \sum_{t=m+1}^{n} \sum_{j} \int_{A_j^e} |f'(z)|^2 G_n(z, \zeta) dx dy \]
\[ \geq C \delta^2 \sum_{t=m+1}^{n} \sum_{j} \omega(\zeta, S_{t,j} \cap K) \]
\[ \geq C'(n - m) \delta^2 \]

and we have a contradiction.

3. A STOPPING-TIME ARGUMENT

We choose $n_\delta = 4^{Mq}$ where $q > 1$ and $M = \left[1 + \frac{C_0}{\delta^2}\right]$. Then because $p > 2$ in the definition of $a_n = (\log(n + 1))^{\frac{1}{p}}$ we have

\[ \lim_{\delta \to 0^+} \delta \cdot a_n M = 0 \]

and

\[ \lim_{\delta \to 0^+} \left(1 - 4^{-M}\right)^{n_\delta} a_n M = 0. \]

By construction, either $\gamma_k^m \prec \gamma_j^n$, $\gamma_j^n \prec \gamma_k^m$, or neither is subordinate to the other. We also write $\gamma_k^m \not\prec F$ if $\gamma_k^m \not\prec \gamma_j^n$ for all $\gamma_j^n \in F$ where $F$ is some family of $\gamma_j^n$. 
Lemma 4. For every $\epsilon > 0$, there exists $\delta > 0$, integer $m$ and two families of sets $F_1$ and $F_2$, such that for some constant $c$:

(a) $F_1 = \{ \gamma_j^n : \gamma_j^n \text{ has condition J} \}$,
(b) $\delta \Lambda_1(\bigcup_{F_1} \gamma_j^n) < c \epsilon$,
(c) $F_2 = \{ \gamma_k^m : \gamma_k^m \neq F_1 \}$,
(d) $\Lambda_1(\bigcup_{F_2} \gamma_k^m) < c \epsilon$,
(e) $F_1 \cup F_2$ has winding number 1 about $K$.

Proof. Given $\epsilon > 0$, choose $\delta > 0$ so that $\delta a_{n,M} < \epsilon$ and $(1 - 4^{-M}) a_{n,M} < \epsilon$. Fix $m = n_\delta M$.

Now define $F_1$ to be the set of $\gamma_k^n$ such that $n \leq m$, $\gamma_k^n$ has condition J, and $\gamma_k^n$ is maximal, i.e. if $K_{n,k} \subset K_{t,j}$ with $t < n$, then $\gamma_t^j$ does not have condition J. Then (a), (c) and (e) hold for $F_1$ and $F_2$.

To prove (b) consider $\gamma_j^n \in F_1$. Since $0 \leq n \leq m$ we may replace $\gamma_j^n$ by $4^{m-n}$ squares of the form $\gamma_k^m$. Consequently,
\[ \Lambda_1(\bigcup_{F_1} \gamma_j^n) \leq 4^{m-n} \cdot \sigma_m = 4^{-n} a_m. \]

Since the $\gamma_j^n \in F_1$ have pairwise disjoint $K_{n,j}$, $\bigcup_{F_1} \gamma_j^n$ has smaller $\Lambda_1$ measure than $\bigcup_{k=1}^{4^m} \gamma_k^m$ and therefore
\[
\delta \Lambda_1 \left( \bigcup_{F_1} \gamma_j^n \right) \leq \delta \Lambda_1 \left( \bigcup_{k=1}^{4^m} \gamma_k^m \right) \\
\leq c \delta \cdot 4^m \cdot 4^{-m} a_m \\\n= c \delta a_{n,M} \\\n\leq c \epsilon,
\]
where $c$ is a universal constant.

To prove (d) we use Lemma 3 to obtain
\[
\Lambda_1 \left( \bigcup_{F_2} \gamma_k^m \right) \leq c(4^M - 1)^m 4^{-m} a_m \\\n\leq c(1 - 4^{-M}) a_{n,M} \\\n\leq c \epsilon.
\]

4. Proof of Theorem 1

Suppose $f \in A(K,1)$ and $\epsilon > 0$ are arbitrary. Let $F_1$ and $F_2$ be the two families provided by Lemma 4. Let $z_k^m$ be an arbitrary point in $\gamma_k^m$. Then
\[
2\pi |f'(\infty)| = \left| \sum_{\gamma_k^m \in F_1} \int_{\gamma_k^m} f(z) dz + \sum_{\gamma_k^m \in F_2} \int_{\gamma_k^m} f(z) dz \right| \\
\leq \left| \sum_{\gamma_k^m \in F_1} \int_{\gamma_k^m} f(z) dz \right| + \left| \sum_{\gamma_k^m \in F_2} \int_{\gamma_k^m} f(z) dz \right| \\
\leq \sum_{\gamma_k^m \in F_1} \int_{\gamma_k^m} |f(z) - f(z_k^m)| dz + \Lambda_1(\bigcup_{F_2} \gamma_k^m) \].
\[ \leq c \sum_{\gamma_k^m \in F_1} \sup_{\gamma_k^m, w \in \gamma_k^m} |f'(w)| 4^{-m} a_m dz + \varepsilon \]

\[ = c \sum_{\gamma_k^m \in F_1} D(\gamma_k^m) \Lambda_1(\gamma_k^m) + \varepsilon \]

\[ \leq c \delta \sum_{\gamma_k^m \in F_1} \Lambda_1(\gamma_k^m) + \varepsilon \]

\[ \leq c \delta \sigma_{\eta s} M + \varepsilon \]

\[ \leq c \varepsilon . \]

Since \( \varepsilon \) was chosen arbitrarily and \( c \) is a universal constant, \( f'(\infty) = 0 \). Therefore, \( \gamma(K) = 0 \).

5. REMARK

We could obtain a better result if we could improve the estimate in Jones’ lemma (Lemma 3). For example, if we could only replace \( M = \frac{C_3}{\delta} \) by \( \frac{C_3}{\delta^q} \) for \( q < 2 \), then in the theorem \( a_n \) could grow like \( (\log n)^{\frac{1}{q}} \). As noted above, Mattila [M96] conjectured that \( \gamma(K) = 0 \) if the Cantor set \( K \) has \( \sum_{(a_n)^{1/2}} = +\infty \). Mattilla’s conjecture would follow from the method here if the Jones’ lemma could be established with \( M = c \log(\frac{1}{\delta}) \) with \( c \) constant.

REFERENCES


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