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LARGE SETS OF ZERO ANALYTIC CAPACITY

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ABSTRACT. We prove that certain Cantor sets with non-sigma-finite one-dimensional Hausdorff measure have zero analytic capacity.

1. INTRODUCTION

In this paper we consider a Cantor set K similar to the $\frac{1}{4}$-Cantor set of [G70] and [I84]. Fix $p > 2$ and for $n > 0$ define

$$\sigma_n = 4^{-n} a_n = 4^{-n} [\log(n+1)]^{1/p}.$$

Set $K_0 = [0,1] \times [0,1]$ and $K_1 = \bigcup_{j=1}^{4} K_{1,j}$, where $K_{1,j} \subset K_0$ is a square of sidelength σ_1 having sides parallel to the axis and containing one of the four corners of K_0. Next take 4^2 squares $K_{2,j}$ of sidelength σ_2, one in each corner of each square $K_{1,j}$, and define $K_2 = \bigcup_{j=1}^{4^2} K_{2,j}$. Continuing we obtain $K_n = \bigcup_{j=1}^{4^n} K_{n,j}$, where $K_{n,j}$ is a square of sidelength σ_n. The Cantor set we study is

$$K = K(p) = \bigcap_{n=1}^{\infty} K_n.$$

If E is a compact plane set define

$$A(E,1) = \{f : f \text{ analytic on } E^c,\ f(\infty) = 0,\ \| f \|_{L^\infty(E^c)} \leq 1\}$$

and define the analytic capacity of E by

$$\gamma(E) = \sup\{| f'(\infty) | : f \in A(E,1)\},$$

where

$$f'(\infty) = \lim_{z \to \infty} z f(z).$$

If $\gamma(E) = 0$, then the only function in $A(E,1)$ is the constant $f \equiv 0$ and in this case E is removable for bounded analytic functions. For more details about analytic capacity see [G72].

Theorem 1. *Let $p > 2$, and let K be the four-corner Cantor set $K(p)$. Then $\gamma(K) = 0$ but K does not have σ-finite one-dimensional measure.*

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The proof of Theorem 1 depends on a lemma of Jones [J89] used for a proof different from [G70] that the $\frac{1}{4}$-Cantor set has zero analytic capacity.

Let $h(t)$ be an increasing continuous function on $t \geq 0$ with $h(0) = 0$, and write $\Lambda_{h(t)}(E)$ for the Hausdorff h-measure of E. Now define an increasing function $h(t)$ so that $h(0) = 0$ and $h(\sigma_n) = 4^{-n}$ for all n. We say $h(t)$ is a **measure function corresponding to the Cantor set** K. For every n define a measure μ_n on K_n by $\mu_n(K_{n,j}) = 4^{-n}$ for all j. Then $\{\mu_n\}$ converges weak-star to a measure μ supported on K and satisfying $\mu(K_{n,j}) = 4^{-n}$. Suppose $\frac{\sqrt{2}}{2}\sigma_n \leq r < \frac{\sqrt{2}}{2}\sigma_{n-1}$ and let $D(z,r)$ be a disk of radius r and center $z \in K$. Then $D(z,r)$ can meet at most 4 squares of sidelength σ_n. Hence

$$\mu(D(z,r)) \leq 4\mu(K_{n,j}) = 4 \cdot 4^{-n} = 4h(\sigma_n) \leq 4h(r),$$

so that $\mu(D(z,r)) \leq 16h(r)$ for any disk $D(z,r)$. Therefore $\Lambda_h(K) > 0$ by Frostman's Theorem [G72]. Since

$$\lim_{t \to 0} \frac{h(t)}{t} = 0,$$

if follows that K has non-σ-finite 1-dimensional measure.

If $h(t)$ is a measure function corresponding to K, then

$$\int_0^1 \frac{h(t)^2}{t^3} dt \sim \sum_{n=1}^{\infty} \frac{1}{(a_n)^2} = \sum_{n=1}^{\infty} \frac{1}{(\log n)^{\frac{2}{p}}} = \infty.$$

