Bell’s spaceships: a useful relativistic paradox

Bell’s spaceship ‘paradox’ [1] in special relativity is a particularly good one to examine with students, because although it deals with accelerated motions, it can be dissolved with elementary space–time diagrams. Furthermore, it forces us to be very clear about the relativity of simultaneity, proper length, and the ‘reality’ of the Lorentz contraction.

**Bell’s spaceships: the paradox**

Bell asks us to imagine three spaceships $A$, $B$ and $C$ that drift freely in a region of space distant from other matter. The three spaceships are originally in a state of relative rest. $B$ and $C$ are equidistant from $A$. When $B$ and $C$ receive a light signal from $A$, they begin to accelerate gently. The ships $B$ and $C$ are assumed to be identical in all relevant respects and to have ‘identical acceleration programmes’. Suppose we tie a string between $B$ and $C$ just long enough to span the distance between them before they begin accelerating. The apparent paradox concerns whether the string breaks. Since $B$ and $C$, and hence the string, are speeding up relative to $A$, the length of the string should Lorentz contract. So it seems the string should break. However, measurements by an observer $O_A$ in $A$ reveal that $B$ and $C$ remain equidistant, since $B$ and $C$ have the same velocity at every instant. So doesn’t this show that the string should not break?

Bell’s solution is that ‘as the rockets speed up, it [the string] will become too short, because of its need to Fitzgerald contract, and must finally break. It must break when, at sufficiently high velocity, the artificial prevention of the natural contraction imposes intolerable stress’. Notice the two elements of the paradox. First, one wants to show that indeed the string breaks, which is enough of a lesson itself, as the string would not break if the spaceships were in Newtonian spacetime. Second, one wants to explain why the string breaks. It is here that we part ways with Bell, for his subtle explanation requires familiarity with relativistic electrodynamics and computer integration, and weighty assumptions about the constitution of matter.

![Space–time diagram showing the configuration of the spaceships for Bell’s paradox. Event a is the emission of the light pulse from ship A (whose worldline is not shown). Events b and c are the reception events, when ships B and C begin to accelerate gently.](image)

**Figure 1.** Space–time diagram showing the configuration of the spaceships for Bell’s paradox. Event $a$ is the emission of the light pulse from ship $A$ (whose worldline is not shown). Events $b$ and $c$ are the reception events, when ships $B$ and $C$ begin to accelerate gently. **Figure 1(b).** Construction of co-moving inertial co-ordinate system $K'$ at a point $p$ on an arbitrary non-inertial worldline $W$.

**Bell’s spaceships: dissolving the paradox**

One can show, using only elementary space–time diagrams, that the following three statements are true:

1. Observers $O_B$ in $B$ and $O_C$ in $C$ each conclude that the string must break because (a) $O_C$ measures $B$ lagging further and further behind and (b) $O_B$ measures $C$ pulling further and further ahead.
2. The distance between the spaceships as measured by $O_A$ does not change.
3. $O_A$ judges that the string will break.

We make three assumptions to draw all our space–time diagrams. First, we suppress two spatial dimensions. Second, we treat the spaceships as idealized point particles and label their worldlines $A$, $B$ and $C$ respectively. Third, we interpret Bell’s requirement that $B$ and $C$ have ‘identical acceleration programmes’ as the condition that the worldlines $B$ and $C$ are parallel but non-inertial paths in space–time. For example, in the reference frame in which the spaceships are originally at rest, the worldline $C$ is
the same as the worldline $B$ but it is shifted some co-
ordinate distance $\Delta x$ in the positive $x$-direction.