On the other hand, Mattila [M96] proved that $\gamma(K) > 0$ if K is a Cantor set built with squares of side σ_n and if

$$\int_0^1 \frac{h(t)^2}{t^3} dt < \infty,$$

where h is any measure function for corresponding to K. Mattila's proof used Menger curvature (see [Me95] and [MMV96]). However, if the Cantor set K has corresponding measure function h satisfying

$$\int_0^1 \frac{h(t)^2}{t^3} dt = \infty,$$

then Eiderman [E98] proved that $\gamma^+(K) = 0$, where

$$\gamma^+(E) = \sup \left\{ \int_E d\mu : \left| \int_E \frac{d\mu(\zeta)}{\zeta - z} \right| < 1, \forall z \in \mathbb{C} \backslash E,\ \mu > 0,\ spt(\mu) \subset\ E \right\}.$$

Since $\gamma^+(E) \leq \gamma(E)$, our result is a partial improvement of Eiderman's theorem. Mattila [M96] has conjectured that for Cantor sets of this type $\gamma(K) = 0$ if and only if

$$\int_0^1 \frac{h(t)^2}{t^3} dt = \infty,$$

when h corresponds to K. This latter condition holds if and only if

$$\sum_{n=1}^{\infty} \frac{1}{(a_n)^2} = \infty.$$

If Matilla's conjecture is true, then together with Eiderman's theorem it gives Cantor set evidence supporting the more ambitious conjecture that $\gamma(E) > 0$ implies $\gamma^+(E) > 0$.

In [G72] it was incorrectly claimed that $\gamma(K) > 0$ if and only if

$$\int_0^1 \frac{h(t)}{t^2} dt < \infty.$$

Eiderman, however, found a mistake in the proof. In fact the result in [M96] shows that the claim is false. See L.D. Ivanov [I84] for the first example of a Cantor set of non-σ-finite linear measure and zero analytic capacity.

2. Two lemmas of Peter Jones

We need the following two lemmas from [J89]. The proofs we give are small variations on [J89] and [C90].

Define $\gamma_j^n = \partial cK_{n,j}$, where $cK_{n,j}$ is the square concentric to $K_{n,j}$ with sidelength $c\sigma_n$ and where $c > 1$ is chosen so that the γ_j^n do not overlap. We refer to γ_k^m as a square, although it is only the boundary of a square. Notice that

$$\Lambda_1(\gamma_k^m) = c\Lambda_1(\partial K_{m,k})$$

for the same constant c. We associate to each γ_k^m a "square annulus"

$$A_k^m = \{w : \operatorname{dist}(w, \gamma_k^m) \leq \varepsilon_0 \sigma_m\} \tag{1}$$

and we choose $\varepsilon_0 > 0$ so small that the annuli A_k^m are pairwise disjoint.

Define $\Omega = \overline{\mathbb{C}} \backslash K$. Since K has positive logarithmic capacity, Green's function $G(z, \zeta)$ exists for $\zeta, z \notin K$, and harmonic measure $\omega(\zeta, E)$ exists for $\zeta \notin K$ and $E \subset K$. We write $\omega(\zeta, K_{m,k})$ for $\omega(\zeta, K_{m,k} \cap K)$.

We also define the slightly larger "squares"

$$S_{m,k} = \{w : \operatorname{dist}(w, K_{m,k}) \leq \varepsilon_1 \sigma_m\}$$

and set

$$S_m = \bigcup_{k=1}^{4^m} S_{m,k},$$

where $\varepsilon_1 > 0$ is so small that $S_{m,k} \cap A_k^m = \emptyset$. Then $K = \bigcap_{m=1}^{\infty} S_m$. Green's function and harmonic measure also exist for the domain $\Omega_m = \overline{\mathbb{C}} \backslash S_m$. Denote these by $G_m(z, \zeta)$ and $\omega_m(\zeta, E)$ respectively.

Lemma 2. *Let $z \in A_k^m$.*

(a) There are constants $0 < c_1 < c_2 < 1$, independent of k and m, such that

$$c_1 \leq \omega_m(z, \partial S_{m,k}) \leq c_2.$$

(b) If $\zeta \in \Omega$ and $1 \geq \operatorname{dist}(\zeta, K) \geq 2 \operatorname{dist}(z, K)$, then

$$G_m(z, \zeta) \sim \omega_m(\zeta, \partial S_{m,k}).$$

Proof. For (a) note that there is $c' > 0$ such that there exists a second square annulus B_k^m so that $A_k^m \subset B_k^m \subset \Omega_m$ and $\operatorname{dist}(z, \partial B_k^m) \geq c'\sigma_m$. The lower bound then follows by a comparison with B_k^m. There is $S_{m,j}$ with $j \neq k$ such that $\operatorname{dist}(S_{m,j}, S_{m,k}) \leq c_4 \sigma_m$ and the upper bound follows by a comparison with $\overline{\mathbb{C}} \backslash (S_{m,k} \cup S_{m,j})$, using symmetry and Harnack's inequality.