All of our constructions begin with a base dia-
gram (figure 1(a)), which illustrates the worldlines
$B$ and $C$, the event $a$ when the light is emitted from
spaceship $A$, and the events $b$ and $c$ when the light is
received by spaceships $B$ and $C$ respectively. Our
demonstration also often requires that we draw the
axes of an inertial co-ordinate system $K'$ moment-
arily co-moving with a particle whose worldline is
a non-inertial path. We will use the following stan-
dard construction and adopt the notation introduced
therein (figure 1(b)). Let $W$ be an arbitrary non-iner-
tial worldline and $p$ an event on that worldline. Let
$v_p$ be the tangent vector to $W$ at $p$. $v_p$ is the four-
vectors of the particle at $p$. The $t'$-axis of $K'$ lies on
$v_p$. The $x'$-axis of $K'$ lies on the plane of simultanei-
ty $\Sigma_{W}(p)$ for $W$ at $p$. To construct $\Sigma_{W}(p)$, we draw
the light-cone $L$ at $p$, measure the angle $\theta$ between
$v_p$ and $L$, and draw a line through $p$ that makes an
angle $\theta$ on the other side of $L$. $\Sigma_{W}(p)$ is the col-
clection of events that are simultaneous with $p$ for an
observer moving with four-velocity $v_p$.

We show that statement (1) is true by drawing
space–time diagrams to illustrate statements (1a)
and (1b). To illustrate (1a), we begin with our base
diagram (figure 1(a)). We select an arbitrary point
$p$ on $C$ and draw the tangent vector $v_p$ (figure 2(a)).

**Figure 2(a).** Space–time diagrams showing that
$O_B$ and $O_C$ both judge the string breaks. Here $O_C$
judges that the trailing spaceship $B$ is falling
further and further behind. $v_p$ is inclined more
toward the light cone $L$ than $v_q$.

We then draw the plane of simultaneity $\Sigma_C(p)$ and
the tangent vector $v_q$ at the point $q$ where $\Sigma_C(p)$ in-
sects $B$. We note that $v_q$ is not parallel to $v_p$.
Consequently, $O_C$ judges that at this ‘instant’, $B$’s
four-velocity is less than $C$’s. As one moves along
$C$, i.e., as $O_C$’s proper time elapses, the planes of simultanei-
ty for the co-moving inertial reference frames tilt, which results in an increasing difference between $C$’s velocity and $B$’s. Thus, $C$ judges that $B$ is lagging further and further behind. If we start
by selecting an arbitrary point $s$ on $B$, construct the
plane of simultaneity $\Sigma_B(s)$ on that point, find the
point of intersection $r$ with $C$, and compare $v_r$ and
$v_s$, we can show that (1b) is true (figure 2(b)).

To show that statement (2) is true, we begin by
superimposing $O_A$’s co-ordinate system $K$ on our
base diagram. For convenience, we shift the origin
away from the centre of the spaceship $A$, and align
the co-ordinate system so that the events $b$ and $c$ lie
on the $x$-axis (figure 3). We select an arbitrary point
$j$ on $B$ and construct the plane of simultaneity $\Sigma_A(j)$
for $O_A$, which is a line parallel to $O_A$’s $x$-axis. We
then draw the tangent vectors $v_j$ and $v_b$ at the events
$j$ and $k$ where $\Sigma_A(j)$ intersects $B$ and $C$ respectively.
We note that $v_j$ and $v_b$ are parallel. Consequently,
$O_A$ judges that the two spaceships have the same
velocity at every instant, and hence they remain the
same distance apart as measured in $K$. So why does
$O_A$ judge that the string will break?

$O_A$ reasons as follows: ‘The taught string spans
the distance between $B$ and $C$ at $t=0$. Since the string moves non-inertially, it is as if it is moving from one Lorentz-boosted frame to another, and each subsequent reference frame has a higher value of the Lorentz factor $\gamma(v)$. Consequently, to keep the string taught, without changing the tension on the string, the distance between the spaceships as measured in $K$ should decrease continuously. Since the distance between the spaceships in $K$ remains constant, the spaceships are exerting a force on the string. Therefore, the string will break. This reasoning shows that (3) is true. However, we can do more.