To prove (b) note first that as in the proof of (a) there are constants C_1 and C_2 such that by Harnack's inequality and a comparison

$$C_1 \leq G_m(z, w) \leq C_2$$

for $w \in \partial B_k^m$. Then using the symmetry of Green's function and (a) for a larger square we obtain

$$C_1\omega_m(\zeta, \partial S_{m,k}) \leq G_m(\zeta, z) \leq C_2\omega_m(\zeta, \partial S_{m,k}).$$

We write $\gamma_k^m \prec \gamma_j^n$ and say γ_k^m is **subordinate** to γ_j^n if γ_j^n has winding number one about γ_k^m. If the winding number is zero, we write $\gamma_k^m \not\prec \gamma_j^n$. For any $f \in A(K,1)$ and γ_k^m define

$$D(\gamma_k^m) = \sup_{w \in \gamma_k^m} |f'(w)|\, \sigma_n.$$

We say a square γ_k^m has **condition J** if

$$D(\gamma_k^m) \leq \delta$$

for some previously defined f and $\delta > 0$.

Lemma 3. *Let $f \in A(K,1)$. For every $\delta > 0$ there exists a $C_0 > 0$ such that for every γ_j^n there exists $\gamma_k^m \prec \gamma_j^n$ such that $m \leq n + C_0\delta^{-2}$ and such that γ_k^m has condition J.*

Proof. Observe that by Harnack's inequality

$$\sup_{\gamma_{n,j}} |f'(z)|^2 \sim \int\int_{A_k^n} |f'|^2 \frac{dxdy}{\sigma_n^2}.$$

Suppose the lemma is false. Choose ζ with $\operatorname{dist}(\zeta, K) = 1$. Then by Green's theorem and the above observation

$$\begin{aligned}
4 &\geq \int_{\partial\Omega_n} |f(z) - f(\zeta)|^2 d\omega_n(\zeta, z) \\
&= \int_{\Omega_n} |f'(z)|^2 G_n(z,\zeta)\, dxdy \\
&\geq \sum_{t=m+1}^{n} \sum_j \int_{A_j^t} |f'(z)|^2 G_n(z,\zeta)\, dxdy \\
&\geq C\delta^2 \sum_{t=m+1}^{n} \sum_j \omega(\zeta, S_{t,j} \cap K) \\
&\geq C'(n-m)\delta^2
\end{aligned}$$

and we have a contradiction.

3. A stopping-time argument

We choose $n_\delta = 4^{Mq}$ where $q > 1$ and $M = \left[1 + \frac{C_0}{\delta^2}\right]$. Then because $p > 2$ in the definition of $a_n = (\log(n+1))^{\frac{1}{p}}$ we have

$$\lim_{\delta \to 0^+} \delta \cdot a_{n_\delta M} = 0$$

and

$$\lim_{\delta \to 0^+} \left(1 - 4^{-M}\right)^{n_\delta} a_{n_\delta M} = 0.$$

By construction, either $\gamma_k^m \prec \gamma_j^n$, $\gamma_j^n \prec \gamma_k^m$, or neither is subordinate to the other. We also write $\gamma_k^m \not\prec F$ if $\gamma_k^m \not\prec \gamma_j^n$ for all $\gamma_j^n \in F$ where F is some family of γ_j^n.

Lemma 4. *For every $\varepsilon > 0$, there exists $\delta > 0$, integer m and two families of sets F_1 and F_2, such that for some constant c:*
(*a*) $F_1 = \{\gamma_j^n : \gamma_j^n$ *has condition* $J\}$,
(*b*) $\delta\Lambda_1(\bigcup_{F_1} \gamma_j^n) < c\varepsilon$,
(*c*) $F_2 = \{\gamma_k^m : \gamma_k^m \not\prec F_1\}$,
(*d*) $\Lambda_1(\bigcup_{F_2} \gamma_k^m) < c\varepsilon$,
(*e*) $F_1 \bigcup F_2$ *has winding number* 1 *about* K.

Proof. Given $\varepsilon > 0$, choose $\delta > 0$ so that $\delta a_{n_\delta M} < \varepsilon$ and $(1 - 4^{-M})^{n_\delta} a_{n_\delta M} < \varepsilon$. Fix $m = n_\delta M$.

Now define F_1 to be the set of γ_k^n such that $n \le m$, γ_k^n has condition J, and γ_k^n is **maximal**, i.e. if $K_{n,k} \subset K_{t,j}$ with $t < n$, then γ_j^t does not have condition J. Then $(a), (c)$ and (e) hold for F_1 and F_2.

To prove (b) consider $\gamma_j^n \in F_1$. Since $0 \le n \le m$ we may replace γ_j^n by 4^{m-n} squares of the form γ_k^m. Consequently,

$$\Lambda_1(\gamma_j^n) \le 4^{m-n} \cdot \sigma_m = 4^{-n} a_m.$$

Since the $\gamma_j^n \in F_1$ have pairwise disjoint $K_{n,j}$, $\bigcup_{F_1} \gamma_j^n$ has smaller Λ_1 measure than $\bigcup_{k=1}^{4^m} \gamma_k^m$ and therefore

$$\begin{aligned} \delta\Lambda_1(\bigcup_{F_1} \gamma_j^n) &\le \delta\Lambda_1(\bigcup_{k=1}^{4^m} \gamma_k^m) \\ &\le c\delta \cdot 4^m \cdot 4^{-m} a_m \\ &= c\delta a_{n_\delta M} \\ &\le c\varepsilon, \end{aligned}$$

where c is a universal constant.

To prove (d) we use Lemma 3 to obtain

$$\begin{aligned} \Lambda_1(\bigcup_{F_2} \gamma_k^m) &\le c(4^M - 1)^m 4^{-m} a_m \\ &\le c(1 - 4^{-M})^{n_\delta} a_{n_\delta M} \\ &\le c\varepsilon. \end{aligned}$$

4. Proof of Theorem 1

Suppose $f \in A(K, 1)$ and $\varepsilon > 0$ are arbitrary. Let F_1 and F_2 be the two families provided by Lemma 4. Let z_k^m be an arbitrary point in γ_k^m. Then

$$\begin{aligned} 2\pi\,|f'(\infty)| &= \left| \sum_{\gamma_k^m \in F_1} \int_{\gamma_k^m} f(z)dz + \sum_{\gamma_k^m \in F_2} \int_{\gamma_k^m} f(z)dz \right| \\ &\le \left| \sum_{\gamma_k^m \in F_1} \int_{\gamma_k^m} f(z)dz \right| + \left| \sum_{\gamma_k^m \in F_2} \int_{\gamma_k^m} f(z)dz \right| \\ &\le \sum_{\gamma_k^m \in F_1} \int_{\gamma_k^m} |f(z) - f(z_k^m)|\, dz + \Lambda_1(\bigcup_{F_2} \gamma_k^m) \end{aligned}$$

$$
\begin{aligned}
&\leq c \sum_{\gamma_k^m \in F_1} \int_{\gamma_k^m} \sup_{w\in\gamma_k^m} |f'(w)|\, 4^{-m} a_m dz + \varepsilon \\
&= c \sum_{\gamma_k^m \in F_1} D(\gamma_k^m)\Lambda_1(\gamma_k^m) + \varepsilon \\
&\leq c\delta \sum_{\gamma_k^m \in F_1} \Lambda_1(\gamma_k^m) + \varepsilon \\
&\leq c\delta a_{n_\delta M} + \varepsilon \\
&\leq c\varepsilon.
\end{aligned}
$$

Since ε was chosen arbitrarily and c is a universal constant, $f'(\infty) = 0$. Therefore, $\gamma(K) = 0$. □

5. Remark

We could obtain a better result if we could improve the estimate in Jones' lemma (Lemma 3). For example, if we could only replace $M = \frac{C_0}{\delta^2}$ by $\frac{C_0}{\delta^q}$ for $q < 2$, then in the theorem a_n could grow like $(\log n)^{\frac{1}{q}}$. As noted above, Mattila [M96] conjectured that $\gamma(K) = 0$ if the Cantor set K has $\sum \frac{1}{(a_n)^2} = +\infty$. Matilla's conjecture would follow from the method here if the Jones' lemma could be established with $M = c\log(\frac{1}{\delta})$ with c constant.

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