We can draw the worldline $C^*$ (approximately) that spaceship $C$ would have to follow to keep the string taught if we leave the worldline $B$ unchanged. To keep the string taught, the distance between the spaceships $C$ and $B$ has to remain constant in successive co-moving inertial reference frames. To generate $C^*$, at arbitrarily selected events $p_i$ on $B$, we draw the co-moving inertial co-ordinate system $K'$ (adding primes as necessary to distinguish different co-ordinate systems), calibrate the $x'$-axis, and find the event $r_j$ where the front of the string would be if its length did not change, i.e., if it remained taught. We then trace a smooth curve through the events $r_j$ to get a sense of $C^*$. Comparing $C^*$ and $C$, it is clear that if the front spaceship follows path $C$, it pulls on the string. Consequently, the string must break.

The details of the construction are as follows (figure 4). We begin with our base diagram (figure 1(a)). We draw a copy of $K$ off to the side. We draw the light cone $L$ at the origin of $K$ and the hyperbola $H$ defined by $x^2 - r^2 = s^2$, where $s$ is the length of the taught string. We label the origin $p_1$ and the intersection of $H$ and the $x'$-axis $q_1$, to anticipate that we will ‘stamp’ this entire diagram, complete with light cone and hyperbola, so that the $p_i$ fall on $B$. The points $q_i$ will fall on $C$. This is our calibrator diagram (figure 4(a)).

To approximate $C^*$, we ‘stamp’ the calibrator diagram several times on our base diagram so that the $p_i$ fall on $B$. At each $p_i$, we draw the $t'$-axis and $x'$-axis of $K'$ (again adding primes as necessary). The intersection of $H$ and $x'$ we label $r_i$. Since $H$ is an invariant, we have $x'^2 - r'^2 = s^2$. Consequently, the intersection of $H$ and $x'$ designates where the front end of the string would be if it was being kept taught. Thus, the path the front ship would have to follow to keep the string taught is approximated by the smooth curve $C^*$ through the events $r_i$ (figure 4(b)). Finally, we can draw planes of simultaneity $\Sigma_A(r_i)$ to illustrate that to keep the string taught, the spaceships must move so that the distance between them, as measured in $K$, must decrease continuously (figure 5).

Figure 3. Space–time diagram showing that $O_A$ judges that the spaceships $B$ and $C$ remain a constant distance apart in her reference frame, since $v_j$ and $v_k$ are parallel.

Figure 4. Space–time diagram showing the path $C^*$ that the spaceship $C$ would have to follow to keep the string taught without breaking. Figure 4(a). The calibrator diagram. Figure 4(b). The construction of $C^*$ from the events $r_i$ a constant distance from events $p_i$ on $B$ at successive co-moving inertial frames.
The photoelectric effect: a useful sporting analogy

Teachers of science routinely use analogies to help students grasp complex ideas; most of those that I use are spontaneous and respond to the needs of the particular group of students at that moment in time. However, there is a pair of analogies that I have used many times to great success when explaining key features of the photoelectric effect, the concept of the liberation of a photoelectron depending on photon energy rather than on total intensity, and how incident photon energy relates to work function and maximum kinetic energy of the photoelectron.

Intensity of radiation and photon energy

I find that students have difficulty with the idea that no matter how bright the source the photoelectric effect will not be observed unless a certain threshold energy is reached, that corresponding to the work function. I point out that one photon can liberate a photoelectron if it is energetic enough, whereas a billion photons whose energy is below the value of the work function will have no effect. The analogy that I use comes from rugby. Imagine a set of rugby goalposts and now take 500 little children with rugby balls. Ask the children to kick the balls over the bar and you will probably find that not one achieves this feat (and in this example they mustn’t!). This is analogous to our sub-threshold photons; in our rugby exercise we have a high intensity, as there are a lot of balls in the air, but none of the balls has enough individual energy to cross the bar. The sum of all the energies may be high and...